197.

NOTE ON THE THEORY OF LOGARITHMS.

[From the *Philosophical Magazine*, vol. XI. (1856), pp. 275–280.]

AN imaginary quantity $x+yi$ may always be expressed in the form

$$x+yi=r(\cos\theta+i\sin\theta)=re^{\theta i},$$

where r is positive, and θ is included between the limits $-\pi$ and $+\pi$. We have in fact

$$r=\sqrt{x^2+y^2};$$

and when x is positive,

$$\theta=\tan^{-1}\frac{y}{x};$$

but when x is negative,

$$\theta=\tan^{-1}\frac{y}{x}\pm\pi;$$

where $\tan^{-1}$ denotes an arc between the limits $-\frac{1}{2}\pi$, $+\frac{1}{2}\pi$, and where the upper or under sign is to be employed according as y is positive or negative. I use for convenience the mark $\equiv$ to denote identity of sign; we may then write

$$\theta=\tan^{-1}\frac{y}{x}+\epsilon\pi,$$

where

$$x\equiv+,\quad \epsilon=0,$$

$$x\equiv-,\quad \epsilon=\pm1\equiv y.$$

It should be remarked that θ has a unique value except in the single case $x\equiv-$, $y=0$, where θ is indeterminately $\pm\pi$. We have, in fact, $\theta=+\pi$ or $\theta=-\pi$

according as x is considered as the limit of $x+yi$, $y \equiv +$, or of $x+yi$, $y \equiv -$. It is natural to write

$$\log (x+yi) = \log r + \theta i,$$

or what is the same thing,

$$\log (x+yi) = \log \sqrt{x^2+y^2} + \left(\tan^{-1}\frac{y}{x} + \epsilon\pi\right) i\,;$$

and I take this equation as the definition of the logarithm of an imaginary quantity. The question then arises, to find the value of the expression

$$\log (x+yi) + \log (x'+y'i) - \log (x+yi)(x'+y'i).$$

The preceding definition is, in fact, in the case of x positive, that given by M. Cauchy in the *Exercises de Mathématique*, vol. I. [1826]; and he has there shown that x, x', $xx'-yy'$ being all of them positive, the above-mentioned expression reduces itself to zero. The general definition is that given in my *Mémoire sur quelques Formules du Calcul Intégral*, Liouville, vol. XII. [1847], p. 231 [49]; but I was wrong in asserting that the expression always reduced itself to zero. We have, in fact, in general

$$\tan^{-1}\alpha + \tan^{-1}\beta = \tan^{-1}\frac{\alpha+\beta}{1-\alpha\beta},$$

when $1-\alpha\beta$ is positive; but when $1-\alpha\beta$ is negative (which implies that α, β have the same sign), then

$$\tan^{-1}\alpha + \tan^{-1}\beta = \tan^{-1}\frac{\alpha+\beta}{1-\alpha\beta} \pm \pi,$$

where the upper or under sign is to be employed according as α and β are positive or negative; or what is the same thing,

$$\tan^{-1}\alpha + \tan^{-1}\beta = \tan^{-1}\frac{\alpha+\beta}{1-\alpha\beta} + \epsilon\pi,$$

where

$$1-\alpha\beta \equiv +, \quad \epsilon = 0,$$
$$1-\alpha\beta \equiv -, \quad \epsilon = \pm 1 \equiv \alpha+\beta \equiv \alpha \equiv \beta.$$

This being premised, then writing

$$\log (x+yi) = \log \sqrt{x^2+y^2} + \left(\tan^{-1}\frac{y}{x} + \epsilon\pi\right) i,$$

$$\log (x'+y'i) = \log \sqrt{x'^2+y'^2} + \left(\tan^{-1}\frac{y'}{x'} + \epsilon'\pi\right) i,$$

$$\log (x+yi)(x'+y'i) = \log [(xx'-yy') + (xy'+yx')\,i]$$

$$= \log \sqrt{x^2+y^2}\,\sqrt{x'^2+y'^2} + \left(\tan^{-1}\frac{xy'+x'y}{xx'-yy'} + \epsilon''\pi\right) i,$$

$$\tan^{-1}\frac{y}{x} + \tan^{-1}\frac{y'}{x'} = \tan^{-1}\frac{xy'+x'y}{xx'-yy'} + \epsilon'''\pi,$$

we find

$$\log(x+yi)+\log(x'+y'i)-\log(x+yi)(x'+y'i)=(\epsilon+\epsilon'-\epsilon''+\epsilon''')\pi i.$$

Hence, considering the different cases:

I. $$x\equiv +,\quad x'\equiv +,\quad xx'-yy'\equiv +,$$

$$\epsilon=0,$$

$$\epsilon'=0,$$

$$\epsilon''=0,$$

$$\epsilon'''=0,$$

and therefore

$$\epsilon+\epsilon'-\epsilon''+\epsilon'''=0.$$

II. $$x\equiv +,\quad x'\equiv +,\quad xx'-yy'\equiv -,$$

$$\epsilon=0,$$

$$\epsilon'=0,$$

$$\epsilon''=\pm 1\equiv (xy'+x'y)\equiv\left(\frac{y}{x}+\frac{y'}{x'}\right),$$

$$\epsilon'''=\pm 1 \equiv\left(\frac{y}{x}+\frac{y'}{x'}\right),$$

and therefore

$$\epsilon+\epsilon'-\epsilon''+\epsilon'''=0.$$

III. $$x\equiv +,\quad x'\equiv -,\quad xx'-yy'\equiv +,$$

$$\epsilon=0,$$

$$\epsilon'=\pm 1\equiv y'\equiv -\frac{y'}{x'}\equiv -\left(\frac{y}{x}+\frac{y'}{x'}\right),$$

$$\epsilon''=0,$$

$$\epsilon'''=\pm 1 \equiv \left(\frac{y}{x}+\frac{y'}{x'}\right),$$

and therefore

$$\epsilon+\epsilon'-\epsilon''+\epsilon'''=0.$$

IV. $$x\equiv +,\quad x'\equiv -,\quad xx'-yy'\equiv -,$$

$$\epsilon=0,$$

$$\epsilon'=\pm 1\equiv y'\equiv -\frac{y'}{x'},$$

$$\epsilon''=\pm 1\equiv xy'+x'y\equiv -\left(\frac{y}{x}+\frac{y'}{x'}\right),$$

$$\epsilon'''=0,$$

and therefore

$$\epsilon + \epsilon' - \epsilon'' + \epsilon''' = 0 \qquad \text{if } \frac{y'}{x'} \equiv \left(\frac{y}{x} + \frac{y'}{x'}\right),$$

but

$$\epsilon + \epsilon' - \epsilon'' + \epsilon''' = \pm 2 \equiv \left(\frac{y}{x} + \frac{y'}{x'}\right) \text{ if } \frac{y'}{x'} \equiv -\left(\frac{y}{x} + \frac{y'}{x'}\right).$$

V. $\quad x \equiv -, \quad x' \equiv +, \quad xx' - yy' \equiv +,$

$$\epsilon = \pm 1 \equiv y \equiv -\frac{y}{x} \equiv -\left(\frac{y}{x} + \frac{y'}{x'}\right),$$

$$\epsilon' = 0,$$

$$\epsilon'' = 0,$$

$$\epsilon''' = \pm 1 \qquad \equiv \left(\frac{y}{x} + \frac{y'}{x'}\right),$$

and therefore

$$\epsilon + \epsilon' - \epsilon'' + \epsilon''' = 0.$$

VI. $\quad x \equiv -, \quad x' \equiv +, \quad xx' - yy' \equiv -,$

$$\epsilon = \pm 1 \equiv y \equiv -\frac{y}{x},$$

$$\epsilon' = 0,$$

$$\epsilon'' = \pm 1 \equiv (xy' + x'y) \equiv -\left(\frac{y}{x} + \frac{y'}{x'}\right),$$

$$\epsilon''' = 0,$$

and therefore

$$\epsilon + \epsilon' - \epsilon'' + \epsilon''' = 0 \qquad \text{if } \frac{y}{x} \equiv \left(\frac{y}{x} + \frac{y'}{x'}\right),$$

but

$$\epsilon + \epsilon' - \epsilon'' + \epsilon''' = \pm 2 \equiv \frac{y}{x} + \frac{y'}{x'} \text{ if } \frac{y}{x} \equiv -\left(\frac{y}{x} + \frac{y'}{x'}\right).$$

VII. $\quad x \equiv -, \quad x' \equiv -, \quad xx' - yy' \equiv +,$

$$\epsilon = \pm 1 \equiv y \equiv -\frac{y}{x},$$

$$\epsilon' = \pm 1 \equiv y' \equiv -\frac{y'}{x'},$$

$$\epsilon'' = 0,$$

$$\epsilon''' = 0;$$

and therefore

$$\epsilon + \epsilon' - \epsilon'' + \epsilon''' = 0 \qquad \text{if } \frac{y}{x} \equiv -\frac{y'}{x'},$$

but

$$\epsilon + \epsilon' - \epsilon'' + \epsilon'' = \pm 2 \equiv -\left(\frac{y}{x} + \frac{y'}{x'}\right) \text{ if } \frac{y}{x} \equiv \frac{y'}{x'}.$$

VIII.
$$x\equiv -,\quad x'\equiv -,\quad xx'-yy'\equiv -,$$
$$\epsilon=\pm 1\equiv y\equiv -\frac{y}{x}\quad \equiv -\left(\frac{y}{x}+\frac{y'}{x'}\right),$$
$$\epsilon'=\pm 1\equiv y'\equiv -\frac{y'}{x'}\quad \equiv -\left(\frac{y}{x}+\frac{y'}{x'}\right),$$
$$\epsilon''=\pm 1\equiv (xy'+x'y)\equiv \left(\frac{y}{x}+\frac{y'}{x'}\right),$$
$$\epsilon'''=\pm 1\qquad \equiv \left(\frac{y}{x}+\frac{y'}{x'}\right),$$

and therefore
$$\epsilon+\epsilon'-\epsilon''+\epsilon'''=\pm 2\equiv -\left(\frac{y}{x}+\frac{y'}{x'}\right).$$

Hence writing
$$\log(x+yi)+\log(x'+y'i)-\log(x+yi)(x'+y'i)=E\pi i,$$

we have $E=0$, except in the following cases, viz.

1. (See IV.)
$$x\equiv +,\ x'=-,\ xx'-yy'\equiv -,\ \frac{y'}{x'}\equiv -\left(\frac{y}{x}+\frac{y'}{x'}\right),$$

where
$$E=\pm 2\equiv \left(\frac{y}{x}+\frac{y'}{x'}\right).$$

2. (See VI.)
$$x\equiv -,\ x'\equiv +,\ xx'-yy'\equiv -,\ \frac{y}{x}\equiv -\left(\frac{y}{x}+\frac{y'}{x'}\right),$$

where
$$E=\pm 2\equiv \left(\frac{y}{x}+\frac{y'}{x'}\right).$$

3. (See VII.)
$$x\equiv -,\ x'\equiv -,\ xx'-yy'\equiv +,\ \frac{y}{x}\equiv \frac{y'}{x'},$$

where
$$E=\pm 2\equiv -\left(\frac{y}{x}+\frac{y'}{x'}\right).$$

4. (See VIII.)
$$x\equiv -,\ x'\equiv -,\quad xx'-yy'\equiv -,$$

where
$$E=\pm 2\equiv -\left(\frac{y}{x}+\frac{y'}{x'}\right).$$

It thus appears that when the real parts x, x', $xx'-yy'$ are all three of them positive, or any two of them positive and the third negative, E is equal to zero, or the logarithm of the product is equal to the sum of the logarithms of the factors; but that if the real parts are one of them positive and the other two of them negative, then if a certain relation between the real and imaginary parts is satisfied, but not otherwise, the property holds; and if the real parts are all three of them negative, the property does not hold in any case.

The preceding results do not apply to the case where any one of the arguments $x+yi$, $x'+y'i$, $(x+yi)(x'+y'i)$ is real and negative, for no definition applicable to such case has been given of a logarithm. If, however, we assume as a definition that the logarithm of a negative real quantity is equal to the logarithm of the corresponding positive quantity, then in the case, $x\equiv -$, $y=0$, we have

$$\log x+\log(x'+y'i)-\log x(x'+y'i)=\epsilon\pi i,\quad \epsilon=\pm 1\equiv y';$$

an equation which is, in fact, equivalent to

$$\log(x'+y'i)-\log[-(x'+y'i)]=\epsilon\pi i,\quad \epsilon=\pm 1\equiv y';$$

and in the case $xy'+x'y=0$, $xx'-yy'\equiv -$, which implies $y\equiv y'$, then

$$\log(x+yi)+\log(x'+y'i)-\log(x+yi)(x'+y'i)=\pi i,\quad \epsilon=\pm 1\equiv y \text{ or } y';$$

an equation which is in fact equivalent to

$$\log(x+yi)+\log(-x+yi)-\log(x^2+y^2)=\epsilon\pi i,\ \epsilon=\pm 1\equiv y.$$

The case where both of the arguments $x+yi$, $x'+y'i$ are real and negative, i.e. $x\equiv -$, $y=0$, $x'\equiv -$, $y'=0$ gives of course $\log x+\log x'-\log xx'=0$, the logarithms of the negative real quantities x, x' being by the definition the same as the logarithms of the corresponding positive quantities. It should, however, be remarked that the definition, $(x\equiv -)$, $\log x=\log(-x)$ not only gives for $\log x$ a different value from that which would be obtained from the general definition of a logarithm, by considering $\log x$ as the limit of $\log(x+yi$, $y\equiv +$, or of $\log(x+yi)$, $y\equiv -$, but gives also a value, which, for the particular case in question, contradicts the fundamental equation $e^{\log x}=x$. It is therefore, I think, better not to establish any definition for the logarithm of a negative real quantity x, but to say that such logarithm is absolutely indeterminate and indeterminable, except in the case where, from the nature of the question, x is considered as the limit of $x+yi$, y positive, or of $x+yi$, y negative.

2, *Stone Buildings, March* 15, 1856.