175.

ON THE PORISM OF THE IN-AND-CIRCUMSCRIBED TRIANGLE.

[From the *Quarterly Mathematical Journal*, vol. I. (1857), pp. 344—354.]

THE porism of the in-and-circumscribed triangle in its most general form relates to a triangle the angles of which lie in fixed curves, and the sides of which touch fixed curves; but at present I consider only the case in which the angles lie in one and the same fixed curve, which for greater simplicity I assume to be a conic. We have therefore a triangle ABC, the angles of which lie in a fixed conic $\mathfrak{S}$, and the sides of which touch the fixed curves $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$; the points of contact may be represented by α, β, γ. And if we consider the conic $\mathfrak{S}$ and the curves $\mathfrak{A}$, $\mathfrak{B}$ as given, the curve $\mathfrak{C}$ will be the envelope of the side AB; to construct this side we have only to take at pleasure a point C on the curve $\mathfrak{S}$ and to draw through this point tangents to the curve $\mathfrak{B}$, $\mathfrak{A}$ respectively meeting the conic $\mathfrak{S}$ in the points A and B; the line joining these points is the required side AB. I may notice that in the case supposed of the curve $\mathfrak{S}$ being a conic, the lines $A\alpha$, $B\beta$, $C\gamma$ meet in a point; which gives at once a construction for γ, the point of contact of AB with the curve C. For the sake however of exhibiting the reasoning in a form which may be modified so as to be applicable to a curve $\mathfrak{S}$ of any order, instead of the conic $\mathfrak{S}$, I shall dispense with the employment of the property just mentioned, which is peculiar to the case of the conic.

Suppose for a moment (figs. 1 and 1 *bis*) that the curves $\mathfrak{A}$, $\mathfrak{B}$ are points, and let the line through $\mathfrak{A}$, $\mathfrak{B}$ meet the conic $\mathfrak{S}$ in the points M, N. If we take the point N for the angle C of the triangle, the points A, B will each of them coincide with M, and the side AB will be the tangent at M to the conic $\mathfrak{S}$: call this tangent T. Consider next a point C in the neighbourhood of N, we shall have two points A, B in the neighbourhood of M, and the point in which AB intersects T will be the point of contact γ of T with the curve $\mathfrak{C}$. To find this point, suppose that $M\mathfrak{A} = a$, $M\mathfrak{B} = b$, $N\mathfrak{A} = a'$, $N\mathfrak{B} = b'$ and let the distance of C from N be ds; the distances parallel to T of A, B from the line MN will be proportional to $\frac{b}{b'}$, ds, $\frac{a}{a'} ds$, and the perpendicular distances of these points from T are consequently proportional to $\frac{b^2}{b'^2}ds^2$, $\frac{a^2}{a'^2}ds^2$. The inclination of AB to T is therefore proportional to

$$\left(\frac{b^2}{b'^2}-\frac{a^2}{a'^2}\right)ds^2 \div \left(\frac{b}{b}-\frac{a}{a'}\right)ds, \text{ i.e. to } \left(\frac{b}{b'}+\frac{a}{a'}\right)ds,$$

which is of the same order as ds, and it is at once seen that the point γ will

Fig. 1.

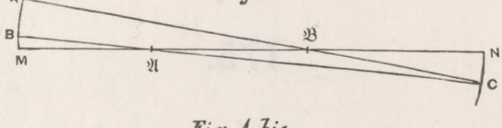
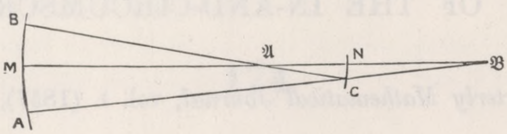


Fig. 1 bis.

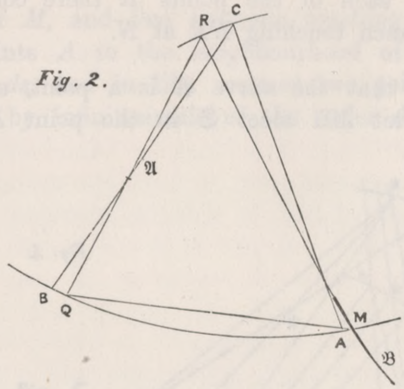


coincide with M, i.e. that the curve $\mathfrak{C}$ touches the conic $\mathfrak{S}$ at the point M. If, however, $\frac{a}{a'}+\frac{b}{b'}=0$, i.e. if the points M, N are harmonically related to $\mathfrak{A}$, $\mathfrak{B}$, then the inclination is in general of the order ds^2, and the point γ will be at a finite distance from M; moreover T is in this case a stationary tangent (i.e. a tangent at a point of inflexion) of the curve $\mathfrak{C}$. Now reverting to the general case of $\mathfrak{A}$, $\mathfrak{B}$ being any two curves, then if there be a common tangent touching these curves in the points α, β, and meeting the curve $\mathfrak{S}$ in the points M, N, the like reasonings apply to this case. Hence

First Lemma. If a common tangent to the curves $\mathfrak{A}$, $\mathfrak{B}$ touch these curves in α, β and meet the conic $\mathfrak{S}$ in the points M, N; the point N gives rise to a branch of the curve $\mathfrak{C}$ which (except in the case after mentioned) touches the conic $\mathfrak{S}$ at the point M. If however M, N are harmonically related with respect to α, β, then the branch of the curve $\mathfrak{C}$ does *not* pass through M, but it has for a *stationary* tangent the tangent M to the conic $\mathfrak{S}$.

Suppose again that the curve $\mathfrak{A}$ (fig. 2) is a point, and let the curve $\mathfrak{B}$ intersect the conic $\mathfrak{S}$ at the point M, and let the tangent to $\mathfrak{B}$ at M meet $\mathfrak{S}$ in a point R, and $R\mathfrak{A}$ meet $\mathfrak{S}$ in a point Q. Then taking the point R for the angle C of the triangle, we shall have A, B coinciding with M, Q respectively, and thence MQ a tangent to the curve $\mathfrak{C}$. To find the point of contact, take C in the neighbourhood of R at a distance from it ds. B will be a point in the neighbourhood of Q and distant from it by an infinitesimal of the order ds, A will be in the neighbourhood of M and distant from it by an infinitesimal of the order ds^2. Hence AB will intersect MQ at the point M, or the curve $\mathfrak{C}$ will pass through M. Reverting to the general case where $\mathfrak{A}$ is a curve, we have only to consider RQ as a tangent to the curve $\mathfrak{A}$ at a point α, and the like reasoning will apply to this case: hence

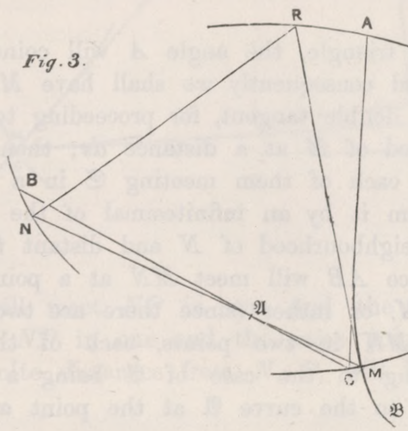
Fig. 2.



Second Lemma. If the curve 𝔅 intersect the conic 𝔖 at a point M, and the tangent to 𝔅 at M intersect 𝔖 in R, if moreover a tangent through R to the curve 𝔄 intersect 𝔖 in Q; then to each of the points Q there corresponds a branch through M of the curve ℭ, viz. a branch touching MQ at the point M.

Suppose as before (fig. 3) that 𝔄 is a point, and let the curve 𝔅 intersect the

Fig. 3.



conic 𝔖 at the point M, and the tangent to 𝔅 at M meet 𝔖 in the point R. And let M𝔄 meet 𝔖 in the point N. Then taking the point M for the angle C of the triangle we shall have A, B coinciding with R, N respectively, and thence NR a tangent to the curve ℭ. To find the point of contact, take A in the neighbourhood of R at a distance from it ds, then C will be in the neighbourhood of M and distant from it by an infinitesimal of the order ds^2, and B will be in the neighbourhood of N and distant from it by an infinitesimal of the same order ds^2, and consequently AB will intersect NR at the point N, or the curve ℭ passes through N. Reverting to the general case where 𝔄 is a curve, we have only to consider MN as a tangent to the curve 𝔄 at a point α and the like reasoning applies: hence,

Third Lemma. If the curve 𝔅 intersect the conic 𝔄 at a point M, and the tangent to 𝔅 at M intersect 𝔖 in R, if moreover a tangent through M to the curve

$\mathfrak{A}$ meet $\mathfrak{S}$ in N; then to each of the points R there corresponds a branch through N of the curve $\mathfrak{C}$, viz. a branch touching NR at N.

Suppose again (fig. 4) that the curve $\mathfrak{A}$ is a point, and let the curve $\mathfrak{B}$ *touch* $\mathfrak{S}$ at the point M. And let $M\mathfrak{A}$ meet $\mathfrak{S}$ in the point N. Then taking the point

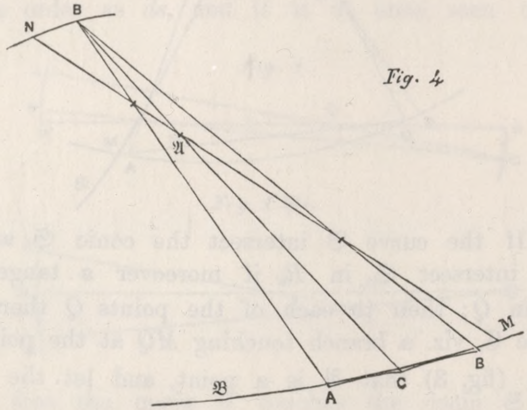


Fig. 4

M for the angle C of the triangle, the angle A will coincide with M and the angle B will coincide with N, and consequently we shall have MN a tangent to the curve $\mathfrak{C}$. And MN is in fact a double tangent, for proceeding to find the point of contact, take C in the neighbourhood of M at a distance ds; then from C we may draw to the curve $\mathfrak{B}$ two tangents each of them meeting $\mathfrak{S}$ in a point $\mathfrak{A}$ in the neighbourhood of M and distant from it by an infinitesimal of the order ds; again CA meets $\mathfrak{S}$ in a point B in the neighbourhood of N and distant from it by an infinitesimal of the same order ds; hence AB will meet MN at a point which will be in general at a finite distance from M, or rather (since there are two positions of the point A) the lines AB will meet MN in two points, each of them in general at a finite distance from M. Reverting to the case of $\mathfrak{A}$ being a curve, we must as before consider MN as a tangent to the curve $\mathfrak{A}$ at the point α, and the same reasoning applies: hence

FOURTH LEMMA. If the curve $\mathfrak{B}$ *touch* the conic $\mathfrak{S}$ at the point M, and if a tangent through M to the curve $\mathfrak{A}$ meet $\mathfrak{S}$ in N, then MN is a double tangent of the curve $\mathfrak{C}$, viz. the line MN has two distinct points of contact with the curve $\mathfrak{C}$.

Suppose (fig. 5) that the curves $\mathfrak{A}$ and $\mathfrak{B}$ meet the conic $\mathfrak{S}$ in one and the same point M, and let the tangent at M to the curve $\mathfrak{A}$ meet $\mathfrak{S}$ in the point P, and the tangent at M to the curve $\mathfrak{B}$ meet $\mathfrak{S}$ in the point N; then if we take the point M for the angle C of the triangle, the angles A and B of the triangle will coincide with N, P respectively, and NP will be a tangent of the curve $\mathfrak{C}$. But NP will be a double tangent, for proceeding to find the point of contact, take C in the neighbourhood of M, and distant from it by an infinitesimal of the second order ds^2; then since from the point C there may be drawn two tangents touching

$\mathfrak{A}$ in the neighbourhood of M, and two tangents touching $\mathfrak{B}$ in the neighbourhood of M, there will be two points A in the neighbourhood of N and distant from it by infinitesimals of the order ds, and in like manner two points B in the neighbourhood of P and distant from it by infinitesimals of the order ds. Call these points A, A';

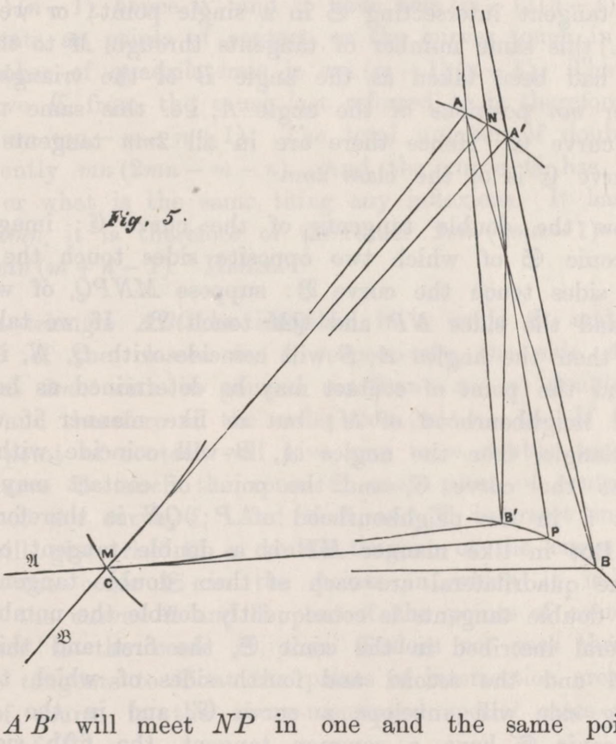


Fig. 5.

B, B'; then AB', $A'B'$ will meet NP in one and the same point, and AB', $A'B'$ will in like manner meet NP in one and the same point; these two points, which will be in general at finite distances from N, P, will be points of contact of NP with the curve $\mathfrak{C}$: hence,

Fifth Lemma. If the curves $\mathfrak{A}$, $\mathfrak{B}$ meet the conic $\mathfrak{S}$ in one and the same point M, and if the tangents at M to the curves $\mathfrak{B}$ and $\mathfrak{A}$ respectively meet $\mathfrak{S}$ in the points N and P, then, joining these points, the line NP will be a double tangent to the curve $\mathfrak{C}$, viz. there will be two distinct points of contact of the line NP with the curve $\mathfrak{C}$.

The double tangents of the fourth and fifth lemmas exist by virtue of the particular relations assumed between the curves $\mathfrak{A}$, $\mathfrak{B}$ and the conic $\mathfrak{S}$, viz. from one or both of the curves $\mathfrak{A}$, $\mathfrak{B}$ touching the conic $\mathfrak{S}$, or from the two curves having a common point of intersection or common points of intersection with $\mathfrak{S}$; there are besides double tangents which exist independently of any such relations, and the theory of which will be presently investigated, but it will be convenient in the first instance to find the class of $\mathfrak{C}$.

The class of $\mathfrak{C}$ is at once determinable from the classes (which I represent by m, n) of the curves $\mathfrak{A}$, $\mathfrak{B}$. In fact, take any point M on the conic $\mathfrak{S}$ for the angle A of the triangle. Through the point M we may draw N tangents to $\mathfrak{B}$, each of which will intersect the conic $\mathfrak{S}$ in a single point, or there will be N positions of the point C, and since from each of these we may draw M tangents to the curve $\mathfrak{A}$, each tangent intersecting $\mathfrak{S}$ in a single point; or we have mn positions of the angle B, i.e. this same number of tangents through M to the curve $\mathfrak{C}$. But if the same point M had been taken as the angle B of the triangle, there would have been in like manner mn positions of the angle A, i.e. this same number of tangents through M to the curve $\mathfrak{C}$. Hence there are in all $2mn$ tangents through M to the curve $\mathfrak{C}$, or the curve $\mathfrak{C}$ is of the class $2mn$.

Considering now the double tangents of the curve $\mathfrak{C}$; imagine a quadrilateral inscribed in the conic $\mathfrak{S}$ of which two opposite sides touch the curve $\mathfrak{A}$, and the other two opposite sides touch the curve $\mathfrak{B}$: suppose $MNPQ$, of which the sides MN and PQ touch $\mathfrak{A}$ and the sides NP and QM touch $\mathfrak{B}$. If we take M for the angle C of the triangle, then the angles A, B will coincide with Q, N, i.e. NQ is a tangent to the curve $\mathfrak{C}$, and the point of contact may be determined as before by considering a point C in the neighbourhood of M; but in like manner if we take P for the angle C of the triangle, then the angles A, B will coincide with N, Q, i.e. QN is again a tangent to the curve $\mathfrak{C}$, and the point of contact may be determined by considering a point C in the neighbourhood of P; QN is therefore a double tangent of the curve $\mathfrak{C}$. But in like manner MP is a double tangent of the curve $\mathfrak{C}$, i.e. the diagonals of the quadrilateral are each of them double tangents of the curve $\mathfrak{C}$, and the number of double tangents is consequently double the number of quadrilaterals. Imagine a pentilateral inscribed in the conic $\mathfrak{S}$, the first and third sides of which touch the curve $\mathfrak{A}$ and the second and fourth sides of which touch the curve $\mathfrak{B}$, the fifth or closing side will envelope a curve $\mathfrak{C}'$, and in the cases in which the curve $\mathfrak{C}'$ and the conic $\mathfrak{S}$ have a common tangent, the fifth side will vanish, and the pentilateral becomes a quadrilateral of the kind before referred to. Now as $\mathfrak{C}$ was shown to be of the class $2mn$, so it may be shown that $\mathfrak{C}'$ is of the class $2m^2n^2$, hence $\mathfrak{C}'$ and $\mathfrak{S}$ have $4m^2n^2$ common tangents. The quadrilateral may reduce itself to a common tangent of the curves $\mathfrak{A}$ and $\mathfrak{B}$: this gives rise to $2mn$ points of contact of $\mathfrak{C}'$ and $\mathfrak{S}$, and the common tangent at a point of contact reckons as two common tangents; the number of the remaining common tangents is therefore $4m^2n^2 - 4mn$. And these are in fact tangents at points of contact of $\mathfrak{C}'$ and $\mathfrak{S}$, i.e. $\mathfrak{C}'$ and $\mathfrak{S}$ touch in $2m^2n^2 - 2mn$ points. And since each angle of the quadrilateral may be taken as the first angle, the number of quadrilaterals is one fourth of this, or $\frac{1}{2}(m^2n^2 - mn)$, and the number of double tangents of the curve $\mathfrak{C}$ from the before-mentioned cause is therefore $m^2n^2 - mn$.

But there is another way in which double tangents arise; we may have a quadrilateral $MNPQ$ inscribed in the conic $\mathfrak{S}$, such that two adjacent sides MN, NP touch the curve $\mathfrak{A}$, and the other two adjacent sides PQ, QM touch the curve $\mathfrak{B}$. In fact in this case one of the diagonals, viz. NQ, is a double tangent of the curve $\mathfrak{C}$; the number of double tangents is therefore equal to the number of quadrilaterals.

Consider a pentilateral inscribed in the conic, and such that the first and second sides touch the curve $\mathfrak{A}$ and the third and fourth sides touch the curve $\mathfrak{B}$, the fifth or closing side will envelope a curve $\mathfrak{C}''$, and in the cases in which the curve $\mathfrak{C}''$ and the conic $\mathfrak{S}$ have a common tangent, the fifth side will vanish, and the pentilateral will become a quadrilateral of the kind last referred to. The curve $\mathfrak{C}''$ is of the class $2mn(m-1)(n-1)$, hence $\mathfrak{C}''$ and $\mathfrak{S}$ have $4mn(m-1)(n-1)$ common tangents; but these are tangents at points of contact, or the curves touch in $2mn(m-1)(n-1)$ points, and the number of quadrilaterals is $mn(m-1)(n-1)$. The number of double tangents of the curve $\mathfrak{C}$ from the cause last referred to is therefore $mn(m-1)(n-1)$, which is equal to $mn(mn-m-n+1)$. The total number of double tangents of the curve $\mathfrak{C}$ is consequently $mn(2mn-m-n)$. And the curve $\mathfrak{C}$ has not in general any stationary tangents or what is the same thing any inflexions. It has been shown that $\mathfrak{C}$ is of the class $2mn$, it is therefore of the order $2mn(2mn-1)-2mn(2mn-m-n)$, which is equal to $2mn(m+n-1)$. Hence

THEOREM. If a triangle ABC be inscribed in a conic $\mathfrak{S}$, and the sides BC, AC touch curves $\mathfrak{A}$, $\mathfrak{B}$ of the classes m, n respectively, the side AB will envelope a curve $\mathfrak{C}$ of the class $2mn$ with in general $mn(2mn-m-n)$ double tangents, but no stationary tangents, and therefore of the order $2mn(m+n-1)$. If the curve $\mathfrak{A}$ touch the conic $\mathfrak{S}$, each point of contact will give rise to n double tangents of the curve $\mathfrak{C}$, and so if the curve $\mathfrak{B}$ touch the conic $\mathfrak{S}$, each point of contact will give rise to m double tangents of the curve $\mathfrak{C}$. And if $\mathfrak{A}$ and $\mathfrak{B}$ intersect on the conic $\mathfrak{S}$, each such intersection will give rise to a double tangent of the curve $\mathfrak{C}$. The curve $\mathfrak{C}$ in general touches the conic $\mathfrak{S}$ in the points in which it is intersected by any common tangent of the curves $\mathfrak{A}$ and $\mathfrak{B}$; but if the points of contact be harmonically situated with respect to the conic $\mathfrak{S}$, then $\mathfrak{C}$ does *not* pass through the points of intersection, but the tangents to $\mathfrak{S}$ at the points of intersection are stationary tangents of $\mathfrak{C}$. There is of course in the above-mentioned special cases a corresponding reduction in the order of $\mathfrak{C}$.

It sometimes happens that the number of double tangents of $\mathfrak{C}$ becomes infinite, i.e. in fact that $\mathfrak{C}$ is made up of two coincident curves; instances of this will be presently mentioned.

Suppose that the curves $\mathfrak{A}$, $\mathfrak{B}$ are points; $\mathfrak{C}$ is in general a conic having double contact with the conic $\mathfrak{S}$ in the points in which it is intersected by the line joining $\mathfrak{A}$, $\mathfrak{B}$. But if the points $\mathfrak{A}$, $\mathfrak{B}$ are harmonically situated with respect to the conic $\mathfrak{S}$, then $\mathfrak{C}$ does *not* pass through the points of intersection, but the tangents to $\mathfrak{S}$ at the points of intersection are stationary tangents of $\mathfrak{C}$. This implies that the curve $\mathfrak{C}$ is made up of two coincident points at the point of intersection of the two tangents of $\mathfrak{S}$: call this the point $\mathfrak{C}$, then $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$ are conjugate points of the conic $\mathfrak{S}$, and we have the well-known theorem that in a conic $\mathfrak{S}$ there may be inscribed an infinity of triangles the sides of which pass through three conjugate points of the conic. It should be remarked that for each position of the side AB, there are two positions of the angle $\mathfrak{C}$, i.e. each side AB is properly a double tangent of the curve (point) $\mathfrak{C}$.

Let 𝔄 be a point and 𝔅 be a conic, the curve ℭ is in general of the class 4, with two double tangents, and therefore of the order 8. It is proper to remark that the double tangents originate in a quadrilateral of the kind first considered, viz. a quadrilateral of which two opposite sides pass through the point 𝔄 and the other two opposite sides touch the conic 𝔅. It is easy in the present case to construct the quadrilateral: consider the point 𝔄 as a pole, and take its polar with respect to the conic 𝔖, take the pole of this polar with respect to the conic 𝔅. Join the two poles, and from the point in which the joining line meets the polar draw tangents to the conic 𝔅, these tangents will meet the conic 𝔖 in four points, lying two and two in lines passing through the point 𝔄: we have thus the quadrilateral, and the diagonals of the quadrilateral (or the other two lines passing through the four points) are the double tangents of the curve ℭ.

There are two particular cases to be considered. First, where the conic 𝔅 has double contact with the conic 𝔖. Here the lines joining the point 𝔄 with the points of contact are double tangents of the curve ℭ, which has therefore in all four double tangents, and being a curve of the fourth class it must break up into two curves of the second class, i.e. into two conics. And these are curves having double contact with the conic 𝔖; for the curve ℭ touches the conic 𝔖 in the four points in which 𝔖 is intersected by the tangents through 𝔄 to the conic 𝔅. Secondly, where 𝔄 is one of the conjugate points of the conics 𝔅, 𝔖. The general construction for the quadrilateral shows that if from any point of the common polar of 𝔄 with respect to 𝔅 and with respect to 𝔖, we draw tangents to 𝔅, these will meet 𝔖 in four points, lying two and two in lines passing through 𝔄, i.e. that the number of the double tangents of the curve ℭ is indefinite; ℭ is therefore made up of two coincident curves of the second class, i.e. of two coincident conics. Moreover, ℭ passes through the points of intersection of 𝔅, 𝔖; hence, disregarding one of the two coincident conics, we may say that the curve ℭ is a conic passing through the points of intersection of the conics 𝔅, 𝔖.

Next, let the curves 𝔄, 𝔅 be each of them a conic. The curve ℭ is of the class 8, with in general 16 double tangents, and therefore of the order 24. But there are two particular cases to be considered: first, where the conics 𝔄, 𝔅 have each of them double contact with the conic 𝔖. Here the tangents drawn from the points of contact of either of the conics 𝔄 or 𝔅 with 𝔖 to the other of the conics, 𝔅 or 𝔄, is a double tangent of the curve ℭ, i.e. there are 8 new double tangents, or in all 24 double tangents of the curve ℭ, which is therefore of the order 8; and being of the class 8 with 24 double tangents, it must break up into four curves of the second class, i.e. into four conics. And the curve ℭ touches the conic 𝔖 in the points in which 𝔖 is intersected by any one of the four common tangents of the conics 𝔄, 𝔅, viz. 8 points in all; hence each of the four conics has double contact with the conic 𝔖. Attending only to one of the four conics of which ℭ is made up, we have thus what (in a restricted sense of the expression, the porism of the in-and-circumscribed triangle) I call the porism (homographic) of the in-and-circumscribed triangle, viz.

If a triangle be inscribed in a conic, and two of the sides touch conics having double contact with the circumscribed conic, then will the third side touch a conic having double contact with the circumscribed conic.

Secondly, the conics $\mathfrak{A}$ and $\mathfrak{B}$ may cut the conic $\mathfrak{S}$ in the same four points. Here it may be seen that there are an infinity of inscribed quadrilaterals of the kind first considered, viz. of which two opposite sides touch the conic $\mathfrak{A}$, and the other two opposite sides touch the conic $\mathfrak{B}$. Hence, the curve $\mathfrak{C}$ is made up of two coincident curves of the class 4. But the curve of the class 4 has in fact 4 double tangents, viz. considering each of the points of intersection of $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{S}$, and drawing tangents to $\mathfrak{A}$ and $\mathfrak{B}$ meeting $\mathfrak{S}$ in two new points, the line joining these points is a double tangent of the curve in question, which is therefore of the 4th order, and being of the class 4 with 4 double tangents, it must break up into two curves of the second class, i.e. into two conics. Each of these conics passes through the points of intersection of $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{S}$, and touches the four lines last referred to, the conics would of course be completely determined by the condition of passing through the four points and touching one of the four lines. Attending only to one of the two conics, we have thus what I call the porism (allographic) of the in-and-circumscribed triangle, viz.

If a triangle be inscribed in a conic, and two of the sides touch conics meeting the circumscribed conic in the same four points, the remaining side will touch a conic meeting the circumscribed conic in the four points.

The *à posteriori* demonstration of these theorems will form the subject of another paper, [178].