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SUPPLEMENTARY REMARKS ON THE PORISM OF THE IN-AND-CIRCUMSCRIBED TRIANGLE.

[From the *Philosophical Magazine*, vol. XIII. (1857), pp. 19—30.]

In my former papers (see *Phil. Mag.* August and November, 1853, [115, 116]) I established (as part of a more general one) the following theorem, viz. the condition that there may be inscribed in the conic $U=0$ an infinity of triangles circumscribed about the conic $V=0$, is, that if we develope in ascending powers of k the square root of the discriminant of $kU+V$, the coefficient of k^2 in this development must vanish. Thus writing

$$\text{disct.}\,(kV+U)=(K,\ \Theta,\ \Theta',\ K'\between k,\ 1)^3,$$

the condition in question is found to be

$$3\Theta^2-4K\Theta'=0.$$

The following investigations, although relating only to particular cases of the theorem, are, I think, not without interest.

If the equation of the conic containing the angles is

$$U=2ayz+2bzx+2cxy=0,$$

and the equation of the conic touched by the sides is

$$V=x^2+y^2+z^2-2yz-2zx-2xy=0,$$

we have

$$\text{disct.}\ (k,\ k,\ k,\ a-k,\ b-k,\ c-k\between x,\ y,\ z)^2=(K,\ \Theta,\ \Theta',\ K'\between k,\ 1)^3,$$

that is,

$$\begin{aligned}K&=-4,\\ \Theta&=\tfrac{4}{3}(a+b+c),\\ \Theta'&=-\tfrac{1}{3}(a+b+c)^2,\\ K'&=2abc.\end{aligned}$$

and the equation $3\Theta^2-4K\Theta'=0$ becomes

$$\tfrac{16}{3}(a+b+c)^2-\tfrac{16}{3}(a+b+c)^2=0,$$

which is satisfied identically. This is as it should be; for it is plain that there exists a triangle, viz. the triangle ($x=0$, $y=0$, $z=0$), inscribed in the conic $U=0$, and circumscribed about the conic $V=0$.

Suppose that the equation of the conic containing the angles is

$$y^2-4zx=0,$$

and the equation of the conic touched by the sides is

$$ax^2+\ by^2+\ cz^2=0,$$

then the tangential equation of the last-mentioned conic is

$$bc\xi^2+ca\eta^2+ab\zeta^2=0;$$

and if we take for the angles of the triangle $x:y:z=1:2\lambda:\lambda^2$, or $1:2\mu:\mu^2$, or $1:2\nu:\nu^2$, then the equation of the line joining the angles (μ), (ν) is

$$2\mu\nu x-(\mu+\nu)\,y+z=0,$$

which will touch the conic $ax^2+by^2+cz^2=0$ if

$$bc\,.\,4\mu^2\nu^2+ca\,(\mu+\nu)^2+ab\,.\,4=0;$$

and it is required to find under what circumstances the equations

$$\begin{aligned}bc\,.\,4\mu^2\nu^2+ca\,(\mu+\nu)^2+ab\,.\,4&=0,\\ bc\,.\,4\nu^2\lambda^2+ca\,(\nu+\lambda)^2+ab\,.\,4&=0,\\ bc\,.\,4\lambda^2\mu^2+ca\,(\lambda+\mu)^2+ab\,.\,4&=0,\end{aligned}$$

become equivalent to two equations only. The condition is of course included in the general formula; and putting

$$\text{disct.}\,(ka,\ kb+1,\ kc,\ 0,\ -2,\ 0\between x,\ y,\ z^2)=(K,\ \Theta,\ \Theta',\ K'\between k,\ 1)^3,$$

we must have

$$3\Theta^2-4K\Theta'=0.$$

The discriminant in question is

$$k^3abc+k^2ac-k\,.\,4b-4=0,$$

where $K=1$, $\Theta=\tfrac{1}{3}ac$, $\Theta'=-\tfrac{4}{3}b$, $K'=-4$; the required condition is therefore $ac+16b^2=0$, or say

$$b=-\tfrac{1}{4}i\sqrt{ac}.$$

Substituting this value, the equations become

$$c\mu^2\nu^2 + i\sqrt{ca}\,(\mu+\nu)^2 + a = 0,$$

$$c\nu^2\lambda^2 + i\sqrt{ca}\,(\nu+\lambda)^2 + a = 0,$$

$$c\lambda^2\mu^2 + i\sqrt{ca}\,(\lambda+\mu)^2 + a = 0;$$

the first and second of these are

$$A + 2H\nu + B\nu^2 = 0$$

$$A' + 2H'\nu + B'\nu^2 = 0,$$

where

$$A = (-i\sqrt{a} + \mu^2\sqrt{c})\,i\sqrt{a},\quad H = i\sqrt{ca}\mu,\quad B = (i\sqrt{a} + \mu^2\sqrt{c})\sqrt{c},$$

$$A' = (-i\sqrt{a} + \lambda^2\sqrt{c})\,i\sqrt{a},\quad H' = i\sqrt{ca}\lambda,\quad B' = (i\sqrt{a} + \lambda^2\sqrt{c})\sqrt{c},$$

$$AB' + A'B - 2HH' = 2i\sqrt{ac}\,(a - i\sqrt{ac}\lambda\mu + c\lambda^2\mu^2),$$

$$AB - H^2 = i\sqrt{ac}\,(a - i\sqrt{ac}\mu^2 + c\mu^4),$$

$$A'B' - H'^2 = i\sqrt{ac}\,(a - i\sqrt{ac}\lambda^2 + c\lambda^4);$$

and the result of the elimination therefore is

$$(a - i\sqrt{ac}\lambda^2 + c\lambda^4)(a - i\sqrt{ac}\mu^2 + c\mu^4) - (a - i\sqrt{ac}\lambda\mu + c\lambda^2\mu^2)^2 = 0,$$

viz.

$$2\sqrt{ca}\,(\lambda-\mu)^2\,(c\lambda^2\mu^2 + i\sqrt{ca}\,(\lambda+\mu)^2 + a) = 0;$$

which agrees, as it should do, with the third equation.

To find the condition that it may be possible in the conic

$$x^2 + y^2 + z^2 = 0$$

to inscribe an infinity of triangles, each of them circumscribed about the conic

$$ax^2 + by^2 + cz^2 = 0:$$

let the equations of the sides be

$$l\sqrt{a}x + m\sqrt{b}y + n\sqrt{c}z = 0,$$

$$l'\sqrt{a}x + m'\sqrt{b}y + n'\sqrt{c}z = 0,$$

$$l''\sqrt{a}x + m''\sqrt{b}y + n''\sqrt{c}z = 0;$$

then the conditions of circumscription are

$$l^2 + m^2 + n^2 = 0,$$

$$l'^2 + m'^2 + n'^2 = 0,$$

$$l''^2 + m''^2 + n''^2 = 0;$$

and the conditions of inscription are

$$bc\,(m'n''-m''n')^2+ca\,(n'l''-n''l')^2+ab\,(l'm''-l''m')^2=0,$$
$$bc\,(m''n-mn'')^2+ca\,(n''l-nl'')^2+ab\,(l''m-lm'')^2=0,$$
$$bc\,(mn'-m'n)^2+ca\,(nl'-n'l)^2+ab\,(lm'-l'm)^2=0.$$

Now

$$\begin{aligned}(mn'-m'n)^2 &= (m^2+n^2)(m'^2+n'^2)-(mm'+nn')^2\\ &= l^2l'^2-(mm'+nn')^2\\ &= -(ll'+mm'+nn')(-ll'+mm'+nn');\end{aligned}$$

and making the like change in the analogous expressions, and putting for shortness

$$-bc+ca+ab=\alpha,$$
$$bc-ca+ab=\beta,$$
$$bc+ca-ab=\gamma,$$

the conditions in question become

$$(l'l''+m'm''+n'n'')(\alpha l'l''+\beta m'm''+\gamma n'n'')=0,$$
$$(l''l+m''m+n''n)(\alpha l''l+\beta m''m+\gamma n''n)=0,$$
$$(ll'+mm'+nn')(\alpha ll'+\beta mm'+\gamma nn')=0.$$

The proper solution is that given by the system of equations

$$l^2+m^2+n^2=0,$$
$$l'^2+m'^2+n'^2=0,$$
$$l''^2+m''^2+n''^2=0,$$

$$\alpha l'l''+\beta m'm''+\gamma n'n''=0,$$
$$\alpha l''l+\beta m''m+\gamma n''n=0,$$
$$\alpha ll'+\beta mm'+\gamma nn'=0;$$

and by writing $l=\dfrac{f}{\sqrt{\alpha}}$, $l'=\dfrac{f'}{\sqrt{\alpha}}$, $l''=\dfrac{f''}{\sqrt{\alpha}}$, $m=\dfrac{g}{\sqrt{\beta}}$, &c., $A=\dfrac{1}{\alpha}$, $B=\dfrac{1}{\beta}$, $C=\dfrac{1}{\gamma}$, these equations become

$$Af^2+Bg^2+Ch^2=0,$$
$$Af'^2+Bg'^2+Ch'^2=0,$$
$$Af''^2+Bg''^2+Ch''^2=0,$$

$$f'f''+g'g''+h'h''=0,$$
$$f''f+g''g+h''h=0,$$
$$ff'+gg'+hh'=0.$$

The first of which systems expresses that the points (f, g, h), (f', g', h'), (f'', g'', h'') are points in the conic

$$Ax^2+By^2+Cz^2=0\,;$$

and the second condition expresses that each of the points in question is the pole with respect to the conic

$$x^2+\;\;y^2+\;\;z^2=0$$

of the line joining the other two points, i.e. that the three points are a system of conjugate points with respect to the last-mentioned conic. The problem is thus reduced to the following one:

To find the condition in order that it may be possible in the conic

$$Ax^2+By^2+Cz^3=0$$

to inscribe an infinity of triangles such that the angles are a system of conjugate points with respect to the conic

$$x^2+\;\;y^2+\;\;z^2=0.$$

Before going further it is proper to remark that if, instead of assuming $\alpha l'l''+\beta m'm''+\gamma n'n''=0$, we had assumed

$$l'l''+m'm''+n'n''=0,$$

this, combined with the equations

$$l'^2+m'^2+n'^2=0,\quad l''^2+m''^2+n''^2=0,$$

would have given $l' : m' : n' = l'' : m'' : n''$, i.e. two of the angles of the triangle would have been coincident: this obviously does not give rise to any proper solution. Returning now to the system of equations in f, g, h, &c., since the equations give only the ratios $f : g : h$; $f' : g' : h'$; $f'' : g'' : h''$, we may if we please assume

$$f^2\;+g^2+h^2\;=1,$$
$$f'^2+g'^2+h'^2=1,$$
$$f''^2+g''^2+h''^2=1,$$

which, combined with the second system of equations, gives

$$f^2\;+f'^2\;+f''^2=1,$$
$$g^2\;+g'^2\;+g''^2=1,$$
$$h^2\;+h'^2\;+h''^2=1.$$

We have, consequently,

$$\begin{aligned}A+B+C&=A\,(f^2+f'^2+f''^2)+B\,(g^2+g'^2+g''^2)+C\,(h^2+h'^2+h''^2)\\&=(Af^2+Bg^2+Ch^2)+(Af'^2+Bg'^2+Ch'^2)+(Af''^2+Bg''^2+Ch''^2),\end{aligned}$$

i.e.

$$A+B+C=0,$$

for the condition that it may be possible in the conic

$$Ax^2 + By^2 + Cz^2 = 0$$

to describe an infinity of triangles the angles of which are conjugate points with respect to the conic $x^2 + y^2 + z^2 = 0$.

The equation of the conic $Ax^2 + By^2 + Cz^2 = 0$ may be written in the form

$$(b^2c^2 - c^2a^2 - a^2b^2 + 2a^2bc)\, x^2 + (c^2a^2 - a^2b^2 - b^2c^2 + 2ab^2c)\, y^2 + (a^2b^2 - b^2c^2 - c^2a^2 + 2abc^2)\, z^2 = 0,$$

which gives the values of A, B, C; or again in the form

$$\begin{aligned} &2\,(bc + ca + ab)\,(bcx^2 + cay^2 + abz^2) \\ &- (bc + ca + ab)^2\,(\;x^2 + \;y^2 + \;z^2) \\ &+ 4abc\,(ax^2 + \;by^2 + \;cz^2) = 0\,; \end{aligned}$$

where it should be observed that $bcx^2 + cay^2 + abz^2 = 0$ is the equation of the conic which is the polar of $ax^2 + by^2 + cz^2 = 0$ with respect to $x^2 + y^2 + z^2 = 0$. It is very easy from the last form to deduce the equation of the auxiliary conic, when the conics $ax^2 + by^2 + cz^2 = 0$, $x^2 + y^2 + z^2 = 0$ are replaced by conics represented by perfectly general equations.

The condition $A + B + C = 0$ gives, substituting the values of A, B, C,

$$b^2c^2 + c^2a^2 + a^2b^2 - 2abc\,(a + b + c) = 0\,;$$

or in a more convenient form,

$$(bc + ca + ab)^2 - 4abc\,(a + b + c) = 0,$$

as the condition in order that it may be possible to inscribe in the conic $x^2 + y^2 + z^2 = 0$ an infinity of triangles, the sides of which touch the conic $ax^2 + by^2 + cz^2 = 0$: this agrees perfectly with the general theorem.

It is convenient to add (as a somewhat more general form of the equation $A + B + C = 0$), that the condition in order that it may be possible in the conic $Ax^2 + By^2 + Cz^2 = 0$ to inscribe an infinity of triangles the angles of which are conjugate points with respect to the conic $A_1x^2 + B_1y^2 + C_1z^2 = 0$, is

$$\frac{A}{A_1} + \frac{B}{B_1} + \frac{C}{C_1} = 0.$$

But the problem to find the condition in order that it may be possible in the conic $x^2 + y^2 + z^2 = 0$ to inscribe an infinity of triangles the sides of which touch the conic $ax^2 + by^2 + cz^2 = 0$, may, by the assistance of the geometrical theorem to be presently mentioned, be at once reduced to the problem:

To find the condition in order that it may be possible in the conic $x^2 + y^2 + z^2 = 0$ to inscribe an infinity of triangles the sides of which are conjugate points with respect to a conic

$$A_1x^2 + B_1y^2 + C_1z^2 = 0.$$

The theorem referred to is as follows:

THEOREM. If the chord PP' of a conic S envelope a conic σ, the points P, P' are harmonics with respect to a conic T which has with S, σ, a system of common conjugate points.

Take for the equation of S,

$$x^2 + y^2 + z^2 = 0,$$

and for the equation of σ,

$$ax^2 + by^2 + cz^2 = 0\,;$$

then if (x_1, y_1, z_1), (x_2, y_2, z_2) are the coordinates of the points P, P' respectively, we have

$$x_1^2 + y_1^2 + z_1^2 = 0,$$
$$x_2^2 + y_2^2 + z_2^2 = 0,$$

and the condition in order that the chord may touch the conic σ is

$$bc\,(y_1z_2 - y_2z_1)^2 + ca\,(z_1x_2 - z_2x_1)^2 + ab\,(x_1y_2 - x_2y_1)^2 = 0.$$

But we have

$$\begin{aligned}(y_1z_2 - y_2z_1)^2 &= (y_1^2 + z_1^2)(y_2^2 + z_2^2) - (y_1y_2 + z_1z_2)^2\\ &= x_1^2x_2^2 - (y_1y_2 + z_1z_2)^2\\ &= (x_1x_2 + y_1y_2 + z_1z_2)(x_1x_2 - y_1y_2 - z_1z_2),\end{aligned}$$

and making the like change in the analogous quantities, and putting for shortness

$$\begin{aligned}\alpha &= -bc + ca + ab,\\ \beta &= bc - ca + ab,\\ \gamma &= bc + ca - ab,\end{aligned}$$

the condition in question becomes

$$(x_1x_2 + y_1y_2 + z_1z_2)(\alpha x_1r_2 + \beta y_1y_2 + \gamma z_1z_2) = 0.$$

But the equation $x_1x_2 + y_1y_2 + z_1z_2 = 0$ must be rejected, as giving with the equations $x_1^2 + y_1^2 + z_1^2 = 0$, $x_2^2 + y_2^2 + z_2^2 = 0$ the relation $x_1 : y_1 : z_1 = x_2 : y_2 : z_2$; we have therefore

$$\alpha x_1x_2 + \beta y_1y_2 + \gamma z_1z_2 = 0,$$

which implies that the points (x_1, y_1, z_1) and (x_2, y_2, z_2) are harmonics with respect to the conic

$$\alpha x^2 + \beta y^2 + \gamma z^2 = 0,$$

which is a conic having with S, σ, a system of common conjugate points. The equation may also be written

$$(-bc + ca + ab)\,x^2 + (bc - ca + ab)\,y^2 + (bc + ca - ab)\,z^2 = 0\,;$$

or, as it may also be written,

$$(bc + ca + ab)(x^2 + y^2 + z^2) - 2(bcx^2 + caz^2 + abx^2);$$

and, as before remarked, $bcx^2 + cay^2 + abz^2 = 0$ is the equation of the conic which is the polar of $ax^2 + by^2 + cz^2 = 0$ with respect to $x^2 + y^2 + z^2 = 0$.

The condition in order that there may be inscribed in the conic $x^2 + y^2 + z^2 = 0$ an infinity of triangles the angles of which are conjugate points with respect to the conic $\alpha x^2 + \beta y^2 + \gamma z^2 = 0$, is

$$\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = 0;$$

or writing this equation under the form $\beta\gamma + \gamma\alpha + \alpha\beta = 0$, and substituting for α, β, γ their values, we have the equation already found, as the condition in order that it may be possible in the conic $x^2 + y^2 + z^2 = 0$ to inscribe an infinity of triangles the sides of which touch the conic $ax^2 + by^2 + cz^2 = 0$.

THEOREM. Let

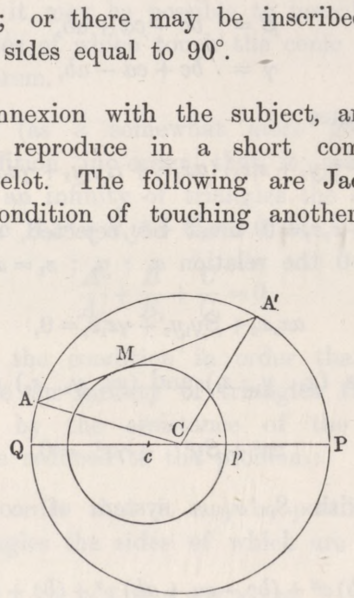
$$ax^2 + by^2 + cz^2 = 0$$

be the equation of a spherical conic, and let $(\xi : \eta : \zeta)$, a point on the conic, be the pole of a great circle cutting the conic in two points; the conic intersects upon the great circle an arc given by the equation

$$\cos\delta = \frac{(a+b+c)\sqrt{\xi^2 + \eta^2 + \zeta^2}}{\sqrt{(a+b+c)^2(\xi^2 + \eta^2 + \zeta^2) - 4(bc\xi^2 + ca\eta^2 + ab\zeta^2)}};$$

hence if $a + b + c = 0$, $\delta = 90°$; or there may be inscribed in the conic an infinity of triangles having each of their sides equal to 90°.

It is worth while, in connexion with the subject, and for the sake of a remark to which they give rise, to reproduce in a short compass some results long ago obtained by Jacobi and Richelot. The following are Jacobi's formulæ for the chords of a circle subjected to the condition of touching another circle; viz. if in the figure we put



$$CP = R,$$
$$cp = r,$$
$$Cc = a,$$
$$\angle ACP = 2\phi,$$
$$\angle A'CP = 2\phi';$$

then it is clear from geometrical considerations that

$$\frac{d\phi}{\overline{MA}} = \frac{d\phi'}{\overline{MA'}}.$$

We have

$$\begin{aligned}\overline{MA}^2 = \overline{cA}^2 - \overline{cM}^2 &= a^2 + R^2 + 2aR\cos 2 - \phi r^2 \\ &= (a+R)^2 - r^2 - 4aR\sin^2\phi \\ &= \{(a+R)^2 - r^2\}(1 - k^2 \sin 2\phi),\end{aligned}$$

or

$$\frac{d\phi}{\sqrt{1-k^2\sin^2\phi}} = \frac{d\phi'}{\sqrt{1-k^2\sin^2\phi'}},$$

where

$$k^2 = \frac{4aR}{(a+R)^2 - r^2},$$

and therefore also

$$k'^2 = \frac{(R-a)^2 - r^2}{(R+a)^2 - r^2}.$$

It will be convenient for comparison with the formulæ of Richelot to write $\angle ACQ = 2\psi$; this gives

$$2\psi = \pi - 2\phi,$$

and the differential equation thus becomes

$$\frac{d\psi}{\sqrt{a^2+R^2-r^2-2aR\cos 2\psi}} = \frac{d\psi'}{\sqrt{a^2+R^2-r^2-2aR\cos 2\psi'}},$$

i.e.

$$\frac{d\psi}{\sqrt{m-n\cos 2\psi}} = \frac{d\psi'}{\sqrt{m-n\cos 2\psi'}};$$

or if

$$\tan\psi = g\tan\theta,$$

and therefore

$$\cos 2\psi = \frac{1-g^2\tan^2\theta}{1+g^2\tan^2\theta},$$

$$m - n\cos 2\psi = \frac{(m-n)\cos^2\theta + (m+n)g^2\sin^2\theta}{\cos^2\theta + g^2\sin^2\theta};$$

or if $g^2 = \dfrac{m-n}{m+n}$, then this is

$$= \frac{m-n}{\cos^2\theta + \dfrac{m-n}{m+n}\sin^2\theta},$$

$$= \frac{m-n}{1-\dfrac{2n}{m+n}\sin^2\theta}$$

and we have also

$$d\psi = \frac{\sqrt{\dfrac{m-n}{m+n}}\,d\theta}{1-\dfrac{2n}{m+n}\sin^2\theta},$$

and thence,

$$\frac{d\psi}{\sqrt{m-n\cos 2\psi}} = \frac{d\theta}{\sqrt{m+n}\sqrt{1-\dfrac{2n}{m+n}\sin^2\theta}},$$

that is,

$$\frac{d\psi}{\sqrt{m-n\cos 2\psi}} = \frac{d\theta}{\sqrt{m+n}\sqrt{1-k^2\sin^2\theta}},$$

where $k^2 = \dfrac{2n}{m+n}$ has the same value as before. Hence the relation between θ, θ' is

$$\frac{d\theta}{\sqrt{1-k^2\sin^2\theta}} = \frac{d\theta'}{\sqrt{1-k^2\sin^2\theta'}},$$

which is identical with that between ϕ and ϕ'; and in fact the equation between θ, ϕ is

$$\tan\theta\tan\phi = \frac{1}{k'},$$

which, if $\phi = \operatorname{am} u$, gives $\theta = \operatorname{am}(K-u)$.

The differential equation contains only a single arbitrary parameter; hence the same differential equation might have been obtained from different values of a, R, the parameters which determine the circle enveloped by the moveable chord. The condition for this of course is $\dfrac{m'}{n'} = \dfrac{m}{n}$, that is

$$\frac{a^2+R^2-r^2}{2aR} = \frac{a'+R^2-r'^2}{2a'R},$$

or as the equation may be written

$$(aa'-R^2)(a-a') - R(a'r^2-ar'^2) = 0;$$

this implies that the enveloped circles intersect the other circle in the same two points, or that all the circles have a common chord.

Suppose for $\psi = 0$, we have $\psi' = \beta$, then it is easy to see geometrically that

$$\tan^2 \beta = \frac{(R-a)^2 - r^2}{r^2};$$

let the corresponding value of θ' be $\theta' = \alpha$, i.e. suppose that for $\theta = 0$, we have $\theta' = \alpha$, then

$$\tan^2 \alpha = \frac{(R+a)^2 - r^2}{(R-a)^2 - r^2} \cdot \frac{(R-a)^2 - r^2}{r^2},$$

i.e.

$$\tan^2 \alpha = \frac{(R+a)^2 - r^2}{r^2};$$

or, what is the same thing,

$$\sec \alpha = \frac{R}{r} + \frac{a}{r};$$

and α having this value, we have for the finite relation between θ, θ',

$$F\theta' = F\theta + F\alpha.$$

Richelot has shown, by precisely similar reasoning, that for circles of the sphere we have

$$\frac{d\psi}{\sqrt{\cos^2 r - (\cos R \cos a + \sin R \sin a \cos 2\psi)^2}} = \frac{d\psi'}{\sqrt{\cos^2 r - (\cos R \cos a + \sin R \sin a \cos 2\psi')^2}}$$

which is of the form

$$\frac{d\psi}{\sqrt{1 - (\lambda + \mu \cos 2\psi)^2}} = \frac{d\psi'}{\sqrt{1 - (\lambda + \mu \cos 2\psi')^2}},$$

where

$$\lambda = \frac{\cos R \cos a}{\cos r},$$

$$\mu = \frac{\sin R \sin a}{\cos r}.$$

It is very important to remark, that this equation contains the two parameters λ, μ, so that the same equation cannot be obtained with any new values of the parameters a, r; or the formulæ *in plano* for three or more circles do not apply to circles of the sphere: the geometrical reason for this is as follows, viz. in the plane a circle is a conic passing through two fixed points (the circular points at ∞), and consequently any number of circles having a common chord are in fact to be considered as conics each of which passes through the same four points. But circles of the sphere are not spherical conics passing through two fixed points, but are merely spherical conics having a double contact with an imaginary spherical conic (viz. the curve of intersection of the sphere with a sphere radius zero); hence circles of the sphere having a common spherical chord are not spherical conics passing through the same four points. I am not sure whether this remark as to the ground of the distinction between the theory of circles *in plano* and that of circles on the sphere has been explicitly made in any of the treatises on spherical geometry.

To reduce the equation, write

$$\tan\psi = \sqrt{\frac{1-(\lambda+\mu)}{1-(\lambda-\mu)}}\tan\theta;$$

then, after a simple reduction,

$$\frac{d\psi}{\sqrt{1-(\lambda+\mu\cos 2\psi)^2}} = \frac{d\theta}{\sqrt{(1+\mu)^2-\lambda^2}\sqrt{1-\dfrac{4\mu}{(1+\mu)^2-\lambda^2}\sin^2\theta}};$$

or the relation between the two values of θ is

$$\frac{d\theta}{\sqrt{1-k^2\sin^2\theta}} = \frac{d\theta'}{\sqrt{1-k^2\sin^2\theta'}},$$

where

$$k^2 = \frac{4\mu}{(1+\mu)^2-\lambda^2},$$

that is

$$k^2 = \frac{4\dfrac{\tan R}{\tan r}\cdot\dfrac{\sin a}{\cos R\sin r}}{\left(\dfrac{\tan R}{\tan r}+\dfrac{\sin a}{\cos R\sin r}\right)^2-1}.$$

Suppose that for $\psi=0$, we have $\psi'=\beta$, it is easy to see that

$$\tan^2\beta = \frac{\sin^2(R-a)-\sin^2 r}{\cos^2 R\sin^2 r};$$

let the corresponding value of θ' be $\theta'=\alpha$, i.e. suppose that for $\theta=0$, we have $\theta'=\alpha$, then

$$\tan^2\alpha = \frac{1-\dfrac{\cos(R+a)}{\cos r}}{1-\dfrac{\cos(R-a)}{\cos r}}\cdot\frac{\sin^2(R-a)-\sin^2 r}{\cos^2 R\sin^2 r}$$

$$= \frac{\cos r-\cos(R+a)}{\cos r-\cos(R-a)}\cdot\frac{\cos^2 r-\cos^2(R-a)}{\cos^2 R\sin^2 r}$$

$$= \frac{(\cos r-\cos(R+a))(\cos r+\cos(R-a))}{\cos^2 R\sin^2 r}$$

$$= \frac{(\cos r+\sin R\sin a)^2-\cos^2 R\cos^2 a}{\cos^2 R\sin^2 r}$$

$$= \frac{(\cos r\sin R+\sin a)^2-\cos^2 R\sin^2 r}{\cos^2 R\sin^2 r},$$

i.e.

$$\tan^2 \alpha = \left(\frac{\tan R}{\tan r} + \frac{\sin a}{\cos R \sin r}\right)^2 - 1,$$

whence

$$\sec \alpha = \left(\frac{\tan R}{\tan r} + \frac{\sin a}{\cos R \sin r}\right);$$

and α having this value, the finite relation between θ, θ' is

$$F\theta' = F\theta + F\alpha.$$

By comparing with the corresponding formula *in plano*, we arrive at Richelot's conclusion, that the formulæ for the sphere may be deduced from those *in plano* by writing in the place of $\frac{R}{r}$, $\frac{a}{r}$, the functions $\frac{\tan R}{\tan r}$, $\frac{\sin a}{\cos R \sin r}$, respectively.

2, *Stone Buildings, October* 1, 1856.