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TOWARZYSTWA NAUKOWEGO WARSZAWSKIEGO

Wydział III

nauk matematyczno-fizycznych

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WARSZAWA
NAKŁADEM TOWARZYSTWA NAUKOWEGO WARSZAWSKIEGO
Z ZASIŁKU MINISTERSTWA WYZNAŃ RELIGIJNYCH I OŚWIECENIA PUBLICZNEGO
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(Table des matières).

SPRAWOZDANIA Z POSIEDZEŃ
TOWARZYSTWA NAUKOWEGO WARSZAWSKIEGO
Wydział III nauk matematyczno-fizycznych.

Posiedzenie

z dnia 15 października 1932 r.

Tadeusz W. Jezierski.

O 9—10-dwuhydroksy-9—10-dwu-α-naftylo-9—10-dwuwodorofenantrenie i 2—2'-dwu-α-naftoilodwufenylu.

Przedstawił L. Szperl na posiedzeniu dn. 15 października 1932 r.

Sur le 9—10-dioxy-9—10-di-α-naphtalène-9—10-dihydrophénantrène et la 2—2'-di-α-naphtoylediphényle.

Mémoire présenté par M. L. Szperl à la séance du 15 Octobre 1932.

Wytwarzanie się odpowiednich glikoli z fenantrenochinonu i odczynnika Grignarda (z rodnikiem fenylowym [1]), metylowym, etylowym, propylowym i benzylowym [2]) opisane jest w literaturze.

Tak sporządzone dwutrzeciorzędowe alkohole poddawano utlenianiu, za pomocą CrO_3 w roztworze kw. octowego, otrzymując odpowiednie dwuketony.

Pragnąc przekonać się, czy i w tym przypadku, gdy na odczynnik Grignarda z rodnikiem naftylowym będzie działać fenantrenochinon, równie dodatni będzie wynik reakcji, działałem na bromek α-naftylomagnezowy fenantrenochinonem, otrzymując, z wydajnością ok. 55%, 9—10-dwuhydroksy-9—10-dwu-α-naftylo-9—10-dwuwodorofenantren:

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[1]) Werner i Grob. Ber. 37, 2892 (1904).
[2]) Zincke i Tropp. Ann. 362, 242 (1908); 363, 302 (1908).

$$\begin{array}{l} C_6H_4\cdot C{=}O \\ | \qquad\quad | \\ C_6H_4\cdot C{=}O \end{array} + 2\,\alpha - C_{10}H_7MgBr \rightarrow \begin{array}{l} \qquad\quad\; OH \\ \qquad\quad\;\; | \\ C_6H_4\cdot C - C_{10}H_7\,\alpha \\ | \qquad\quad\; | \\ C_6H_4\cdot C - C_{10}H_7\,\alpha \\ \qquad\quad\;\; | \\ \qquad\quad\; OH \end{array}$$

który jest substancją stałą, krystaliczną, bezbarwną, o temperat. t. 263—264^0, trudnorozpuszczalną w alkoholu, kw. octowym, łatwiej w benzenie, acetonie.

Otrzymany w powyższy sposób pinakon poddawałem, po rozpuszczeniu go w kw. octowym, utlenianiu za pomocą CrO_3 (w postaci 10% roztworu wodnego); powstał wtedy, z wydajnością ok. 60%, 2—2′-dwu-α-naftoylodwufenyl:

$$3\begin{array}{l} \qquad\quad\; OH \\ \qquad\quad\;\; | \\ C_6H_4\cdot C - C_{10}H_7\,\alpha \\ | \qquad\quad\; | \\ C_6H_4\cdot C - C_{10}H_7\,\alpha \\ \qquad\quad\;\; | \\ \qquad\quad\; OH \end{array} + 2\,CrO_3 \rightarrow 3\begin{array}{l} C_6H_4\cdot CO\cdot C_{10}H_7\,\alpha \\ | \\ C_6H_4\cdot CO\cdot C_{10}H_7\,\alpha \end{array}$$

Dwuketon ten jest ciałem stałem, bezbarwnem, krystalicznem, topiącem się w temp. 204, 5—205^0; trudno rozpuszcza się w alkoholu, łatwiej w wrzącym kw. octowym, acetonie i benzenie.

Praca in extenso będzie drukowana w „Rocznikach Chemji”.

Antoni Łaszkiewicz.

O miedzi rodzimej.

Przedstawił St. J. Thugutt dn. 15 października 1932 r.

Sur le cuivre natif.

Mémoire présenté par M. St. J. Thugutt dans la séance du 15 Octobre 1932.

Praca niniejsza zawiera opis miedzi rodzimej z Miedzianej Góry pod Kielcami, oraz zajmuje się ustaleniem związku pomiędzy morfologją i strukturą miedzi. Ukaże się w IX tomie Archiwum Mineralogicznego.

Posiedzenie

z dnia 26 listopada 1932 r.

Marjan Polaczek.

O kwasach akrydonosulfonowych.

Przedstawił L. Szperl dnia 26 listopada 1932 r.

Sur les acides acridone-soulfoniques.

Mémoire présenté par M. L. Szperl dans la séance du 26 Novembre 1932.

Streszczenie.

Przez działanie kwasu siarkowego (1,84) w 100^0 na akrydon w ciągu 5 godzin, otrzymano po oczyszczeniu żółty osad krystaliczny. Próby jakościowe i analiza ilościowa wykazały, że jest to kwas akrydonosulfonowy.

Celem bliższego poznania tego związku i ustalenia pozycji grupy sulfonowej wykonano badania następujące:

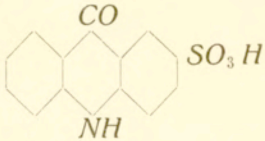
1. Z reakcji otrzymanego produktu z dwumetyloaniliną i tlenochlorkiem fosforu wyodrębniono związek o temper. topn. 250^0—251^0 o składzie: $C_{21} H_{18} O_3 N_2 S$.

2. Dokonano syntezy kwasu akrydono-2-sulfonowego z kwasu dwufenyloamino-2-karboksy-4-sulfonowego.

3. Reakcja powyższego kwasu z dwumetyloaniliną i tlenochlorkiem fosforu dała w wyniku produkt o temp. top. 250^0—251^0 identyczny z wyżej zaznaczonym.

Sporządzono i zanalizowano sole baru i ołowiu kwasu akrydono-2-sulfonowego.

Badania powyższe świadczą, że grupa sulfonowa w otrzymanym przez sulfonowanie akrydonu kwasie zajmuje pozycję 2:



Przez działanie kwasu siarkowego (1,84) w 100^0 na kwas fenyloantranilowy w ciągu 7—8 godzin został otrzymany również kwas akrydono-2-sulfonowy, który utożsamiono zapomocą reakcji z dwumetyloaniliną i tlenochlorkiem fosforu.

Badania nad budową drugiego, prawdopodobnie izomerycznego kwasu, otrzymanego w małych ilościach w reakcji sulfonowania akrydonu i kwasu fenyloantranilowego, są w toku.

Kwas akrydono-2-sulfonowy poddano działaniu pięciochlorku fosforu. Otrzymano produkt krystaliczny o własnościach zbliżonych do sulfochlorku, którego budowy jeszcze ostatecznie nie ustalono.

Reakcja sulfonowania akrydonu dymiącym kwasem siarkowym (80% SO_3) w 100^0 w ciągu 3 godzin dała żółty krystaliczny produkt. Na zasadzie analizy ilościowej stwierdzono, że jest to kwas akrydonodwusulfonowy.

Dalsze badania nad oznaczeniem pozycyj grup sulfonowych oraz nad rozdzieleniem izomerów i opracowaniem pochodnych kwasów akrydonosulfonowych będą kontynuowane.

J. H. Kusner.

Krzywe o cyklicznej spójności wyższych rzędów.

Przedstawił K. Kuratowski dn. 26 listopada 1932 r.

Streszczenie.

Autor zajmuje się badaniem krzywych dwojakiego rodzaju: 1) krzywych, w których każda para punktów daje się połączyć przez n łuków rozłącznych, 2) krzywych, w których przez każdych n punktów daje się przeciągnąć krzywa zamknięta zwyczajna.

J. H. Kusner.

On continuous curves with cyclic connection of higher order.

Mémoire présenté par M. K. Kuratowski dans la séance du 26 Novembre 1932.

J. R. Kline has raised the question whether for $n > 2$ there exists a continuous curve [1]) such that between every two points of it there are exactly n simple continuous arcs which are mutually exclusive except for end points. For $n = 1, 2$, the acyclic [2]) continuous curve and the simple closed curve respectively have this property. It is here shown that for $n = 3, 4$ no such curve exists.

The consideration of this question leads to a class of continuous curves which involve an extension of the notion of cyclic connection [3]) introduced by G. T. Whyburn [4]) in 1927. A continuous curve M is said to be n-cyclicly connected if every n points of it lie together on a simple closed curve contained in M.

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[1]) A closed, connected, connected im kleinen point set.

[2]) A continuous curve which contains no simple closed curve.

[3]) A continuous curve is cyclicly connected if every two points of it lie on a simple closed curve in it.

[4]) Cyclicly connected continuous curves, Proceedings of the National Academy of Sciences, vol. 13 (1927), pp. 31—38.

Regular curves [1]) of this type are exhibited and a few of their properties are developed. Necessary and sufficient conditions for a continuous curve to be 3-cyclicly connected are derived.

The curves discussed in this paper are assumed to lie in euclidean space of n dimensions.

I take this opportunity to acknowledge my indebtedness so Professor J. R. Kline for his kindly encouragement and assistance in the preparation of this paper.

Theorem 1. *For $n = 3, 4$, there exists no continuous curve M such that between every two points of M there are exactly n independent arcs.*

Proof for $n = 3$. Suppose such a curve M exists. Take two points P_1, P_2 in M. Then, by Rutt's theorem [2]) there are three points A_1, A_2, A_3 in M which separate [3]) P_1 and P_2 in M.

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[1]) The point P of a continuum M is said to be a regular point of M if for each $\epsilon > 0$ an open subset of M exists which contains P and is of diameter $< \epsilon$ and whose boundary with respect to M is finite.

If R is an open subset of M (i. e., no point of R is a limit point of $M - R$), then the boundary of R with respect to M is the set of all those points of $M - R$ which are limit points of R.

A continuum consisting of regular points is a regular curve. Cf. K. Menger, Grundzüge einer Theorie der Kurven, Mathematische Annalen, vol. 95 (1925), pp. 287—306.

[2]) If two points A and B of a continuous curve M are separated in M by no n points, then there are at least $n+1$ independent (i. e., mutually exclusive except for endpoints) arcs AB in M. Cf. N. E. Rutt, Concerning the cut points of a continuous curve when the arc curve, AB, contains exactly N independent arcs, American Journal of Mathematics, vol. 51 (1929), pp. 217—246.

Rutt proved this theorem for the plane. It has been extended to euclidean n-space by G. Nöbeling, Eine Verschärfung des n-Beinsatzes, Fundamenta Mathematicae, vol. 18 (1931), pp. 23—38, and independently by L. Zippin, in a paper soon to appear.

[3]) A set K separates P_1 and P_2 in M if $K + P_1 + P_2 \subset M$, and $M - K \supset C_1 + C_2$ where $C_i \supset P_i$, $C_1 \cdot C_2 = 0$ and neither contains a limit point of the other. C_1 and C_2 are said to be mutually separated.

Then $M - \sum_1^3 A_i \supset C_1 + C_2$ where C_i is the component [1]) which contains P_i. Consider the sets $C_i + \sum_1^3 A_i = D_i$. Each is a continuous curve. If one of them, say D_1, contains only one independent arc $A_1 A_2$, then there is a point B which separates A_1 and A_2 in D_1. Then $D_1 - B \supset C_{11} + C_{12}$ where C_{1i} is the component which contains A_i, for $i = 1, 2$. Now $C_{11} \cdot C_{12} = 0$. Hence one of these components, say C_{11}, does not contain A_3.

Then consider any point $P \subset C_{11} - A_1$ and any point $Q \subset C_2$. The only boundary points possessed by the set $C_{11} - A_1$ are A_1 and B. Hence P and Q are separated in M by A_1 and B, and consequently M contains only two independent arcs PQ, contrary to hypothesis. Hence each set D_1 and D_2 contains at least two independent arcs $A_1 A_2$. For every choice of these arcs in D_1 and D_2, one arc in each set contains A_3, for otherwise there would be four independent arcs $A_1 A_2$ in M. For the same reason there are no more than two independent arcs $A_1 A_2$ in each set D_1, D_2. Now consider the independent arcs $A_1 A_3 A_2$ and $A_1 B A_2$ in D_1. If every arc BA_3 in D_1 contains either A_1 or A_2, then $D_1 - (A_1 + A_2)$ consists of two mutually separated sets and since $C_1 = D_1 - (A_1 + A_2) - A_3$, C_1 likewise consists of two mutually separated sets, which is impossible.

Hence there is an arc BA_3 which contains neither A_1 nor A_2.

Let BA_3 in the direction A_3 to B first meet $A_1 B A_2$ in a point B_1 and last meet $A_1 A_3 A_2$ in B_2. If necessary, reletter the points A_1, A_2 so that $B_2 \subset A_1 A_3$ of $A_1 A_3 A_2$. Then between A_2 and A_3 in M there are the following independent arcs:

1. $A_2 A_3$ (of $A_1 A_3 A_2 \subset C_1$).
2. $A_2 B_1$ (of $A_1 B A_2 \subset C_1$) $+ B_1 B_2$ (of $BA_3 \subset C_1$) $+$ $+ B_2 A_3$ (of $A_1 A_3 A_2 \subset C_1$).
3. $A_2 A_3$ (of $A_1 A_3 A_2 \subset C_2$).
4. $A_2 A_1$ (the arc in C_2 independent of $A_1 A_3 A_2$) $+$ $+ A_1 A_3$ (of $A_1 A_3 A_2 \subset C_2$).

[1]) A connected subset of $M - \sum_1^3 A_i$ contained in no other connected subset of $M - \sum_1^3 A_i$.

This contradicts the hypothesis concerning M, hence no such curve exists.

Proof for $n=4$. Suppose such a curve M exists. Take two points P_1, P_2 in M. Then, by Rutt's theorem, there are four points $\sum_1^4 A_i$ which separate P_1 and P_2 in M. It is desired to prove that $M-\sum_1^4 A_i$ contains only two components. Let $M-\sum_1^4 A_i=\sum_1^n C_i$ where C_i is a component and suppose $n>2$.

There are four points $\sum_1^4 B_i \subset M$ such that $M-\sum_1^4 B_i=D_1+D_2$ where $D_i \supset A_i$, for $i=1, 2$ and D_1 and D_2 are mutually separated. As $C_i+A_1+A_2$ is connected it follows that each C_i contains a point B_i, for otherwise there would be a connected subset of $M-\sum_1^4 B_i$ containing a point of each of the mutually separated sets D_i, which is impossible. Hence $n \leqslant 4$, and $\sum_1^4 A_i \cdot \sum_1^4 B_i=B_4$ at most. One component, say C_1, contains only one point of $\sum_1^4 B_i$, since there are more than two components. Now if $C_1+A_1+A_2+A_3-B_1=T_1+T_2$, mutually separated sets, then one of them, say T_1, contains at most one point of $\sum_1^4 A_i$ and hence the only boundary points of $T_1-T_1\cdot\sum_1^3 A_i$ are the three points $T_1\cdot\sum_1^3 A_i+A_4+B_1$ at most. Therefore any points $P\subset T_1-T_1\cdot\sum_1^3 A_i$ and $Q\subset C_2$ are separated in M by three points and there are consequently at most three independent arcs PQ in M, contrary to hypothesis. Hence $C_1+\sum_1^3 A_i-B_1$ is connected. Furthermore it lies in $M-\sum_1^4 B_i$ and contains a point of each of the mutually separated sets, D_1, D_2, which is impossible. Therefore n cannot be greater than 2.

The remainder of the proof will be divided into three cases:

1. For no values of i, j, k, does D_i contain three independent arcs $A_j A_k$.
2. For at least one set of values of i, j, k, D_i contains three independent arcs $A_j A_k$, two of which are contained in $C_i + A_j + A_k$.
3. For every set of values of i, j, k for which D_i contains three independent arcs $A_j A_k$, (and there is at least one set) only one independent arc $A_j A_k$ is contained in $C_i + A_j + A_k$.

Case 1. Since $M - \sum_1^4 A_i = C_1 + C_2$, each set D_i contains exactly two independent arcs $A_1 A_2$. Then there are two points B_1, B_2 which separate A_1 and A_2 in D_1. Hence $D_1 - (B_1 + B_2) \supset C_{11} + C_{12}$ where C_{1i} is the component which contains A_i, for $i = 1, 2$.

Now since $C_{11} \cdot C_{12} = 0$, if one of them, say C_{11}, contains both A_3 and A_4 then any point $P \subset C_{12} - A_2$ and any point $Q \subset C_2$ are separated in M by A_2, B_1, B_2 and hence M contains only three independent arcs PQ, contrary to hypothesis. Therefore each set $C_{1i} \supset$ two points of $\sum_1^4 A_i$.

Let $C_{11} \supset A_1 + A_3$ and $C_{12} \supset A_2 + A_4$. Now in a similar manner $D_1 - (\bar{B}_1 + \bar{B}_2) \supset C_{11}^* + C_{13}^*$ where C_{1i}^* is the component which contains A_i, for $i = 1, 3$ and each component contains two points of $\sum_1^4 A_i$. Hence one of them contains A_2 and the other A_4. That is, A_1 and A_3 are separated in D_1 by $\bar{B}_1 + \bar{B}_2$ and A_2 and A_4 are separated in D_1 by $B_1 + B_2$.

Now, if C_{11} contains only one independent arc $A_1 A_3$, then $C_{11} + B_1 + B_2$ contains a point B which separates A_1 and A_3 in it. Then $C_{11} + B_1 + B_2 - B \supset C_{111} + C_{113}$ where C_{11i} is the component which contains A_i, for $i = 1, 3$. Now, since $C_{111} \cdot C_{113} = 0$, one of them, say C_{111}, contains at most one point of $B_1 + B_2$. Then consider any point $P \subset C_{111} - \left(A_1 + C_{111} \cdot \sum_1^2 B_i\right)$ and any point $Q \subset C_2$. The only boundary points of the set $C_{111} - \left(A_1 + C_{111} \cdot \sum_1^2 B_i\right)$ are the points $A_1 + C_{111} \cdot \sum_1^2 B_i + B$. Hence P and Q are separated in M by three points and there are only three independent arcs PQ in M, contrary to hypothesis.

Therefore $C_{11} \supset$ two independent arcs $A_1 A_3$ and similarly C_{12} two independent arcs $A_2 A_4$. But $\overline{B}_1$ and $\overline{B}_2$ separate A_1 and A_3 in D_1. Hence the two independent arcs $A_1 A_3$ in C_{11} contain $\overline{B}_1$ and $\overline{B}_2$ respectively. Similarly the two independent arcs $A_2 A_4$ contain $\overline{B}_1$ and $\overline{B}_2$ respectively. But $C_{11} \cdot C_{12} = 0$. Hence case 1 is impossible.

Case 2. Let D_1 contain three independent arcs $A_1 A_2$, two of which are in $C_1 + A_1 + A_2$. If D_2 contains only one independent arc $A_1 A_2$, then there is a point B such that $D_2 - B \supset C_{21} + C_{22}$ where C_{2i} is the component which contains A_i, for $i = 1$, 2. Now one component, say C_{21}, contains at most two points of $\sum_1^4 A_i$, since $C_{21} \cdot C_{22} = 0$.

Consider any point $P \subset C_{21} - C_{21} \cdot \sum_1^4 A_i$ and any point $Q \subset C_1$. The only boundary points of $C_{21} - C_{21} \cdot \sum_1^4 A_i$ are $C_{21} \cdot \sum_1^4 A_i + B$. Hence P and Q are separated in M by three points, which as before leads to a contradiction. Therefore D_2 contains at least two independent arcs $A_1 A_2$. One of them and one of the arcs $A_1 A_2$ in D_1 have a common point other than endpoints, for otherwise there would be five independent arcs $A_1 A_2$ in M.

Also since $C_1 + A_1 + A_2$ contains two independent arcs $A_1 A_2$, D_2 contains only two independent arcs, for otherwise there would be five independent arcs $A_1 A_2$ in M. Hence there are two points B_1 and B_2 which separate A_1 and A_2 in D_2. Then $D_2 - (B_1 + B_2) \supset C_{21} + C_{22}$ where C_{2i} is the component which contains A_i, for $i = 1$, 2. If one set, say C_{21}, contained only one point of $\sum_1^4 A_i$, then any point $P \subset C_{21} - C_{21} \cdot \sum_1^4 A_i$ and any point $Q \subset C_1$ would be separated in M by three points, which leads to a contradiction as before. Hence each set C_{2i} contains two points of $\sum_1^4 A_i$. Let $C_{21} \supset A_1 + A_3$ and $C_{22} \supset A_2 + A_4$. Choose any point P an inner point of one of the arcs $A_1 A_2$ in D_1 which does not contain A_3. If $A_3 A_2$ of $A_1 A_3 A_2$ in D_1 does not contain A_4, choose an inner point Q in $A_3 A_4$ of it. Then, by an argument similar to that in the proof for $n=3$, P and Q are joined in D_1 by an arc α which contains no point of $\sum_1^4 A_i$. Let Q_1 be the last point of $A_3 A_2$ of $A_1 A_3 A_2$ on α in the direction Q to P. Now if $Q_1 P$ of α in the direction Q_1 to P meets $A_1 A_3$ of $A_1 A_3 A_2$ before it meets either of the arcs $A_1 A_2$ which are independent of $A_1 A_3 A_2$ then $C_1 + A_1 + A_2$ contains three independent arcs $A_1 A_2$ and hence D_2 contains only one independent arc $A_1 A_2$, which in case 1 was proved impossible. Therefore $Q_1 P$ in the direction Q_1 to P meets an arc $A_1 A_2$ independent of $A_1 A_3 A_2$ in a point P_1 such that $P_1 Q_1$ of α contains no point of $A_1 A_3$ of $A_1 A_3 A_2$. Then clearly $C_1 + A_1 + A_3$ contains two independent arcs $A_1 A_3$. If $A_3 A_2$ of $A_1 A_3 A_2$ in D_1 contains A_4, choose an inner point Q in $A_3 A_4$ of it. If the arc α joining P and Q in D_1 and not containing any point of $\sum_1^4 A_i$ meets $A_1 A_3 A_2 - (A_3 A_4$ of $A_1 A_3 A_2)$ in the direction Q to P before it meets either of the arcs $A_1 A_2$ in D_1 independent of $A_1 A_3 A_2$, then this reduces to the case just considered. If this situation does not obtain, then clearly there are two independent arcs $A_2 A_4$ in $C_1 + A_2 + A_4$. Therefore either $C_1 + A_1 + A_3$ contains two independent arcs $A_1 A_3$ or $C_1 + A_2 + A_4$ contains two independent arcs $A_2 A_4$. Suppose the former. Then D_2 contains at most two independent arcs $A_1 A_3$. It cannot contain only one such arc, as was shown in case 1 for A_1 and A_2, hence it contains exactly two. Then there are two points B_1, B_2 which separate A_1 and A_3 in D_2. Hence $D_2 - (B_1 + B_2) \supset C_{21}^* + C_{23}^*$ where C_{2i}^* is the component which contains A_i, for $i=1, 3$. Here again, each set C_{2i}^* must contain two points of $\sum_1^4 A_i$, for otherwise two points could be found which are separated in M by three points, which leads to a contradiction. Hence $\bar{B}_1$ and $\bar{B}_2$ separate A_2 and A_4 in D_2 as well as A_1 and A_3 in D_2. As in case 1, if one of the components, say C_{21}, contains only one arc $A_1 A_3$, a similar contradiction follows. Hence C_{21} contains two independent arcs $A_1 A_3$ and C_{22} contains two independent arcs $A_2 A_4$. Also as in case 1, $\bar{B}_1$ and $\bar{B}_2$ must be contained in the two arcs respectively in each of these sets, which contradicts the fact that $C_{21} \cdot C_{22} = 0$. Hence case 2 is impossible.

Case 3. Let D_1 contain three independent arcs $A_1 A_2$. Since only one of them, α_1, is in $C_1 + A_1 + A_2$, one, β_1, contains A_3 and another, γ_1, A_4. D_2 cannot contain only one arc $A_1 A_2$, as was shown in case 1. Since D_2 contains three independent arcs $A_1 A_2$, $C_2 + A_1 + A_2$ cannot contain two such arcs. Hence D_2 contains at least two such arcs, exactly one of which contains either A_3 or A_4. If D_2 contains only two such arcs, depending on the manner of their choice there are three possibilities:

a) α_2 does not contain A_4, $\beta_2 \supset A_3$ but not A_4.

b) $\beta_2 \supset A_3 + A_4$.

c) $\alpha_2 \supset A_4$, $\beta_2 \supset A_3$.

In case a) there is an arc $E_1 E_2$ in D_2 such that $E_1 \subset A_3 A_2$ of β_2 $E_2 \subset \alpha_2$ and $E_1 E_2 \cdot (A_1 A_3 [\text{of } \beta_2] + A_2) = 0$. There are then two arcs $A_1 A_3$ in $C_2 + A_1 + A_3$:

1. $A_1 A_3$ (of β_2).
2. $A_1 E_2$ (of α_2) $+ E_2 E_1 + E_1 A_3$ (of β_2).

Also in D_1 there is an arc $F_1 F_2$ such that $F_1 \subset A_3 A_2$ of β_1, $F_2 \subset \alpha_1$ and $F_1 F_2 \cdot (A_2 + A_1 A_3 [\text{of } \beta_1] + A_1 A_4 [\text{of } \gamma_1]) = 0$.

Hence there are two arcs $A_1 A_3$ in $C_1 + A_1 + A_3$:

1. $A_1 A_3$ (of β_1).
2. $A_1 F_2$ (of α_1) $+ F_2 F_1 + F_1 A_3$ (of β_1).

Now as in case 2, since D_2 contains only two independent arcs $A_1 A_2$, $D_2 \supset \bar{B}_1 + \bar{B}_2$ such that $D_2 - (\bar{B}_1 + \bar{B}_2) \supset C_{21} + C_{22}$ where $C_{21} \supset A_1 + A_3$ and $C_{22} \supset A_2 + A_4$ and $C_{21} \cdot C_{22} = 0$. Also $D_2 - (\bar{B}_1 + \bar{B}_2) \supset C_{21}^* + C_{23}^*$ where $C_{21}^* \supset A_1$ and one other point of $\sum_1^4 A_i$ and $C_{23}^* \supset A_3$ and the remaining point of $\sum_1^4 A_i$. Hence $\bar{B}_1 + \bar{B}_2 \subset C_{21}$, $\bar{B}_1 + \bar{B}_2 \subset C_{22}$, which contradicts $C_{21} \cdot C_{22} = 0$.

Hence case a) is impossible. In case b) there is an arc $E_1 E_2$ in C_2 such that $E_1 \subset A_3 A_4$ (of β_2) and $E_2 \subset \alpha_2$. If $E_1 E_2$ meets either $A_1 A_3$ or $A_4 A_2$ (of β_2), this reduces to the previous case. If not, the arcs $A_1 A_3$ of the previous case exist here also in both D_1 and D_2 and the argument is precisely the same. Hence case b) is impossible.

In case c) consider any inner point E_1 of $A_1 A_4$ of α_2 and any inner point E_2 of $A_3 A_2$ of β_2. There is an arc $E_1 E_2$ in D_2 which contains no point of $\sum_1^4 A_i$ and which has only E_1 in $A_1 A_4$ of α_2. If $E_1 E_2$ in the direction E_1 to E_2 meets $A_2 A_4$ of α_2 before meeting β_2, then this reduces to case a). If it meets $A_3 A_2$ of β_2 before meeting $A_1 A_3$ of β_2 then there are two arcs $A_1 A_3$ in $C_2 + A_1 + A_3$ and the argument proceeds as in case a). If it meets $A_1 A_3$ of β_2 before meeting $A_3 A_2$ of β_2 and then meets $A_3 A_2$ of β_2 before meeting $A_4 A_2$ of α_2, then this reduces to case a).

If it meets $A_1 A_3$ of β_2 before meeting $A_3 A_2$ of β_2 and then meets $A_4 A_2$ of α_2 before meeting $A_3 A_2$ of β_2 then $C_2 + A_2 + A_3$ contains two arcs $A_2 A_3$ and the argument proceeds as in case a) with A_2 replacing A_1. Therefore case c) is impossible also, and D_2 contains three independent arcs $A_1 A_2$, one of which contains A_3 and another of which contains A_4.

Now there is in C_1 an arc α joining an inner point of α_1 to an inner point of $A_1 A_3$ of β_1 such that α has only an endpoint B_1 on α_1 and an endpoint B_2 on $A_1 A_3$ of β_1. α does not meet $A_3 A_2$ of β_1 or $A_4 A_2$ of γ_1, for otherwise $C_1 + A_1 + A_2$ would contain two independent arcs $A_1 A_2$. Similarly there is an arc $\beta \subset C_1$ with one endpoint $\bar{B}_1$ on α_1 and the other $\bar{B}_2$ on $A_2 A_4$ of γ_1 and not meeting $A_1 A_3$ of β_1 or $A_1 A_4$ of γ_1 for the same reason. Then $C_1 + A_2 + A_3$ contains two independent arcs $A_2 A_3$:

1. $A_3 B_2$ (of β_1) $+ B_2 B_1 + B_1 A_2$ (of α_1) $= \delta_1$.
2. $A_3 A_2$ (of β_1) $= \delta_2$.

Also $C_1 + A_1 + A_4$ contains two independent arcs $A_1 A_4$:

1. $A_1 \bar{B}_1$ (of β_1) $+ \bar{B}_1 \bar{B}_2 + \bar{B}_2 A_4$ (of γ_1) $= \eta_1$.
2. $A_1 A_4$ (of γ_1) $= \eta_2$.

By a similar argument $C_2 + A_2 + A_3$ contains two independent arcs δ_1' and δ_2' from A_2 to A_3 and $C_2 + A_1 + A_4$ contains two independent arcs η_1' and η_2' from A_1 to A_4. Now D_1 can contain at most two independent arcs $A_2 A_3$, for otherwise M would contain five arcs $A_2 A_3$ since $C_2 + A_2 + A_3$ contains two such arcs. For a similar reason D_1 can contain at most two independent arcs $A_1 A_4$. Hence there are two points E_1 and E_2 which separate A_2 and A_3 in D_1, so that $D_1 - (E_1 + E_2) \supset K_2 + K_3$ where K_i is the component which contains A_i, for $i = 2$, 3. By the same argument used in cases 1 and 2, each component K_i contains two points of $\sum_1^4 A_i$. Hence E_1 and E_2 separate A_1 and A_4 in D_1 also. Therefore δ_1 and δ_2 contain E_1 and E_2 respectively, as do η_1 and η_2. Now $\delta_2 \cdot \eta_2 = 0$. Hence $\delta_1 \cdot \eta_2 = E_1$ and $\delta_2 \cdot \eta_1 = E_2$. Now $\delta_1 \cdot \eta_2 = (A_3 B_2) \cdot \eta_2 + (B_2 B_1) \cdot \eta_2 + (B_1 A_2) \cdot \eta_2$. But the first and last terms are 0. Hence E_1 is an inner point of $B_2 B_1$. Also $\delta_2 \cdot \eta_1 = \delta_2 \cdot (A_1 \bar{B}_1) + \delta_2 \cdot (\bar{B}_1 \bar{B}_2) + \delta_2 \cdot (\bar{B}_2 A_4)$. Here also the first and last terms are 0 and E_2 is an inner point of $\bar{B}_1 \bar{B}_2$. But the arc $A_1 A_2 + A_2 A_4$ in D_1 contains neither E_1 nor E_2 and hence lies in $D_1 - (E_1 + E_2)$. This contradicts the fact that E_1 and E_2 separate A_1 and A_4 in D_1. Hence case 3 is impossible and no such curve exists.

Theorem 2. *A continuous curve M, every two points of which are joined in M by exactly n independent arcs is a regular curve, for finite n.*

Proof. Take any point P i M and consider any region R about P. R may be chosen of diameter $< \epsilon$, for any $\epsilon > 0$. Let $F(R)$ be the boundary of R with respect to M.

Take any point Q in $F(R)$. Now P and Q are separated in M by a set A consisting of n points. There is a region R_Q about Q such that every point of $M \cdot R_Q$ is separated from P by the set A. For since A is a finite set of points, there is a region R_Q' about Q which contains no point of A and since M is connected im kleinen there is a region $R_Q \subset R_Q'$ such that every point of $M \cdot R_Q$ lies with Q in a connected subset of M within R_Q' and hence does not contain any point of A. Now take any point $T \subset M \cdot R_Q - Q$. If A does not separate P and T, then there is a connected subset of $M - A$ containing them, and since there is a connected subset of $M - A$ containing T and Q, there is a connected subset of $M - A$ containing P and Q, contrary to hypothesis. Hence A does separate P and T. Now, for every point $Q \subset F(R)$ there is a region R_Q such that every point of $M \cdot R_Q$ is separated from P by the same set of points which separate P and Q. Hence, by the Heine-Borel theorem [1]) there is a finite subset of such regions, $R_1, R_2, R_3, \ldots, R_k$, such that $\sum_1^k R_i \supset F(R)$ and every point in $R_i \cdot F(R)$ is separated from P by a set of n points A_i. P, then, is separated from every point of $F(R)$, and consequently from every point of $M - M \cdot \bar{R}$, by the finite set of points $\sum_1^k A_i$. Let $M - \sum_1^k A_i = C_p + D$ where C_p is the component which contains P. Now, 1) C_p is an open set. 2) The diameter of C_p is $\leqslant \epsilon$. For suppose C_p is of diameter $> \epsilon$. Then there is in C_p a point Q not in $\bar{R}$. Any two points in C_p can be joined by an arc entirely in C. Now any arc from P to a point not in $\bar{R}$ must pass through a point of $F(R)$. Hence there is an arc from P to a point of $F(R)$ not containing any point of $\sum_1^k A_i$ which is impossible. 3) $F(C_p) \subset \sum_1^k A_i$ and hence $F(C_p)$ is finite, which shows that any point P can be ϵ — separated in M by a finite set. Therefore M is a regular curve.

Now, since every regular curve is topologically contained in euclidean space of three dimensions [2]), it follows from theorem 2 that if a curve of the type under consideration exists it is so contained. It may be remarked in passing that it is possible to prove that no such curve exists in the euclidean plane.

Theorem 3. *A continuous curve M every two points of which are joined in M by exactly $n > 1$ independent arcs has the property that every three of its points lie together on a simple closed curve in M.*

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[1]) If every point of a closed, bounded point set is contained in at least one of a set of open sets, then there is a finite subset of the set of open sets such that every point of the closed, bounded set is contained in at least one of them. Cf. F. Hausdorff, Grundzüge der Mengenlehre, Leipzig, 1914, p. 231.

[2]) Cf. K. Menger, Zur allgemeinen Kurventheorie, Fundamenta Mathematicae, vol. 10 (1927), pp. 96—115.

Proof. If $n = 2$, M is a simple closed curve and the theorem s obviously true. If $n > 2$, no two points of M are separated in M by one point, hence M is cyclicly connected.

Ayres has proved [1]) that of any three points of a cyclicly connected continuous curve M, either every two points are separated in M by two points or the three points lie together on a simple closed curve in M. Since no two points of M are separated in M by two points, every three points of M lie on a simple closed curve in it.

The question now arises whether there exists a regular curve other than the simple closed curve which has the property that every three points of it lie on a simple closed curve in it. Sierpiński [2]) has constructed a regular curve which has the property that every point of it is a ramification [3]) point. To do this he formed a hexagon of six triangle curves constructed as follows:

Join the midpoints of the sides of a triangle T^0 by straight lines, and let those of the new triangles formed which have a vertex in common with T^0 constitute the set of triangles T^1. For any n, in each triangle of T^n join the midpoints of the sides by straight lines and let those of the new triangles formed which have a vertex in common with a triangle of T^n constitute the set of triangles T^{n+1}. Continue this process indefinitely. The limiting curve M is Sierpiński's triangle curve. Obviously every three points of M lie on a simple closed curve in it. In fact, M has a much stronger property.

Theorem 4. *Any n points (n finite) of the Sierpiński triangle curve lie together on a simple closed curve in it.*

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[1]) W. L. Ayres, Continuous curves which are cyclicly connected, Bulletin de l'Académie Polonaise des Sciences et des Lettres, (1928), Pp. 127—142.

[2]) W. Sierpiński, Sur une courbe dont tout point est un point de ramification, Comptes Rendus, vol. 160 (1915), pp. 302—305.

[3]) If an integer n exists such that for each ϵ, an open subset of M of diameter $< \epsilon$ and containing the point P can be found such that its boundary with respect to M contains at most n points, and if n is the smallest integer such that this property holds, then P is said to be of order n in M. Cf. Menger, loc. cit.

A ramification point is a point of order > 2.

Proof. The proof is based upon a lemma, which is as follows:

Lemma. *For any finite set K consisting of n points of M, there is an arc of M containing the points of the set K and having any two of the vertices of the triangle T^0 of M as endpoints.*

Proof of the lemma. Choose the two vertices V_1 and V_2 and let the third vertex be V_3. Place the curve so that V_1 is at the top and V_2 is at the lower right. Since K consists of a finite number of points, there is a region S about V_1 such that no point of K other than V_1 (whether or not $V_1 \subset K$) is in S. The line segments which join sides of T^0 of M will be called entire segments. There is within S a horizontal entire segment H such that all the points of K other than V_1 are below H. That is, all the points of K other than V_1 lie in that component of $M - H$ which contains V_2. Now the number of horizontal entire segments below H is finite, from the nature of the construction of M. Therefore there is a last one, H_1, below H such that all the points of K other than V_1 lie below H_1. Let H_2 be the next horizontal entire segment below H_1. Now, if between H_1 and the entire segment $V_2 V_3$ there are an even (odd) number of entire segments, we will traverse the left (right) side of T^0, that is $V_1 V_3$ $(V_1 V_2)$, from V_1 to its intersection V_{11} with H_1. The argument from here on will be made on the assumption that the number of horizontal entire segments between H_1 and $V_2 V_3$ of T^0 is even, namely $2k$. The modifications necessary for the odd case will be obvious. Now let V_{21} be the right endpoint of H_1; let V_{12} be the midpoint, V_{22} the right endpoint, V_{13} the left endpoint of H_2. From the construction of M, the points V_{11}, V_{13}, V_{12} are the vertices of a constituent triangle of M, and V_{21}, V_{12}, V_{22} are the vertices of another constituent triangle of M. Let T_1 be the triangle with vertices V_{11}, V_{12}, V_{13}, together with that part of M which is formed from it, and let T_2 be the corresponding set formed from the triangle with vertices V_{21}, V_{12}, V_{22}. Now it is desired to show that a simple continuous arc from V_{11} to V_{12} exists which contains all the points of $T_1 \cdot K$. If $T_1 \cdot K = 0$, the side $V_{11} V_{12}$ of T_1 is traversed and there is sought an arc joining V_{12} to V_{22} in T_2 and containing all the points of $T_2 \cdot K$, which from the choice of H_1, $\neq 0$. The argument which will be given on the assumption that $T_1 \cdot K \neq 0$

will, if $T_1 \cdot K = 0$, hold by replacing T_1 by T_2. If $T_1 \cdot K \neq 0$, then there are two cases to be considered.

1. $T_1 \cdot K \neq K$.
2. $T_1 \cdot K = K$.

In case 1, $T_1 \cdot K$ is a proper subset of K and therefore contains at most $n-1$ points. Now T_1 is itself a Sierpiński triangle curve. Therefore, if the lemma is true for $n-1$ points, there is in this case an arc from V_{11} to V_{12} containing all the points of $T_1 \cdot K$. Then $T_2 \cdot K \neq K$ and therefore contains at most $n-1$ points. Then there is an arc from V_{12} to V_{22} containing all points of $T_2 \cdot K$. There is thus an arc from V_1 to V_{11} to V_{22} containing all points of K which are not below H_2. That is, the arc from V_{11} to V_{22}, which is from the left endpoint of H_1 to the right endpoint of H_2 includes all points of K which lie in and between the entire segments H_1 and H_2. Repeating this process in the reverse direction for the triangles T_3 and T_4 contained between H_2 and H_3 will carry the arc to the left endpoint of H_3, all the points of K not below H_3 having been traversed.

Thus $2k$ performances of the process bring the arc to the left end of H_{2k+1}. There being $2k$ entire segments between H_1 and the entire segment $V_2 V_3$ of T^0, H_{2k+1} is the last entire segment before coming to $V_2 V_3$ of T^0. Therefore another repetition of the process carries the arc to the right endpoint of $V_2 V_3$, namely V_2, which is the desired result. Thus the truth of the lemma for $n-1$ implies its truth for n, in case 1. The lemma being obviously true for the initial value 1, there remains merely to show that its truth for $n-1$ implies its truth for n in case 2.

In case 2, $T_1 \cdot K = K$. T_1 being itself a Sierpiński triangle curve, it can be treated in exactly the same manner in which M was treated above. If, as the set T_1 was arrived at, there is reached a triangle set which will be designated T_2 such that $T_2 \cdot K = K$, then the process will be repeated, treating T_2 as M was treated. Thus there is found a monotone decreasing sequence of triangle sets $T_1, T_2, T_3, \ldots$, each containing the set K. If this sequence were infinite it would obviously define a point P of M which would be a limit point of K, which is impossible since K is a finite set. Hence there is a last set T_a such that $T_i \cdot K = K$ for $i \leqslant a$, and no triangle set which is a proper subset of T_a includes K. Therefore there is reached a triangle set which is a proper subset of T_a and which contains at most $n-1$ points of K. Within this set case 1 applies, and the remainder of the construction is obvious. This proves the lemma.

Remainder of proof of the theorem. M is the sum of three triangle sets T_1, T_2, T_3, each of which is a Sierpiński curve, and each pair of which have a common vertex. Let $T_1 \cdot T_2 = V_1$, $T_2 \cdot T_3 = V_2$, $T_1 \cdot T_3 = V_3$. Now K consists of n points.

Therefore each set T_i contains at most n points of K. By the lemma, there is an arc a of T_1 containing $T_1 \cdot K$ and joining V_3 to V_1, and an arc b of T_2 containing $T_2 \cdot K$ and joining V_1 to V_2, and an arc c of T_3 containing $T_3 \cdot K$ and joining V_2 to V_3. $a+b+c$ is a simple closed curve of M which contains the set K. This proves the theorem.

Curves analogous to the Sierpiński triangle curve may be formed by constructing at each vertex of a polygon P^0 of more than four sides a similar polygon with sides parallel to those of P^0 in such a manner that polygons at adjacent vertices of P^0 have a vertex in common, forming a central polygon also parallel to P^0. Letting those of the new polygons which have a vertex in common with P^0 be the set P^1, repeat the process on each one of them. The limit curve formed by repeating this process indefinitely obviously has the same cyclic connection properties as the Sierpiński triangle curve. The construction of the higher dimensional analogs of these curves is obvious.

Definition. *A continuous curve every n points of which lie together on a simple closed curve in it will be said to be n-cyclicly connected.*

Theorem 5. *The set of ramification points of an n-cyclicly connected continuous curve* $(n > 2)$ *is dense in itself* [1]).

Proof. Suppose there is a ramification point P of an n-cyclicly connected continuous curve M which is not a limit point of ramification points of M. Then there is a region R about P such that $\bar{R}$ (R together with its boundary) contains no ramification point of M other than P. Then there are three arcs emanating from P and which contain no ramification points other than P and since M is cyclicly connected meet the boundary

1) Every point of it is a limit point of it.

of R in points A_1, A_2, A_3 respectively, such that $PA_i - A_i \subset R$. Now let P_i be a point of $< PA_i >$ [1]). Every arc from P_1 to P_2 in M contains either P or $(A_1 + A_2)$. Every arc from P_2 to P_3 in M contains either P or $(A_2 + A_3)$. Every arc from P_3 to P_1 in M contains either P or $(A_3 + A_1)$. Hence no simple closed curve in M contains P_1, P_2, P_3, contrary to hypothesis. Therefore P is a limit point of ramification points.

Theorem 6. *Every ramification point of an n-cyclicly connected continuous curve M $(n > 2)$ is an im kleinen cycle point* [2]).

Proof. Suppose a ramification point P of M is not an im kleinen cycle point. By a theorem of Whyburn's [3]), P is of finite order. Hence a finite number of arcs emanate from P. Consider any three such arcs α_1, α_2, α_3. Whyburn has proved [4]) that if a point P is not an im kleinen cycle point of a continuous curve M, then P is an endpoint of each component into which P ϵ-cuts M [5]). Then for any $\epsilon > 0$, since P is not an im kleinen cycle point, there is a region R of diameter $< \epsilon$ about P which contains no arc joining two of the sets $R \cdot \alpha_i$ and not containing P.

Also, for $i = 1, 2, 3$, there is within R a region R_i about P such that the boundary of R_i contains only one point Q_i of the component C_i of $M \cdot R_i - P$ which contains that portion of $\alpha_i \cdot R_i$ which has P as limit point. Then let P_i be any point of C_i. Every arc from P_1 to P_2 in M contains either P or $(Q_1 + Q_2)$. Every arc from P_2 to P_3 in M contains either P or $(Q_2 + Q_3)$. Every arc from P_3 to P_1 in M contains either P or $(Q_3 + Q_1)$. Hence it is impossible to find a simple closed curve of M which contains P_1, P_2, P_3, which is contrary to hypothesis. Therefore P is an im kleinen cycle point.

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[1]) $< AB >$ will henceforth designate the arc AB less its endpoints A, B.

[2]) A point which lies, for each ϵ, on some simple closed curve in M of diameter $< \epsilon$. Cf. G. T. Whyburn, Concerning points of continuous curves defined by certain im kleinen properties, Mathematische Annalen, vol. 102 (1929), pp. 313—336.

[3]) Theorem 6, loc. cit.

[4]) Theorem 7, loc. cit.

[5]) That is, if R is an open set of diameter $< \epsilon$ containing P and N is the component of $\bar{R} \cdot M$ containing P, then P is an endpoint of each continuum obtained by adding it to each component of $N - P$.

Corollary. *Any point A on any one of at least $m-2$ of the m arcs emanating from a point of order $m>2$ of an n-cyclicly connected continuous curve M $(n>2)$ is an endpoint of an arc which does not contain P and which meets another arc emanating from P.*

Theorem 7. *In a continuous curve which is n-cyclicly connected there is no set of $n-1$ points $\sum_1^{n-1} A_i$ such that $M-\sum_1^{n-1} A_i$ is the sum of more than $n-1$ mutually separated sets.*

Proof. Suppose a set $\sum_1^{n-1} A_i$ exists such that $M-\sum_1^{n-1} A_i \supset \sum_1^{n} M_i$ where M_i and M_j for $i \neq j$, are mutually separated sets. Then let P_i be a point of M_i, for $i=1$ to n. By hypothesis $\sum_1^{n} P_i$ lie on a simple closed curve J in M. Let the point sets M_i be so designated that, for every value of i from 1 to $n-1$, one of the two arcs from P_i to P_{i+1} into which those two points divide J contains no points of $\sum_1^{n} P_i$ other than P_i and P_{i+1}. Denote this arc by a_i. Now since M_i and M_{i+1} are mutually separated, a_i must contain one of the points of $\sum_1^{n-1} A_i$. Suppose $a_i \supset \bar{A}_i$. Then since $a_1 \cdot a_2 = P_2$, a_2 does not contain $\bar{A}_1$, but contains some other point $\bar{A}_2$. Continuing in this manner, $a_i \supset \bar{A}_i$, for $i=1$ to n. But $J=\sum_1^{n} a_i$, and $\sum_1^{n} A_i = \sum_1^{n} \bar{A}_i = n-1$ points. Hence J is not a simple closed curve, contrary to hypothesis. Therefore no such set exists.

Theorem 8. *In order that a continuous curve M be 3-cyclicly connected it is necessary and sufficient that every four points of M which do not lie together on a simple closed curve in M lie in all possible orders on arcs in M.*

Proof. The condition is necessary. For let $\sum_1^4 P_i$ be points of a 3-cyclicly connected continuous curve M which lie on no simple closed curve in M. Now P_1, P_2, P_3 lie together on a simple closed curve J in M which is the sum of three arcs $P_1 P_2$, $P_2 P_3$, $P_3 P_1$. Also P_4 lies together with any two points of $\sum_1^3 P_i$ on a simple closed J_i which does not contain the remaining point P_i. Hence there must be an arc $A_i P_4 B_{i+1}$ [1]) of J_i such that $A_i \subset < P_{i-1} P_i >$ and $A_{i+1} \subset < P_i P_{i+1} >$ and $(A_i P_4 B_{i+1}) \cdot J = A_i + B_{i+1}$. $J + J_i$ then contains arcs with the points $\sum_1^4 P_i$ on them in the following orders:

$$P_{i-1} P_{i+1} P_i P_4 \qquad P_{i+1} P_{i-1} P_i P_4 \qquad P_i P_{i-1} P_{i+1} P_4$$
$$P_{i-1} P_{i+1} P_4 P_i \qquad P_{i+1} P_{i-1} P_4 P_i \qquad P_i P_{i+1} P_{i-1} P_4$$

If i be given the values 1, 2, 3, all possible orders of the points $\sum_1^4 P_i$ will have been obtained.

The condition is sufficient. For suppose there are three points P_1, P_2, P_3 of a continuous curve M having the property of the theorem which are not on a simple closed curve in M. Ayres has proved [2]) that if any three points of a continuous curve lie in all possible orders on arcs of the curve then it is cyclicly connected. Any three points of M which are not on a simple closed curve in it satisfy this condition by hypothesis. Any three points of M which are on a simple closed curve in it obviously satisfy the condition also. Hence M is cyclicly connected. Therefore P_1 and P_2 lie on a simple closed curve J_1 and P_2 and P_3 lie on a simple closed curve J_2, and $J_1 + J_2$ contain three independent arcs $A_1 P_i A_2$, for $i = 1, 2, 3$, the sum of which will be called Θ. Now since P_1, P_2, P_3 do not lie on a simple closed curve and M is cyclicly connected, by the theorem of Ayres mentioned in the proof of theorem 3, for each

[1]) All subscripts are to be reduced mod 3.

[2]) Loc. cit.

two points P_i, P_{i+1} there are two points Y_i, Z_i which separate P_i from P_{i+1} in M. Now each one of these pairs Y_i, Z_i also separate one of the points P_i, P_{i+1} from the other two of the points P_1, P_2, P_3. For suppose this were not so. There are two cases:

a) Two of the point pairs, say Y_1, Z_1 and Y_2, Z_2, lie on the same arc, say $A_1 P_2 A_2$ of Θ.

b) No two point pairs Y_i, Z_i lie on the same arc $A_1 P_j A_2$ of Θ.

In case a) the inner pair of the four points Y_1, Z_1, Y_2, Z_2 obviously separate P_2 from both P_1 and P_3.

In case b) for $i=1, 2, 3$, $Y_i + Z_i \subset A_1 P_i A_2$ and for no value of i do both of the points Y_i, Z_i coincide with A_1, A_2 respectively. Then if Y_1, Z_1 do not separate P_1 from P_3 as well as from P_2, there is a connected set $a \subset M - (Y_1 + Z_1)$ which contains $P_1 + P_3$. Then $a + \langle A_1 P_3 A_2 \rangle +$ one of the arcs $A_i P_2$ is a connected set containing P_1 and P_2 and not containing Y_1 or Z_1. Hence $Y_1 + Z_1$ do not separate P_1 from P_2, contrary to hypothesis. Hence $Y_1 + Z_1$ separate P_1 from both P_2 and P_3.

Now choose any point $P_4 \neq P_1$ such that $P_4 \subset \langle Y_1 P_1 Z_1 \rangle$, and consider the order $P_1 P_2 P_4 P_3$ of the four points $\sum_1^4 P_i$. If an arc $b \subset M$ exists containing $\sum_1^4 P_i$ in this order it is the sum of three arcs $P_1 P_2$, $P_2 P_4$, $P_4 P_3$. Now every arc $P_1 P_2$ in M contains either Y_1 or Z_1 as an inner point, and every arc $P_2 P_4$ in M contains either Y_1 or Z_1 as an inner point, and every arc $P_4 P_3$ in M contains either Y_1 or Z_1 as an inner point. Then two of these arcs would have *a* common inner point, which is impossible. Hence no such arc b exists, contrary to hypothesis. Therefore P_1, P_2, P_3 do lie on a simple closed curve in M.

Lemma. *If three points of a cyclicly connected continuous curve M are separated in M by no two distinct simple closed curves, then the three points lie on a simple closed curve in M.*

Proof. Suppose there are three points P_1, P_2, P_3 of M which satisfy the condition of the lemma and lie on no simple closed curve in M. Then they lie respectively on the three arcs $Q_1 P_1 Q_2$, $Q_1 P_2 Q_2$, $Q_1 P_3 Q_2$ of a Θ curve Θ, as was shown in the proof of theorem 8. By hypothesis Q_1 and Q_2 do not separate P_1, P_2, P_3 in M, since a point is a degenerate simple closed curve, hence there is at least one arc α, which will be called a cross arc, such that $\alpha \cdot \Theta = B_i + B_j$, which are two points such that $B_i \subset Q_1 P_i Q_2$ of Θ and $B_j \subset Q_1 P_j Q_2$ of Θ, for some set of values of i and j. B_i and B_j are respectively in $< Q_1 P_i >$ and $< Q_1 P_j >$ or in $< Q_2 P_i >$ and $< Q_2 P_j >$, for otherwise P_1, P_2, P_3 would lie on a simple closed curve in M. There are two cases to consider:

1. On every segment $< P_i Q_j >$ of the Θ curve which contains an endpoint of a cross arc, there is in the direction P_i to Q_j a first point which is the endpoint of a cross arc.

2. On at least one segment $< P_i Q_j >$ of Θ which contains an endpoint of a cross arc, there is in the direction P_i to Q_j no first point which is the endpoint of a cross arc.

In case 1, let $< P_1 Q_1 >$ be a segment which contains an endpoint of a cross arc, and let B_1 be the first point of $< P_1 Q_1 >$ in the direction P_1 to Q_1 which is the endpoint of a cross arc. Let a cross arc with B_1 as endpoint have its other endpoint B_2 on the segment $< P_2 Q_1 >$. If there is a cross arc joining $< P_2 Q_1 >$ to $< P_3 Q_1 >$, then let C_2 be the first endpoint of such a cross arc on $< P_2 Q_1 >$ in the direction P_2 to Q_1 and let C_3 be its other endpoint on $< P_3 Q_1 >$. Then denote by J_1 the simple closed curve $Q_1 B_1$ (of $Q_1 P_1$ of Θ) $+ B_1 B_2$ (cross arc) $+ B_2 C_2$ (of $Q_1 P_2$ of Θ) $+ C_2 C_3$ (cross arc) $+ CQ$ (of $Q_1 P_3$ of Θ). If there is no cross arc joining $< P_2 Q_1 >$ to $< P_3 Q_1 >$, then denote by J_1 the simple closed curve $Q_1 B_1$ (of $Q_1 P_1$ of Θ) $+ B_1 B_2$ (cross arc) $+ B_2 Q_1$ (of $Q_1 P_2$ of Θ). Now if there is a cross arc joining two of the segments $< P_i Q_2 >$ and $< P_j Q_2 >$, denote by J_2 the simple closed curve determined analogously to J_1. If there is no such cross arc, then denote by J_2 the point Q_2. Clearly, J_1 and J_2 separate P_1, P_2, P_3 in M, contrary to hypothesis. Hence case 1 is impossible.

In case 2, suppose $< P_1 Q_1 >$ is a segment which contains endpoints of cross arcs and on which, in the direction P_1 to Q_1, there is no first endpoint of a cross arc. Then P_1 is a limit point of endpoints of cross arcs and clearly is itself an endpoint of a cross arc which meets another segment, from which it immediately follows that P_1, P_2, P_3 lie on a simple closed curve in M. Hence case 2 is impossible also and the lemma is proved.

Definition. Two simple closed curves J_1 and J_2 will be said to disperse three points of a continuous curve M if $J_1 \cdot J_2 = 0$, $J_1 + J_2 \subset M$, and the points lie respectively in three mutually separated subsets of $M - (J_1 + J_2)$ but in a connected subset of $M - J_i$, for $i = 1, 2$.

Theorem 9. *In order that a cyclicly connected continuous curve M be 3-cyclicly connected it is necessary and sufficient that, if two simple closed curves disperse three points in M, either a) one of the simple closed curves is proper*[1] *and together with one of the dispersed points is contained in a cyclic curve*[2] *of the complement of the other two points and the other curve, or b) the two simple closed curves are joined in M by four arcs which are mutually exclusive except for endpoints and no two of which have more than one endpoint in common.*

Proof. The condition is necessary. For consider any three points P_1, P_2, P_3 of a cyclicly connected continuous curve M which are dispersed in M by two simple closed curves J_1 and J_2. They lie on a simple closed curve which is the sum of three arcs P_1P_2, P_2P_3, P_3P_1 such that $P_{i-1}P_i$ does not contain P_{i+1}. Now one of the simple closed curves, say J_1, meets two of the arcs at least, say P_1P_2 and P_1P_3. Let P_1P_3 in the direction P_1 to P_3 first meet J_1 in a point A_1 and let P_1P_2 in the direction P_1 to P_2 first meet J_1 in a point A_2. Now there is at least one arc P_2P_3 which does not meet J_1 and every such arc meets J_2, since $J_1 + J_2$ disperse P_1, P_2, P_3. If $J_2 \cdot (A_2P_1A_1) = 0$, then $J_1 + A_2P_1A_1$ is the cyclic curve of condition a). If $J_2 \cdot (A_2P_1A_1) \neq 0$, and if J_2 meets only one of the arcs P_1A_1, P_1A_2, say the former, then let P_1A_1 in the direction P_1 to A_1 meet J_2 first in B_1 and last in B_2. Then condition b) is fulfilled. If $J_2 \cdot (A_2P_1A_1) \neq 0$ and J_2 meets both arcs P_1A_1 and P_1A_2, then let P_1A_2 in the direction P_1 to A_2 first meet J_2 in B_4. Then $J_2 + B_4P_1B_1$ is the cyclic curve of condition a).

The condition is sufficient. For consider any three points P_1, P_2, P_3 of a cyclicly connected continuous curve M. If they are not on a simple closed curve in M, they are respectively on the three arcs $Q_1P_1Q_2$, $Q_1P_2Q_2$, $Q_1P_3Q_2$, of a Θ curve

[1]) not a point.

[2]) a cyclicly connected continuous curve.

Θ in M. Now according to the condition of the theorem, Q_1 and Q_2 do not disperse P_1, P_2, P_3 in M. Hence there must be a cross arc joining two segments $< Q_i P_j >$ and $< Q_i P_k >$.

By a process similar to that in the proof of the lemma, there can be found two simple closed curves J_1 and J_2, one of which, say not J_1, may be a point, which disperse P_1, P_2, P_3 in M. Let the arcs joining J_1 and J_2 and containing P_1, P_2, P_3 respectively be $A_1 P_1 B_1$, $A_2 P_2 B_2$, $A_3 P_3 B_3$, where $A_i \subset J_1$ and $B_i \subset J_2$, for $i = 1, 2, 3$. Then $A_i \neq A_j$, for $i \neq j$. Now, since J_1 and J_2 disperse P_1, P_2, P_3 in M, no arc of $M - (J_1 + J_2)$ joins any two of the segments $< A_i P_i B_i >$. If condition a) obtains, one point, say P_1, and one of the simple closed curves, say J_1, lie in a cyclic curve of $M - (P_2 + P_3 + J_2)$. Hence there is an arc $A_4 C_1$ where $A_1 \neq A_4 \subset J_1$, $C_1 \subset P_1 B_1 >$, and $A_4 C_1 \cdot \left(\sum_1^3 A_i P_i B_i \right) = C_1 + D$ where D may be A_2 or A_3 or null. In either case $J_1 + J_2 + \sum_1^3 A_i P_i B_i + A_4 C_1$ contains a simple closed curve containing P_1, P_2, P_3. If condition b) obtains, then neither J_1 nor J_2 is a point and $B_i \neq B_j$, for $i \neq j$, and then $J_1 + J_2 + \sum_1^3 A_i P_i B_i +$ the fourth arc which condition b) requires contains a simple closed curve containing P_1, P_2, P_3. This proves the theorem.

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Uwagi o zbiorach i funkcjach względnie mierzalnych.

Przedstawił W. Sierpiński dn. 26 listopada 1932 r.

Streszczenie.

Autor rozważa zagadnienia, dotyczące rozszerzania funkcji względnie mierzalnych oraz superpozycji funkcji, a następnie bada pewną klasę zbiorów, która — przy pewnych hypotezach — pokrywa się z klasą zbiorów miary zero.

Samuel Eilenberg.

Remarques sur les ensembles et les fonctions relativement mesurables.

Note présentée par M. W. Sierpiński dans la séance du 26 Novembre 1932.

M. Ruziewicz a construit une fonction réelle d'une variable réelle $\psi(x)$ telle que

(*) chaque fonction réelle d'une variable réelle est une fonction mesurable de la fonction $\psi(x)$ [1]).

Nous allons donner dans cette communication une simple condition nécessaire et suffisante pour la fonction $\psi(x)$.

Nous considérons la condition suivante pour un ensemble X [2]):

(P) *Tout sous-ensemble de X est mesurable relativement à X* [3]).

Or, pour que $\psi(x)$ satissfasse à la condition (*) il faut et il suffit que la fonction $\psi(x)$ soit une fonction à valeurs distinctes [4]) et que l'ensemble des valeurs de $\psi(x)$ jouisse de la propriété (P) (th. 3).

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[1]) Communication faite au IIᵉ Congrès des Mathématiciens Roumains à Turnu Severin (Mai 1932). La fonction construite par M. Ruziewicz est une fonction de la première classe de Baire.

[2]) Les ensembles considérés sont situés dans l'espace euclidienne à un nombre quelconque de dimensions.

[3]) C.-à-d. il est un produit de X et d'un ensemble mesurable. Hausdorff: *Grundzüge de Mengenlehre*. Leipzig 1914, p. 415.

[4]) C.-à-d. $\psi(x_1) = \psi(x_2)$ implique $x_1 = x_2$.

Il est à remarquer, qu'en supposant l'hypothèse du continu (de même que certaines hypothèses plus faibles), on peut démontrer que la propriété (P) est caractéristique pour les ensembles de mesure nulle (th. 5—7).

Théorème 1. *Pour toute fonction réelle $\varphi(p)$ définie sur X et mesurable relativement à X*[5]) *il existe une fonction mesurable $f(p)$ (définie pour tout l'espace) telle que $f(p)=\varphi(p)$ pour $p \varepsilon X$.*

Démonstration. La fonction $\varphi(x)$ étant mesurable relativement à X, il existe pour tout a réel un ensemble mesurable $H(a)$, tel que

$$(1) \qquad \underset{p}{\mathrm{E}}[\varphi(p) \geqslant a] = X \cdot H(a).$$

Définissons la fonction $f(p)$ comme il suit:

$f(p)=$ borne supérieure des nombres rationels r tels que $p \varepsilon H(r)$.

Soit $p \varepsilon X$, on a d'après (1) et notre définition $f(p)=\varphi(p)$.

On a évidemment, pour chaque a réel

$$\underset{p}{\mathrm{E}}[f(p) \geqslant a] = \sum_{r \leqslant a} H(r)$$

la sommation étant étendue à tous les nombres rationnels $r \leqslant a$. Les ensembles $H(r)$ étant mesurables, la fonction $f(x)$ est mesurable.

Lemme 1. [6]) Pour que

$$|X_1 + X_2| = |X_1| + |X_2| \quad ^{7})$$

ou X_1 et X_2 sont des ensembles disjoints de mesure extérieure finie, il faut et suffit que X_1 soit mesurable relativement à X_1+X_2.

Lemme 2. $\{X_n\}$ étant une suite de sous-ensembles disjoints de X, mesurables relativement à X, on a

$$(2) \qquad \left|\sum_{n=1}^{\infty} X_n\right| = \sum_{n=1}^{\infty} |X_n|.$$

---

[5]) C.-à-d. que l'ensemble $\underset{p}{\mathrm{E}}[\varphi(p) \geqslant a]$ est mesurable relativement à X, pour chaque a réel.

[6]) Cf. Hausdorff l. c.

[7]) $|X|$ désigne la mesure lebesguienne extérieure de X.

D é m o n s t r a t i o n. Chaque X_n étant mesurable relativement à X, il est mesurable relativement à $\sum_{k=1}^{n} X_k$. D'où, en vertu du lemme 1,

$$\left|\sum_{k=1}^{n} X_k\right| = \left|\sum_{k=1}^{n-1} X_k\right| + |X_n|$$

et pour chaque N naturel

$$\left|\sum_{n=1}^{N} X_n\right| = \sum_{n=1}^{N} |X_n|,$$

ce qui donne

$$\sum_{n=1}^{\infty} |X_n| \geq \left|\sum_{n=1}^{\infty} X_n\right| \geq \left|\sum_{n=1}^{N} X_n\right| = \sum_{n=1}^{N} |X_n|,$$

d'où (2).

Théorème 2. *La propriété* (P) *est équivalente à chacune des propriétés suivantes:*

(P_1) *La mesure extérieure est une fonction additive de sous-ensembles de* X.

(P_2) *La mesure extérieure est une fonction complètement additive de sous-ensembles de* X.

(P_3) *Chaque fonction réelle définie sur* X *admet une extension à une fonction mesurable.*

D é m o n s t r a t i o n. Designons par $\{M_n\}$ une suite, recouvrant l'espace, d'ensembles disjoints mesurables de mesure finie. En vertu des lemmes 1 et 2 les propriétés (P), (P_1) et (P_2) sont equivalentes pour chaque $X \cdot M_n$, donc le même pour X.

Si X jouit de la propriété (P), alors chaque fonction réelle définie sur X est mesurable relativement à X et le théorème 1 entraîne la propriété (P_3).

D'autre part, la condition (P_3) étant satisfaite, posons

$$\varphi(p) = 1 \qquad \text{pour} \quad p \in X_1,$$

$$\varphi(p) = 0 \qquad \text{pour} \quad p \in X - X_1,$$

où X_1 désigne un sous-ensemble quelconque de X. Soit $f(p)$ une fonction mesurable, telle que $f(p) = \varphi(p)$ pour $p \in X$. La mesurabilité relative de X_1 resulte d'égalité

$$X_1 = X \cdot \mathop{\mathrm{E}}_{p} [f(p) \geqslant 1],$$

X jouit donc de la propriété (P).

Notre théorème est ainsi démontré.

Théorème 3. *Pour qu'il existe pour toute fonction réelle $f(p)$ définie sur un ensemble E* [8]*, une fonction mesurable $\varphi(x)$ telle que*

$$f(p) = \varphi(\psi(p)),$$

où $\psi(p)$ est une fonction réelle, fixe, définie sur E, il faut et il suffit que

1⁰) *ψ soit une fonction à valeurs distinctes* [4];

2⁰) *$\psi(\mathrm{E})$ jouisse de la propriété (P).*

Démonstration. Les conditions sont suffisantes. Soit $f(p)$ une fonction réelle définie sur E. $\psi(p)$ étant une fonction à valeurs distinctes, posons $\varphi_0(x) = f(\psi^{-1}(x))$ pour tout $x \in \psi(E)$.

On a

$$f(p) = \varphi_0(\psi(p))$$

pour chaque $p \in E$.

$\psi(E)$ jouit de la propriété (P), donc en vertu du théorème 2 (propriété (P_3)) il existe une fonction mesurable de variable réelle $\varphi(x)$ telle que $\varphi(x) = \varphi_0(x)$ pour $x \in \psi(E)$, ce qui donne pour chaque $p \in E$

$$f(p) = \varphi(\psi(p)).$$

Les conditions sont nécessaires. En effet, s'il existe $p_1 \neq p_2$ pour lesquels $\psi(p_1) = \psi(p_2)$, on a $f(p_1) = f(p_2)$ pour chaque fonction réelle $f(p)$ ce qui est impossible. La fonction $\psi(p)$ est ainsi une fonction à valeurs distinctes.

Supposons donc que $\psi(E)$ ne satisfasse pas à la condition (P). Soit $\varphi_0(x)$ une fonction réelle définie sur $\psi(E)$ qui n'est pas extensible à une fonction mesurable — une telle fonction existe en vertu du théorème 2. Posons $f(p) = \varphi_0(\psi(p))$ pour $p \in E$. Admettons qu'il existe une fonction mesurable $\varphi(x)$ telle que pour $p \in E$

$$f(p) = \varphi(\psi(p)),$$

ce qui donne $\varphi(x) = \varphi_0(x)$ pour $x \in \psi(E)$, contrairement à l'hypothèse faite sur $\varphi_0(x)$.

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[8]) E peut être un ensemble abstrait arbitraire.

Théorème 4. *Si chaque sous-ensemble de mesure extérieure positive de X a la puissance du continu, il existe des ensembles X_1 et X_2 tels que*

$$X_1 + X_2 \subset X$$

$$X_1 \cdot X_2 = 0$$

$$|X_1| = |X_2| = |X|.$$

Démonstration. Si $|X| = 0$, posons $X_1 = X_2 = 0$. Supposons donc: $|X| > 0$, et par conséquent en vertu de notre hypothèse:

$$\overline{\overline{X}} = 2^{\aleph_0}.$$

Soit

(3) $$G_1 . G_2 \ldots G_\omega \ldots G_\xi \ldots, \quad \xi < \omega_\eta, \quad \overline{\omega_\eta} = 2^{\aleph_0}$$

une suite transfinie du type ω_η, composée de tous les ensembles ouverts tels que

(4) $$\overline{\overline{X - G}} = 2^{\aleph_0}.$$

Nous allons définir à présent, deux suites d'ensembles: $X_1(\xi)$ et $X_2(\xi)$, où $\xi < \omega_\eta$. Soit $p_1, p_2 \varepsilon X - G_1$ et $p_1 \neq p_2$, posons

$$X_1(1) = \{p_2\} \quad X_2(1) = \{p_2\}.$$

Supposons les ensembles $X_1(\zeta)$ et $X_2(\zeta)$ déjà définis pour chaque $\zeta < \xi$, où $\xi < \omega_\eta$ et

(5) $$X_1(\zeta) + X_2(\zeta) \subset X \quad \text{pour} \quad \zeta < \xi$$

(6) $$X_1(\zeta) \cdot X_2(\zeta) = 0 \quad \text{„} \quad \zeta < \xi$$

(7) $$X_i(\zeta_1) \subset X_1(\zeta_2) \quad \text{pour} \quad i = 1, 2 \quad \text{et} \quad \zeta_1 < \zeta_2 < \xi$$

(8) $$\overline{\overline{X_i(\zeta)}} = \overline{\zeta} \quad \text{„} \quad \text{„} \quad \text{„} \quad \zeta < \xi$$

(9) $$X_i(\zeta_1) - G_{\xi_2} \neq 0 \quad \text{„} \quad \text{„} \quad \text{„} \quad \zeta_2 \leqslant \zeta_1 < \xi.$$

(7) et (8) nous donnent pour $i = 1, 2$

$$\overline{\overline{\sum_{\zeta<\xi} X_i(\zeta)}} = \overline{\xi} < 2^{\aleph_0}$$

et en vertu de (4)

$$\overline{\overline{X - G_\xi}} = 2^{\aleph_0}$$

ce qui entraîne l'existence des deux points q_1 et q_2 différents,

$$q_1, q_2 \varepsilon (X - G_\xi) - \sum_{\zeta<\xi} [X_1(\zeta) + X_2(\zeta)].$$

Posons donc pour $i=1, 2$

$$X_i(\xi)=\{q_i\}+\sum_{\zeta<\xi} X_i(\zeta).$$

On voit facilément que les conditions (5) — (9) restent verifiées.

Posons enfin pour $i=1, 2$

$$X_i=\sum_{\xi<\omega_{\cdot\eta}} X_i(\xi).$$

Il resulte de (5), (7), (6) et (9) que

$$X_1+X_2\subset X$$

$$X_1\cdot X_2=0$$

(10) $$X_i-G_\xi\neq 0 \quad \text{si} \quad \xi<\omega_{\cdot\eta} \quad \text{et} \quad i=1, 2.$$

Il reste à démontrer que $|X_i|=|X|$. En effet, si $|X_1|<|X|$, il existe un ensemble ouvert G tel que

(11) $$X_1\subset G$$

et

$$|X-G|>0,$$

ce qui donne (4) d'àprès notre hypothèse et G appartient donc à la suite (3). Par conséquent $G=G_\xi$ pour un certain $\xi<\omega_{\cdot\eta}$ et dans ce cas (10) et (11) donnent une contradiction.

Les théorèmes 2 (propriété (P_1)) et 4 entraînent le suivant

Théorème 5. *Si chaque sous-ensemble de mesure extérieure positive de X a la puissance du continu, la propriété (P) est équivalente à l'hypothèse que $|X|=0$.*

Théorème 6. *Si chaque nombre cardinal $\leqslant\overline{\overline{X}}$ est accessible*[9]*, la propriété (P) équivaut à l'hypothèse que $|X|=0$.*

La démonstration resulte immédiatement du théorème 2 et d'un théorème de M. Ulam[10]) d'après lequel, si $|X|\neq 0$, l'hypothèse de notre théorème est incompatible avec la propriété (P_2).

Le théorème 5 (de même que le th. 6) entraîne le

Corollaire 7. *La propriété que X est de mesure nulle et la propriété (P) sont équivalentes s'il on admet l'hypothese de continu $(2^{\aleph_0}=\aleph_1)$.*

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[9]) Ulam: *Zur Masstheorie in der allgemenen Mengenlehre.* Fund. Math. 16 (1930), p. 140 note.

[10]) Ulam l. c., p. 141.

W. Opalski.

Zakrycia gwiazd obserwowane w Obserwatorjum Astronomicznem Uniwersytetu Warszawskiego od lipca 1931 r. do października 1932 r.

Przedstawił M. Kamieński dn. 26 listopada 1932 r.

W. Opalski.

Sternbedeckungen beobachtet an der Warschauer Universitäts-Sternwarte seit Juli 1931 bis Oktober 1932.

Mémoire présenté par M. M. Kamieński dans la séance du 26 Novembre 1932.

Bedeutung der Abkürzungen

Refraktor:

Gb = Grubb ⌀ 207mm
M = Merz 162
H = Heyde 162
G = Goerz 110
U. F. = Utzschneider-Fraunhofer ⌀ 97mm

Beobachter:

M. B. = M. Bielicki
J. G. = J. Gadomski
M. K. = M. Kamieński
L. O. = L. Orkisz
E. R. = E. Rybka
L. Z. = L. Zajdler

Erscheinungen:

I. = Verschwinden
o. = dunkler Rand
E. = Wiedererscheinen
c. = heller Rand

Koordinaten der Beobachtungstellen:

Instrument	λ_E Gr.	φ
Gb	$1^h\,24^m\,7^s.21$	$+52^0\,13'\,4''.4$
H	1 24 7 .29	+52 13 4 .5
M	1 24 7 .40	+52 13 4 .5
G, U. F.	1 24 7 .24	+52 13 4 .3

Bemerkung: 1932 September 14 Beobachtungen während der Mondfinsternis.

Nr.	Datum	Weltzeit	Stern	Stern-grösse	Erschei-nung	Instru-ment	Vergrösse-rung	Beobachter	Bemerkungen
	1931	h m s		m					
1	Juli 21	19 41 43.9	α Vir	1.2	I. o.	G	139	L. O.	durch dünne Wolken
2	Septemb. 4	22 5 59.6	χ Tau	5.3	E. o.	H	73	E. R.	
3	Dezemb. 17	19 10 26.5	98 B Psc	6.3	I. o.	Gb	97	M. K.	
		26.7				H	73	E. R.	
		26.8				U. F.	41	M. B.	
4	18	21 35 22.1	ε Psc	4.4	I. o.	H	73	E. R.	gut; durch Wolken
		22.2				M	44	J. G.	mittelmässig
	1932								
5	April 22	21 47 14.7 *)	π Sco	3.0	I. c.	H	73	L. O.	unsicher; Mondrand stark unruhig
6	22	22 46 43.2	π Sco	3.0	E. o.	H	73	L. O.	gut
7	Mai 11	20 40 51.4	35 B Cnc	6.4	I. o.	G	93	J. G.	beobachtet ohne Ephemeride
8	11	20 57 9.8 *)	35 B Cnc	6.4	E. c.	G	93	J. G.	ungefähr 3^s verspätet
9	17	20 24 51.0	ι Vir	5.7	I. o.	G	93	M. B.	
		51.5				G b	97	M. K.	gut
		51.6				H	73	L. O.	durch Zirruswolken

10	Mai 20	22 56 2.0*)	τ Sco	2.8	I. c.	Gb	97	M. B.	
		6.8*)				H	73	L. O.	Mondrand stark unruhig
11	Septemb. 14	20 34 31.5	BI — 3⁰.5657	9.0	I.	H	73	L. O.	Stern flimmerte stark
			A. G. Str. 8085						vor dem Verschwinden
12	14	20 45 30.5	BD — 4⁰.5907	9.2	I.	G	67	L. Z.	
		32.7*)				Gb	97	M. K.	vielleicht einige Sek. zu spät
		32.8*)				H	73	L. O.	sehr schwacher Stern
13	14	20 56 14.6*)	BD — 4⁰.5900	9.7	E.	G	67	L. Z.	
14	14	21 12 6.7*)	BD — 3⁰.5654	9.4	E.	Gb	97	M. K.	
			A. G. Str. 8081						
15	14	21 39 44.2	BD — 3⁰.5657	9.0	E.	Gb	97	M. K.	gut
		44.4	A. G. Str. 8085			M	44	J. G.	gut
		44.6				G	67	L. Z.	
		45.0				H	73	L. O.	gut
16	14	21 54 32.6*)	BD — 4⁰.5907	9.2	E.	H	73	L. O.	schwacher Stern
		34.7*)				Gb	97	M. K.	

*) Beobachtet mit Genauigkeit 1ˢ.

Posiedzenie

z dnia 15 grudnia 1932 r.

W. Sierpiński.

O zbiorze linjowym nieprzeliczalnym, który jest pierwszej kategorji na każdym zbiorze doskonałym.

Komunikat, przedstawiony na posiedzeniu dnia 15 grudnia 1932 r.

Streszczenie.

W komunikacie tym autor podaje, bez użycia hipotezy continuum, nowy, prostszy od dotychczas znanych, dowód istnienia zbioru linjowego nieprzeliczalnego, który jest pierwszej kategorji Baire'a na każdym zbiorze doskonałym.

W. Sierpiński.

Sur un ensemble linéaire non dénombrable qui est de première catégorie sur tout ensemble parfait.

Note présentée dans la séance du 15 décembre 1932.

Parmi les cinq démonstrations connues de l'existence d'un ensemble linéaire non dénombrable qui est de 1re catégorie de Baire sur tout ensemble parfait [1]) deux seulement n'utilisent pas l'hypothèse du continu: ce sont chronologiquement la deuxième et la quatrième.

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[1]) 1. N. Lusin, *Comptes Rendus* t. 158, p. 1259 (1914).
2. N. Lusin, *Fundamenta Mathematicae* t. II, p. 155 (trouvé en 1917, publié en 1921).
3. N. Lusin, *Fundamenta Mathematicae* t. IX, p. 116 (1927).
4. N. Lusin et W. Sierpiński, *Rend. Accad. Lincei*, vol. VII, ser. 6e, p. 214 (1928).
5. S. Saks, *Fundamenta Mathematicae* t. IX, p. 217 (1928).

Or, la deuxième fait usage de fractions continues et de suites transfinies formées de suites infinies de nombres naturels, et la quatrième est basée sur la théorie des ensembles analytiques.

Dans cette Note je donnerai encore une autre démonstration d'existence d'un ensemble linéaire non dénombrable qui est de 1re catégorie sur tout ensemble parfait. Elle n'utilise ni l'hypothèse du continu, ni les fractions continues, ni les ensembles analytiques, et elle peut être regardée comme plus élémentaire que les démonstrations connues qui n'utilisent pas l'hypothèse du continu. Cependant c'est une analyse de la seconde démonstration de M. Lusin qui m'a conduit à la trouver. Comme toutes les démonstrations connues, notre démonstration fait usage du théorème de M. Zermelo *(Wohlordnungssatz)*.

D'après ce théorème il existe une suite transfinie

(1) $$x_1, x_2, x_3, \ldots, x_\omega, x_{\omega+1}, \ldots, x_\alpha, \ldots$$

formée de tous les nombres réels et une suite transfinie

(2) $$\Gamma_1, \Gamma_2, \Gamma_3, \ldots, \Gamma_\omega, \Gamma_{\omega+1}, \ldots, \Gamma_\alpha, \ldots,$$

formée de tous les ensembles linéaires G_δ de mesure nulle.

Nous définirons maintenant par l'induction transfinie une suite transfinie du type Ω de nombres réels $t_\alpha (\alpha < \Omega)$ et une suite transfinie du type Ω des ensembles linéaires $E_\alpha (\alpha < \Omega)$ qui seront des G_δ de mesure nulle comme il suit.

Posons $t_1 = x_1$ et soit E_1 l'ensemble formé d'un seul point t_1. Soit maintenant α un nombre ordinal donné $< \Omega$ et supposons que nous avons déjà défini tous les points t_ξ, où $\xi < \alpha$ et tous les ensembles G_δ de mesure nulle E_ξ, où $\xi < \alpha$.

D'après $\alpha < \Omega$, la somme S_α de tous les ensembles E_ξ, où $\xi < \alpha$, en tant qu'une somme d'un ensemble au plus dénombrable d'ensembles de mesure nulle, est un ensemble de mesure nulle et il existe des termes de la suite (1) qui n'appartiennent pas à S_α: soit t_α le premier d'entre eux. Or, l'ensemble T_α de tous les points t_ξ, où $\xi \leqslant \alpha$ étant au plus dénombrable, il existe un G_δ de mesure nulle, donc un terme de la suite (2), contenant T_α: soit E_α le premier de tels termes de la suite (2).

Les suites transfinies $t_\alpha (\alpha < \Omega)$ et $E_\alpha (\alpha < \Omega)$ sont ainsi définies par l'induction transfinie et on a $t_\xi \in E_\alpha$ pour $\xi \leqslant \alpha < \Omega$. L'ensemble T de tous les termes de la suite $t_\alpha (\alpha < \Omega)$ est évidemment non dénombrable (puisque, pour $\xi < \alpha$, on a $t_\xi \in E_\xi \subset S_\alpha$ et t_α non $\in S_\alpha$, donc $t_\xi \neq t_\alpha$).

L'ensemble T jouit, comme on voit sans peine, de la propriété P suivante:

(P). *Quel que soit le sous-ensemble dénombrable D de T, il existe un ensemble G_δ contenant D et contenant seulement un ensemble dénombrable de points de T* [1]).

En effet, soit

$$t_{\alpha_1},\ t_{\alpha_2},\ t_{\alpha_3},\ \ldots$$

une suite infinie (dénombrable) quelconque de points de T. Les indices $\alpha_1, \alpha_2, \alpha_3, \ldots$ étant des nombres ordinaux $< \Omega$, il existe, comme on sait, un nombre ordinal $\alpha < \Omega$, tel que $\alpha_n < \alpha$ pour $n = 1, 2, 3, \ldots$, et de la définition de l'ensemble E_α résulte que E_α est un ensemble G_δ contenant tous les points t_ξ, où $\xi < \alpha$, donc contenant D. Or, de la définition des points t_ξ résulte que t_β non $\in S_\beta$ pour $\beta < \Omega$, et d'autre part on a $E_\alpha \subset S_\beta$ pour $\alpha < \beta < \Omega$: on a donc t_β non $\in E_\alpha$ pour $\alpha < \beta < \Omega$. Les points t_ξ de T qui appartiennent à l'ensemble E_α ont donc des indices $\xi \leqslant \alpha$: d'après $\alpha < \Omega$ nous concluons donc que leurs ensemble est au plus dénombrable.

Or, on voit sans peine que tout ensemble T qui jouit de la propriété (P) est de 1[re] catégorie sur tout ensemble parfait.

En effet, soit Q un ensemble parfait donné.

Pour démontrer que T est de 1[re] catégorie sur Q, il suffit évidemment de démontrer que T est de 1[re] catégorie sur toute portion de Q sur laquelle T est dense. Soit donc Q_1 une telle portion de Q, et soit D un sous-ensemble dénombrable de T, dense dans Q_1. D'après la propriété (P) de l'ensemble T, il existe un ensemble G_δ, soit E, contenant D et contenant seulement un ensemble dénombrable de points de T. L'ensemble E étant un G_δ, nous pouvons poser

$$E = G_1 G_2 G_3 \ldots, \tag{3}$$

où $G_n (n = 1, 2, 3, \ldots)$ sont des ensembles ouverts.

Posons

$$H_n = T - G_n, \quad \text{pour} \quad n = 1, 2, 3, \ldots \tag{4}$$

[1]) D'après M. Kuratowski la propriété P est équivalente à la suivante: *Tout sous-ensemble dénombrable de T est un G_δ relativement à T.*

L'ensemble D est dense dans Q_1 et on a, d'après (3): $D \subset E \subset G_n$, pour $n = 1, 2, 3, \ldots$: l'ensemble ouvert G_n est donc dense dans Q_1 et par suite l'ensemble (4) est non dense dans Q_1 (pour $n = 1, 2, 3, \ldots$).

Or, on a évidemment, d'après (3) et (4):

$$T = ET + H_1 + H_2 + H_3 + \ldots \tag{5}$$

Les ensembles H_n ($n = 1, 2, \ldots$) étant non denses dans Q_1 et l'ensemble ET étant dénombrable, la formule (5) prouve que l'ensemble T est de 1re catégorie sur Q_1, c. q. f. d.

L'ensemble T est donc de 1re catégorie sur tout ensemble parfait et notre assertion est démontrée.

St. Gołąb.

O funkcjach jednorodnych I. Równanie Eulera.

Przedstawił W. Sierpiński dn. 15 grudnia 1932 r.

Sur les fonctions homogènes I. Equation d'Euler.

Mémoire présenté par M. W. Sierpiński dans la séance du 15 décembre 1932.

Les fonctions homogènes jouent un rôle important dans le calcul des variations, dans la géometrie de Finsler[1]) et dans les applications à la mécanique. Néanmoins on n'a pas sacrifié une attention spéciale à quelques propriétés de ces fonctions. Dans la présente note je m'occupe de l'analyse de la formule classique d'Euler, à laquelle satisfont les fonctions homogènes. Cette formule, trouvée et demontrée pour les polynômes homogènes par Euler, a été ensuite generalisée pour les fonctions homogènes quelconques. C'est sous cette forme qu'elle se trouve exposée dans la plupart des Cours d'Analyse. Les démonstrations qu'on y trouve- très simples d'ailleurs-adoptent quelques hypothèses accessoires. Ces hypothèses renferment des prémisses dans lesquelles subsiste la formule donnant la dérivée d'une

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[1]) P. Finsler, Über Kurven und Flächen in allgemeinen Räumen, Dissertation Göttingen (1918).

fonction composée de plusieures variables. Ainsi, contrairement à ce qui se présente au cas de la fonction composée d'une variable, on est obligé, conformement aux conditions suffisantes connues, à supposer la continuité (de quelques au moins [1])) dérivées partielles. Je me suis posé le problème de savoir si la formule d'Euler subsiste lorsqu'on suppose seulement l'existence des dérivées partielles. La réponse à cette question est le sujet de la note présente. Je montre que pour les fonctions homogènes de deux variables la formule d'Euler subsiste lorsqu'on suppose seulement l'existence des deux dérivées partielles du 1-er ordre ou d'une d'elles (on suppose que $\frac{\partial f}{\partial x}$ existe sur l'axe y et que $\frac{\partial f}{\partial y}$ existe sur l'axe x. Pour les autres points (x, y) on suppose l'existence de l'une quelconque de ces dérivées, sans spécifier de laquelle). Il n'en est pas ainsi pour $n \geqslant 3$. Nous construisons pour tout n de cette sorte l'exemple d'une fonction à n variables qui est continue et possède toutes les dérivées du premier ordre et qui ne remplit pas la formule d'Euler.

La fonction

(1) $$f(x_1, x_2, \ldots, x_n)$$

s'appelle *homogène* d'ordre m lorsqu'elle vérifie pour tous les t l'identité:

(2) $$f(tx_1, tx_2, \ldots, tx_n) = t^m f(x_1, x_2, \ldots, x_n).$$

Elle s'appelle *positivement homogène* lorsqu'on suppose seulement que cette identité a lieu pour tous les t positifs. Une fonction peut être positivement homogène sans être homogène.

Théorème. *Si $f(x, y)$ est une fonction homogène (positivement homogène) d'ordre m et si en un certain point (x_0, y_0) elle possède les dérivées $f'_x(x_0, y_0)$ et $f'_y(x_0, y_0)$, alors on a pour ce point*

(3) $$x_0 f'_x(x_0, y_0) + y_0 f'_y(x_0, y_0) = m f(x_0, y_0).$$

Démonstration. Il suffit de se borner au cas

(4) $$y_0 \neq 0. \text{ [2])}$$

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[1]) Cf. p. ex. W. Sierpiński, Analiza, T. I, cz. IV, Warszawa (1925), p. 248, Théorème 225.

[2]) Pour le point $x_0 = 0$, $y_0 = 0$ la relation (3) est remplie évidemment. L'existence de deux dérivées partielles $f'_x(0, 0)$ et $f'_y(0, 0)$ entraîne dans ce cas l'une de deux relations suivantes: $m = 0$ ou $f(0, 0) = 0$.

Formons l'expression

(5) $$I = \frac{f(x_0, y_0 + h) - f(x_0, y_0)}{h}$$

et posons:

(6) $$\omega = \frac{y_0}{y_0 + h}.$$

En remarquant que $\omega > 0$ lorsque h est suffisamment petit nous obtenons d'après l'identité (2):

(7) $$I = \frac{\omega^{-m}}{h}[f(\omega x_0, y_0) - f(x_0, y_0)] + \frac{\omega^{-m} - 1}{h} f(x_0, y_0).$$

Si $x_0 = 0$ le premier terme du second membre de (7) disparait et d'après un calcul élémentaire on a:

(8) $$\lim_{h=0} \frac{\omega^{-m} - 1}{h} = + \frac{m}{y_0}.$$

Le passage à la limite appliqué à la formule (7) conduit donc à l'identité (3). Il reste à démontrer notre théorème dans le ces

(9) $$x_0 \neq 0.$$

En posant dans ce cas

(10) $$k = - \frac{h x_0}{y_0 + h}$$

on peut écrire (7) sous la forme

(11) $$I = - \frac{x_0}{y_0} \omega^{-m+1} \frac{f(x_0 + k, y_0) - f(x_0, y_0)}{k} + \frac{\omega^{-m} - 1}{h} f(x_0, y_0).$$

Comme $\omega \to 1$ on obtient en passant à la limite ($h \to 0$)

(12) $$f_y'(x_0, y_0) = - \frac{x_0}{y_0} f_x'(x_0, y_0) + \frac{m}{y_0} f(x_0, y_0),$$

d'où résulte l'identité (3). Notre théorème se trouve ainsi démontré.

Du raisonnement précédent résultent les observations suivantes: Si $x_0 \neq 0$, $y_0 \neq 0$, l'existence d'une des dérivées $f_x'(x_0, y_0)$, $f_y'(x_0, y_0)$ implique l'existence de l'autre. Dans le cas $x_0 = 0$ ($y_0 \neq 0$) l'existence de $f_x'(x_0, y_0)$ entraîne l'existence de $f_y'(x_0, y_0)$. Si enfin $y_0 = 0$ ($x_0 \neq 0$), alors de l'existence de $f_y'(x_0, y_0)$ résulte l'existence de $f_x'(x_0, y_0)$. En général, d'après une remarque due à M. T. Ważewski, en supposant qu'une fonction positivement homogène possède en un point $(x_0, y_0) \neq (0, 0)$ une dérivée calculée dans une direction (λ, μ), $(\lambda y_0 - \mu x_0) \neq 0$, on démontre l'existence de la dérivée, prise dans n'importe quelle direction pour tous les points de la demi-droite issue de $(0, 0)$ et passant par (x_0, y_0). En particulier f_x' et f_y existent alors pour tout point de la dite droite (excepté peut-être le point $(0, 0)$).

Nous construisons maintenant l'exemple d'une fonction homogène de n variables ($n \geqslant 3$) possédant toutes les dérivées du premier ordre qui est en outre continue et qui malgré cela ne remplit pas partout la formule d'Euler.

Nous définirons d'abord une fonction auxiliaire F de $(n-1)$ variables par les formules

$$(13) \quad \begin{cases} F\left(\frac{\pi}{4}, \frac{\pi}{4}, \ldots, \frac{\pi}{4}\right) = 0 \\ F(\varphi_2, \varphi_3, \ldots, \varphi_n) = \dfrac{\sin^3\left(\varphi_2 - \frac{\pi}{4}\right)}{\sum\limits_{i=2}^{n} \sin^2\left(\varphi_i - \frac{\pi}{4}\right)} \text{ pour } \sum\limits_{i=2}^{n} \left|\varphi_i - \frac{\pi}{4}\right| > 0. \end{cases}$$

Ensuite nous définirons la fonction

$$(14) \qquad f(x_1, x_2, \ldots, x_n) = F\left(\text{arc } tg \frac{x_2}{x_1}, \ldots, \text{arc } tg \frac{x_n}{x_1}\right).$$

Cette fonction sera définie partout à l'exception des points pour lesquels $x_1 = 0$. De la formule (14) on voit que f est homogène du degré 0. Nous affirmons qu'elle est continue en tout point où elle est définie. Les fonctions arc $tg \frac{x_i}{x_1}$ sont évidemment continues. Il reste à prouver que F est continue en $(\varphi_2, \ldots, \varphi_n)$. Ce n'est pas évident seulement dans le cas où tous les φ_i sont égaux à $\frac{\pi}{4}$. Dans ce cas on a

$$(15) \qquad \left|F(\varphi_2, \ldots, \varphi_n) - F\left(\frac{\pi}{4}, \ldots, \frac{\pi}{4}\right)\right| \leqslant \left|\sin\left(\varphi_2 - \frac{\pi}{4}\right)\right| \leqslant \left|\varphi_2 - \frac{\pi}{4}\right| \leqslant \sum_{i=2}^{n} \left|\varphi_i - \frac{\pi}{4}\right|,$$

d'où résulte la continuité de F au point en question.

Considérons maintenant le point

(16) $$x_1 = x_2 = \cdots = x_n = 1 .$$

Nous affirmons qu'en ce point f possède toutes les dérivées partielles du premier ordre. Désignons à cet effet par R_i la valeur de f au point dont toutes les coordonnées sont égales à 1 à l'exception de x_i qui est égal à $1 + h$. Avant de calculer R_i remarquons que des relations

(17) $$tg\, z = a, \quad a > 0, \quad 0 < z < \frac{\pi}{2}$$

il s'ensuit que

(18) $$\sin z = \frac{a}{\sqrt{1 + a^2}}, \quad \cos z = \frac{1}{\sqrt{1 + a^2}} .$$

Nous avons donc

(19) $$R_1 = F\left(\text{arc } tg \frac{1}{1 + h}, \ldots, \text{arc } tg \frac{1}{1 + h}\right).$$

En posant

(20) $$\varphi_i = \text{arc } tg \frac{1}{1 + h} \qquad (i = 2, \ldots, n),$$

nous obtenons

(21) $$\sin \varphi_i = \frac{1}{\sqrt{H}}, \quad \cos \varphi_i = \frac{1 + h}{\sqrt{H}}$$

où

(22) $$H = 2 + 2h + h^2$$

et par conséquent

(23) $$\sin\left(\varphi_i - \frac{\pi}{4}\right) = \alpha (\sin \varphi_i - \cos \varphi_i) = \frac{-\alpha h}{\sqrt{H}}, \quad \alpha = \frac{1}{\sqrt{2}} .$$

On a donc

(24) $$R_1 = -\frac{\alpha h}{(n - 1)\sqrt{H}} .$$

Calculons maintenant R_2. On a

(25) $$R_2 = F(\text{arc } tg (1 + h), \text{arc } tg\, 1, \ldots, \text{arc } tg\, 1) = \sin\left(\varphi_2 - \frac{\pi}{4}\right).$$

Afin de calculer φ_2 remarquons que $tg\, \varphi_2 = 1 + h$, et par suite

(26) $$\sin \varphi_2 = \frac{1 + h}{\sqrt{H}}, \quad \cos \varphi_2 = \frac{1}{\sqrt{H}}, \quad \sin\left(\varphi_2 - \frac{\pi}{4}\right) = \frac{\alpha h}{\sqrt{H}},$$

d'où

$$R_2 = \frac{\alpha\, b}{\sqrt{H}}. \tag{27}$$

On a enfin pour $j \geqslant 3$:

$$R_j = 0. \tag{28}$$

D'après la définition de la dérivée partielle $\frac{\partial f}{\partial x_k} = f_k$ nous aurons

$$f_k = \lim_{b=0} \frac{R_k}{b}. \tag{29}$$

Mais $\sqrt{H} \to \sqrt{2}$ si $b \to 0$. Nous aurons donc

$$\begin{cases} f_1 = -\dfrac{1}{2(n-1)} \\ f_2 = \dfrac{1}{2} \\ f_j = 0 \quad \text{pour} \quad 3 \leqslant j \leqslant n. \end{cases} \tag{30}$$

La fonction f possède, par conséquent, au point (16) toutes les dérivées partielles du premier ordre. Nous affirmons maintenant que la formule d'Euler n'est pas remplie pour le point (16). On a en effet

$$\sum_{i=1}^{n} x_i f_i = \sum_{i=1}^{n} f_i = \frac{1}{2} \cdot \frac{n-2}{n-1} \neq 0, \tag{31}$$

parce que $n \geqslant 3$, tandis que la formule d'Euler donnerait $\sum_{i=1}^{n} x_i f_i = 0$.

A partir de cette fonction homogène d'ordre 0 il est aisé de déduire l'exemple d'une fonction homogène analogue d'ordre quelconque m. Il suffit de poser à cet effet

$$\omega(x_1, x_2, \ldots, x_n) = x_1^m f(x_1, x_2, \ldots, x_n), \tag{32}$$

où f désigne la fonction précédemment construite.