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ON HANSEN'S LUNAR THEORY.

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THE following paper was written in order to exhibit, in as clear a form as may be, the investigation of the remarkable equations for the motion of the moon established in Hansen's "Fundamenta Nova Investigationis Orbitæ veræ quam Luna perlustrat," &c., Gothæ, 1838. I have availed myself for this purpose of the remarks in Jacobi's two letters in answer to a letter of Hansen's, *Crelle*, t. XLII. [1851], p. 12; it may be convenient to remark that the quantity there represented by Λ, and which does not occur in Hansen's own investigation, is in this paper represented by Θ.

The position of the moon referred to the earth as centre is determined by

r, the radius vector,
L, the longitude,
Λ, the latitude.

Suppose, moreover, that the attractive force at distance unity, $=\kappa(M+E)$, is represented by n^2a^3, then the principal function will be $V=-\dfrac{n^2a^3}{r}$, and the disturbing function R may be represented by $n^2a^3\Omega$; the expression for the half of the vis viva is

$$T=\tfrac{1}{2}\left\{\left(\frac{dr}{dt}\right)^2+r^2\cos^2\Lambda\left(\frac{dL}{dt}\right)^2+r^2\left(\frac{d\Lambda}{dt}\right)^2\right\},$$

and the equations of motion are therefore

$$\frac{d}{dt}\frac{dr}{dt}-r\cos^2\Lambda\left(\frac{dL}{dt}\right)^2-r\left(\frac{d\Lambda}{dt}\right)^2+\frac{n^2a^3}{r^2}=n^2a^3\frac{d\Omega}{dr},$$

$$\frac{d}{dt}\left(r^2\cos^2\Lambda\frac{dL}{dt}\right)=n^2a^3\frac{d\Omega}{dL},$$

$$\frac{d}{dt}\left(r^2\frac{d\Lambda}{dt}\right)+r^2\cos\Lambda\sin\Lambda\left(\frac{dL}{dt}\right)^2=n^2a^3\frac{d\Omega}{d\Lambda},$$

where Ω is considered as a function of r, L, Λ.

Consider the orbit as an ellipse, then putting

a, the mean distance,

n, the mean motion $= \sqrt{\dfrac{\kappa\,(M+E)}{a^3}}$,

e, the excentricity,

c, the mean anomaly at epoch,

ω, the distance of perigee from node,

θ, the longitude of node,

i, the inclination,

Φ, the distance from node,

Ψ, the reduced distance from node, $= L - \theta$,

U, the excentric anomaly,

f, the true anomaly, $= \Phi - \omega$,

the elements of the orbit are a, e, c, ω, θ, i, and we have from the theory of elliptic motion,

$$nt + c = U - e \sin U,$$

$$f = \tan^{-1} \frac{\sqrt{(1-e^2)} \sin U}{\cos U - e},$$

$$r = a\,(1 - e \cos U) = \frac{a\,(1-e^2)}{1 + e \cos f}.$$

Moreover i is the angle at the base of a right-angled spherical triangle, the base, perpendicular, and hypothenuse of which are Ψ, Λ, Φ, hence

$$\tan \Psi = \cos i \tan \Phi,$$

$$\sin \Lambda = \sin i \sin \Phi,$$

$$\sin \Psi = \cot i \tan \Lambda,$$

$$\cos \Phi = \cos \Lambda \cos \Psi.$$

Considering the elements as constant, we have

$$\frac{dr}{dt} = \frac{nae \sin f}{\sqrt{(1-e^2)}},$$

$$\frac{df}{dt} = \frac{na^2 \sqrt{(1-e^2)}}{r^2},$$

$$\frac{d\Phi}{dt} = \frac{na^2 \sqrt{(1-e^2)}}{r^2},$$

$$\frac{d\Psi}{dt} = \frac{\cos i}{\cos^2 \Lambda} \frac{na^2 \sqrt{(1-e^2)}}{r^2},$$

$$\frac{dL}{dt} = \frac{\cos i}{\cos^2 \Lambda} \frac{na^2 \sqrt{(1-e^2)}}{r^2},$$

$$\frac{d\Lambda}{dt} = \sin i \cos \Psi \frac{na^2 \sqrt{(1-e^2)}}{r^2}.$$

Hence also

$$\frac{d}{dt}\left(\frac{dr}{dt}\right) = \frac{n^2a^3e\cos f}{r^2},$$

$$\frac{d}{dt}\left(r^2\cos^2\Lambda\frac{dL}{dt}\right) = 0,$$

$$\frac{d}{dt}\left(r^2\frac{d\Lambda}{dt}\right) = -\frac{\cos^2 i\sin\Lambda}{\cos^3\Lambda}\frac{n^2a^4(1-e^2)}{r^2},$$

$$\frac{d}{dt}\left(\frac{df}{dt}\right) = -\frac{2n^2a^3}{r^3}e\sin f.$$

The foregoing values show that the equations of motion, neglecting the terms which involve the disturbing functions, are satisfied by the elliptic values of r, L, Λ: and in order to satisfy the actual equations of motion, we have only to consider the elements as variable and to write

$$dr = 0,$$

$$dL = 0,$$

$$d\Lambda = 0,$$

$$d\frac{dr}{dt} = n^2a^3\frac{d\Omega}{dr}dt,$$

$$d\left(r^2\cos^2\Lambda\frac{dL}{dt}\right) = n^2a^3\frac{d\Omega}{dL}dt,$$

$$d\left(r^2\frac{d\Lambda}{dt}\right) = n^2a^3\frac{d\Omega}{d\Lambda}dt,$$

where the differentiations relate only to the elements, or, what is the same thing, to t in so far only as it enters through the variable elements: the system is at once transformed into

$$dr = 0,$$

$$dL = 0,$$

$$d\Lambda = 0,$$

$$d\frac{nae\sin f}{\sqrt{(1-e^2)}} = n^2a^3\frac{d\Omega}{dr}dt,$$

$$d\,na^2\sqrt{(1-e^2)}\cos i = n^2a^3\frac{d\Omega}{dL}dt,$$

$$d\,na^2\sqrt{(1-e^2)}\sin i\cos\Psi = n^2a^3\frac{d\Omega}{d\Lambda}dt.$$

Now $\Psi = L - \theta$, or (supposing, as before, that the differentiations relate to t, only in so far as it enters through the variable elements) $d\Psi = -d\theta$, and thence $d\theta = \frac{\tan\Phi}{\sin i} di$; we have also $d\Phi = -\cos i\, d\theta$. The equations containing $\frac{d\Omega}{dL}$ and $\frac{d\Omega}{d\Lambda}$ give

$$\cos i\, d\, na^2\sqrt{(1-e^2)} - na^2\sqrt{(1-e^2)}\sin i\, di \qquad = n^2a^3\frac{d\Omega}{dL}dt,$$

$$\cos\Psi\sin i\, d\, na^2\sqrt{(1-e^2)} + na^2\sqrt{(1-e^2)}(\cos\Psi\cos i\, di + \sin\Psi\sin i\, d\theta) = n^2a^3\frac{d\Omega}{d\Lambda}dt;$$

or, expressing $d\theta$ by means of di and reducing, the second of these equations becomes

$$\sin i\, d\, na^2\sqrt{(1-e^2)} + na^2\sqrt{(1-e^2)}\frac{\cos i}{\cos^2\Lambda\cos^2\Psi}di \qquad = \frac{1}{\cos\Psi}n^2a^3\frac{d\Omega}{d\Lambda}dt,$$

and combining this with the first of the two equations, and observing that

$$\frac{\cos^2 i}{\cos^2\Lambda\cos^2\Psi} + \sin^2 i = \frac{1}{\cos^2\Psi},$$

we find

$$d\, na^2\sqrt{(1-e^2)} = \quad n^2a^3\left(\frac{\cos i}{\cos^2\Lambda}\qquad\frac{d\Omega}{dL} + \sin i\cos\Psi\frac{d\Omega}{d\Lambda}\right)dt,$$

$$di \qquad = \frac{na^2}{\sqrt{(1-e^2)}}\left(-\sin i\cos^2\Psi\frac{d\Omega}{dL} + \cos i\cos\Psi\frac{d\Omega}{d\Lambda}\right)dt.$$

Now, considering Ω as a function of r, θ, i, Φ, then Λ, L are given as functions of θ, i, Φ by the equations $\sin\Lambda = \sin i\sin\Phi$, $\tan\Psi = \cos i\tan\Phi$, $\Psi = L - \theta$, and after some simple reductions,

$$\frac{d\Omega}{dr} = \qquad \frac{d\Omega}{dr},$$

$$\frac{d\Omega}{d\theta} = \qquad \frac{d\Omega}{dL},$$

$$\frac{d\Omega}{di} = \tan\Phi\left(-\sin i\cos^2\Psi\frac{d\Omega}{dL} + \cos i\cos\Psi\frac{d\Omega}{d\Lambda}\right),$$

$$\frac{d\Omega}{d\Phi} = \qquad\left(\qquad\frac{\cos i}{\cos^2\Lambda}\frac{d\Omega}{dL} + \sin i\cos\Psi\frac{d\Omega}{d\Lambda}\right);$$

whence also

$$\frac{d\Omega}{d\theta} = \cos i\frac{d\Omega}{d\Phi} - \sin i\cot\Phi\frac{d\Omega}{di}.$$

We have therefore

$$d\, na^2\sqrt{(1-e^2)} = n^2a^3\frac{d\Omega}{d\Phi}\,dt,$$

$$di = \frac{na\cot\Phi}{\sqrt{(1-e^2)}}\,\frac{d\Omega}{di}\,dt,$$

$$d\theta = \frac{na}{\sqrt{(1-e^2)}\sin i}\,\frac{d\Omega}{di}\,dt,$$

$$d\Phi = \frac{-na\cot i}{\sqrt{(1-e^2)}}\,\frac{d\Omega}{di}\,dt,$$

$$dr = 0,$$

$$d\,\frac{nae\sin f}{\sqrt{(1-e^2)}} = n^2a^3\frac{d\Omega}{dr}\,dt.$$

Suppose now that we have

ρ, a radius vector, τ for t,

ϕ, a true anomaly, do.,

υ, an excentric anomaly, do.,

i.e. let ρ, ϕ, υ, be what the radius vector, the true anomaly and the excentric anomaly become when the time t, in so far as it enters directly, and not through the variable elements, is replaced by a new variable τ. We have

$$n\tau + c = \upsilon - e\sin\upsilon,$$

$$\phi = \tan^{-1}\frac{\sqrt{(1-e^2)}\sin\upsilon}{\cos\upsilon - e},$$

$$\rho = a(1-e\cos\upsilon) = \frac{a(1-e^2)}{1+e\cos\phi};$$

and of course the differential coefficients of ρ, ϕ with respect to τ may be at once deduced from the corresponding expressions for the differential coefficients of r, f with respect to t, the elements being considered as constant. Now, using l to denote a logarithm, and supposing that the differentiations affect only the elements, we have

$$dl\rho = \frac{da}{a} - \frac{2ede}{1-e^2} - \frac{\rho\cos\phi de}{a(1-e^2)} + \frac{\rho e\sin\phi d\phi}{a(1-e^2)},$$

$$dlr = \frac{da}{a} - \frac{2ede}{1-e^2} - \frac{r\sin fde}{a(1-e^2)} + \frac{re\sin fdf}{a(1-e^2)};$$

and putting for shortness

$$X_{,} = dl\rho - \frac{\rho e\sin\phi d\phi}{a(1-e^2)},$$

$$X = dlr - \frac{re\sin fdf}{a(1-e^2)},$$

we find

$$X_{,} - \frac{\rho\sin\phi}{r\sin f}X = \left(1 - \frac{\rho\sin\phi}{r\sin f}\right)\left(\frac{da}{a} - \frac{2ede}{1-e^2}\right) - \frac{\rho}{a(1-e^2)}\left(\cos\phi - \frac{\sin\phi\cos f}{\sin f}\right)de,$$

or reducing

$$X_{,} - \frac{\rho \sin \phi}{r \sin f} X =$$

$$\frac{1}{\sin f (1 + e \cos \phi)} \left\{ \left(\sin f - \sin \phi + e \sin (f - \phi) \right) \left(\frac{da}{a} - \frac{2ede}{1 - e^2} \right) - \sin (f - \phi)\, de \right\}.$$

Write for a moment

$$P = na^2 \sqrt{(1 - e^2)}, \quad Q = \frac{nae \sin f}{\sqrt{(1 - e^2)}}, \quad R = \frac{a (1 - e^2)}{1 + e \cos f},$$

so that

$$a (1 - e^2) = \frac{P^2}{n^2 a^3},$$

$$e^2 = \frac{1}{n^4 a^6} P^2 Q^2 + \frac{1}{n^4 a^6} \frac{P^4}{R^2} - \frac{2}{n^2 a^3} \frac{P^2}{R} + 1.$$

We have therefore

$$\frac{da}{a} - \frac{2ede}{1 - e^2} = \frac{2dP}{P} = \frac{2}{na^2 (1 - e^2)} dP,$$

$$ede = \left(\frac{1}{n^4 a^6} PQ^2 - \frac{2}{n^4 a^6} \frac{P^3}{R^2} - \frac{2}{n^2 a^3} \frac{P}{R} \right) dP + \frac{1}{n^4 a^6} P^2 Q\, dQ + \left(- \frac{1}{n^4 a^6} \frac{P^4}{R^3} + \frac{2}{n^2 a^3} \frac{P^2}{R^2} \right) dR,$$

which, after reduction, becomes

$$de = \frac{1}{na^2 (1 - e^2)} \left(e \sin^2 f + 2 (\cos f + e \cos^2 f) \right) dP + \frac{1}{na} \sqrt{(1 - e^2)} \sin f\, dQ - \frac{(1 + e \cos f)^2 \cos f}{a (1 - e^2)} dR,$$

and substituting these values,

$$\begin{aligned} X_{,} - \frac{\rho \sin \phi}{r \sin f} X = \frac{1}{1 + e \cos \phi} \Bigg\{ & \left(2 - 2 \cos (f - \phi) + e \sin f \sin (f - \phi) \right) \frac{1}{na^2 \sqrt{(1 - e^2)}} dP \\ & - a (1 - e^2) \sin (f - \phi) \frac{1}{na^2 \sqrt{(1 - e^2)}} dQ \\ & + \frac{(1 + e \cos f)^2 \cot f \sin (f - \phi)}{\sqrt{(1 - e^2)}} \frac{1}{na^2 \sqrt{(1 - e^2)}} dR \Bigg\}, \end{aligned}$$

or substituting for X, $X_{,}$, P, Q, R, their values

$$dl\rho - \frac{\rho \sin \phi}{r \sin f} dlr + \frac{\rho e \sin \phi}{a (1 - e^2)} (df - d\phi) =$$

$$\begin{aligned} \frac{\rho}{a (1 - e^2)} & \left(2 - 2 \cos (f - \phi) + e \sin f \sin (f - \phi) \right) \frac{1}{na^2 \sqrt{(1 - e^2)}}\, dna^2 \sqrt{(1 - e^2)} \\ & - \rho \sin (f - \phi) \frac{1}{na^2 \sqrt{(1 - e^2)}}\, d \frac{nae \sin f}{\sqrt{(1 - e^2)}} \\ & + \frac{\rho a \sqrt{(1 - e^2)}}{r^2} \cot f \sin (f - \phi) \frac{1}{na^2 \sqrt{(1 - e^2)}} dr. \end{aligned}$$

Now

$$-\left\{\frac{\rho}{r}\cos(f-\phi)-1+\frac{\rho}{a(1-e^2)}\Big(\cos(f-\phi)-1\Big)\right\}$$

$$=\frac{\rho}{a(1-e^2)}\Big(2-2\cos(f-\phi)+e\sin f\sin(f-\phi)\Big);$$

therefore

$$dl\rho-\frac{\rho\sin\phi}{r\sin f}\,dlr+\frac{\rho e\sin\phi}{a(1-e^2)}(df-d\phi)=$$

$$-\left\{\frac{\rho}{r}\cos(f-\phi)-1+\frac{\rho}{a(1-e^2)}\Big(\cos(f-\phi)-1\Big)\right\}\frac{1}{na^2\sqrt{(1-e^2)}}\,dna^2\sqrt{(1-e^2)}$$

$$-\rho\sin(f-\phi)\,\frac{1}{na^2\sqrt{(1-e^2)}}\,d\,\frac{nae\sin f}{\sqrt{(1-e^2)}}$$

$$+\frac{\rho a\sqrt{(1-e^2)}}{r^2}\cot f\sin(f-\phi)\,\frac{1}{na^2\sqrt{(1-e^2)}}\,dr.$$

So far the variations of the elements have, in fact, been treated as independent; but if we substitute for $dna^2\sqrt{(1-e^2)}$, $d\,\dfrac{nae\sin f}{\sqrt{(1-e^2)}}$, dr, their values in the disturbed motion, the equation becomes

$$dl\rho+\frac{\rho e\sin\phi}{a(1-e^2)}(df-d\phi)=-\left\{\frac{\rho}{r}\cos(f-\phi)-1+\frac{\rho}{a(1-e^2)}\Big(\cos(f-\phi)-1\Big)\right\}\frac{na}{\sqrt{(1-e^2)}}\frac{d\Omega}{d\Phi}\,dt$$

$$-\rho\sin(f-\phi)\,\frac{na}{\sqrt{(1-e^2)}}\frac{d\Omega}{dr}\,dt.$$

Consider now the point in which the orbit is intersected by any orthogonal trajectory to the successive positions of the orbit, or to fix the ideas, the orthogonal trajectory passing through ♈, the point in question may, for want of a recognised name, be called the "departure point;" and the angular distances in the orbit measured from this point may be termed "departures;" the expression, "the departure," is to be understood as meaning the departure of the moon. Write now

χ, the departure of the perigee,

$v_{,}$, the departure, $=f+\chi$,

σ, the departure of the node, $=\chi-\omega$,

Θ, the longitude in orbit of departure point, $=\theta-\sigma$.

It should be remarked that χ is not properly an element, i.e. it is not a function of a, e, c, ω, θ, i without t, and in like manner σ and Θ (which depend upon χ) are not elements.

We have from the geometrical definition

$$d\chi=d\omega+\cos i\,d\theta;$$

and therefore

$$d\sigma = \cos i \, d\theta,$$
$$d\Theta = (1-\cos i)\, d\theta.$$

Moreover $v_, = \Phi + \sigma$, which gives (assuming that the differentiations are performed with respect to t, only in so far as it enters through the variable elements) $dv_, = d\Phi + d\sigma = d\Phi + \cos i \, d\theta$, i.e. $dv_, = 0$, an equation which might have been assumed for the purpose of defining the departure point; the equation, in fact, expresses that the departure $v_,$ is measured from a point not actually fixed, but such that the increment of $v_,$ in the interval of time dt is the angular distance between two consecutive positions of the moon.

We have, as above noticed, $d\sigma = \cos i \, d\theta$, and thence and from what has preceded

$$di = \frac{na \cot \Phi}{\sqrt{(1-e^2)}} \frac{d\Omega}{di},$$

$$d\sigma = \frac{na \cot i}{\sqrt{(1-e^2)}} \frac{d\Omega}{di}.$$

Now the position of the moon can be determined by means of the quantities r, $v_,$, Θ, σ, i; hence Ω (which has been considered as a function of r, Φ, θ, i) may, if we please, be considered as a function of r, $v_,$, Θ, σ, i and from the differential relations

$$dr = dr,$$
$$dv_, = d\Phi + \cos i \, d\theta,$$
$$d\Theta = (1-\cos i)\, d\theta,$$
$$d\sigma = d\omega + \cos i \, d\theta,$$
$$di = di,$$

we find

$$\frac{d\Omega}{dr} = \frac{d\Omega}{dr},$$

$$\frac{d\Omega}{d\Phi} = \frac{d\Omega}{dv_,},$$

$$\frac{d\Omega}{d\theta} = \cos i \left(\frac{d\Omega}{dv_,} + \frac{d\Omega}{d\sigma}\right) + \left(1-\cos i\right)\frac{d\Omega}{d\Theta},$$

$$\frac{d\Omega}{di} = \frac{d\Omega}{di};$$

we have therefore

$$\frac{d\Omega}{d\sigma} + \frac{1-\cos i}{\cos i}\frac{d\Omega}{d\Theta} = -\frac{d\Omega}{d\Phi} + \frac{1}{\cos i}\frac{d\Omega}{d\theta},$$

or in virtue of a preceding equation

$$\frac{d\Omega}{d\sigma} + \frac{1-\cos i}{\cos i}\frac{d\Omega}{d\Theta} = -\tan i \cot \Phi \frac{d\Omega}{di};$$

and effecting the substitutions, and collecting the results,

$$d\ na^2\sqrt{(1-e^2)} = n^2a^3\frac{d\Omega}{dv_{,}}\,dt,$$

$$dr = 0,$$

$$d\ \frac{nae\sin f}{\sqrt{(1-e^2)}} = n^2a^3\frac{d\Omega}{dr}\,dt,$$

$$di = -\frac{na\cot i}{\sqrt{(1-e^2)}}\left(\frac{d\Omega}{d\sigma}+\frac{1-\cos i}{\cos i}\,\frac{d\Omega}{d\Theta}\right)dt,$$

$$d\sigma = \frac{na\cot i}{\sqrt{(1-e^2)}}\,\frac{d\Omega}{di}\,dt,$$

where Ω is considered as a function of r, $v_{,}$, Θ, σ, i.

Instead of σ, i we may introduce the new quantities p, q defined by the equations

$$p = \sin i\sin\sigma,$$

$$q = \sin i\cos\sigma\,;$$

this gives $\sin^2 i = p^2+q^2$, $\sigma = \tan^{-1}\frac{p}{q}$ and retaining in the formulæ the sine and cosine of i, to avoid the introduction of irrational functions of p^2+q^2, we have

$$d\Theta = (1-\cos i)\,d\theta = \frac{1-\cos i}{\cos i}\,d\sigma = \frac{1-\cos i}{\cos i\sin^2 i}.(qdp-pdq),$$

i.e.

$$d\Theta = \frac{qdp-pdq}{\cos i\,(1+\cos i)},$$

which determines Θ by means of p and q. We have moreover

$$dp = \sin i\cos\sigma\,d\sigma+\cos i\sin\sigma\,di,$$

$$dq = -\sin i\sin\sigma\,d\sigma+\cos i\cos\sigma\,di,$$

$$\frac{d\Omega}{dp} = \frac{\sin\sigma}{\cos i}\,\frac{d\Omega}{di}+\frac{\cos\sigma}{\sin i}\,\frac{d\Omega}{d\sigma},$$

$$\frac{d\Omega}{dq} = \frac{\cos\sigma}{\cos i}\,\frac{d\Omega}{di}-\frac{\sin\sigma}{\sin i}\,\frac{d\Omega}{d\sigma},$$

from which equations and the foregoing values of di and $d\sigma$ we find the values of dp and dq; the other equations of the system remain unaltered, and we have therefore

$$d\ na^2\sqrt{(1-e^2)} = n^2a^3\,\frac{d\Omega}{dv_{,}}\,dt,$$

$$dr = 0,$$

$$d\ \frac{nae\sin f}{\sqrt{(1-e^2)}} = n^2a^3\,\frac{d\Omega}{dr}\,dt,$$

$$dp = \frac{na \cos^2 i}{\sqrt{(1-e^2)}} \left(\frac{d\Omega}{dq} - \frac{p}{\cos i\,(1+\cos i)} \frac{d\Omega}{d\Theta}\right) dt,$$

$$dq = -\frac{na \cos^2 i}{\sqrt{(1-e^2)}} \left(\frac{d\Omega}{dp} + \frac{q}{\cos i\,(1+\cos i)} \frac{d\Omega}{d\Theta}\right) dt,$$

where Ω is considered as a function of r, $v_{,}$, Θ, p, q. The symbols $\frac{d\Omega}{dp}$, $\frac{d\Theta}{dq}$, as employed by Hansen, mean that the differentiations are to be performed as if Θ was a function of p, q, such that

$$d\Theta = \frac{d\Theta}{dp} dp + \frac{d\Theta}{dq} dq = \frac{qdp - pdq}{\cos i\,(1+\cos i)};$$

the last two equations being therefore nothing else than what Hansen represents by

$$dp = \frac{na \cos^2 i}{\sqrt{(1-e^2)}} \frac{d\Omega}{dq} dt,$$

$$dq = \frac{na \cos^2 i}{\sqrt{(1-e^2)}} \frac{d\Omega}{dp} dt.$$

Write now

$$\lambda, \text{ the departure}, \tau \text{ for } t,$$

i.e. λ is what $v_{,}$ becomes when t, in so far as it enters explicitly, and not through the variable elements, is replaced by the new variable τ; so that, in fact $\lambda = \phi + \chi$. The values of r, $v_{,}$ could be at once found from those of ρ, λ by changing τ into t; and to determine the values of ρ, λ, Hansen proceeds as follows:

writing

$$\lambda = \Pi\,(\zeta,\ t),$$

$$l\rho = \Gamma\,(\zeta,\ t) + \beta,$$

where ζ and β are new variables functions of τ and t, and Π, Γ are arbitrary functional symbols; so that if z, w are what ζ, β become when τ is changed into t, we should have

$$v_{,} = \Pi\,(z,\ t),$$

$$lr = \Gamma\,(z,\ t) + w;$$

then the foregoing equations give

$$\frac{d\lambda}{d\tau} = \Pi'(\zeta,\ t) \frac{d\zeta}{d\tau},$$

$$\frac{d\lambda}{dt} = \Pi'(\zeta,\ t) \frac{d\zeta}{dt} + \Pi_{,}(\zeta,\ t),$$

$$\frac{dl\rho}{d\tau} = \Gamma'(\zeta,\ t) \frac{dl\rho}{d\tau} + \frac{d\beta}{d\tau},$$

$$\frac{dl\rho}{dt} = \Gamma'(\zeta,\ t) \frac{dl\rho}{dt} + \Gamma_{,}(\zeta,\ t) + \frac{d\beta}{dt};$$

[where the accents and strokes denote differentiation in regard to ζ, t respectively].

Hence eliminating $\Pi'(\zeta, t)$ and $\Gamma'(\zeta, t)$ we have

$$\frac{d\lambda}{d\tau}\,\frac{d\zeta}{dt}-\frac{d\lambda}{dt}\,\frac{d\zeta}{d\tau}=-\Pi_{,}(\zeta,\,t)\frac{d\zeta}{d\tau},$$

$$\frac{dl\rho}{d\tau}\,\frac{d\zeta}{dt}-\frac{dl\rho}{dt}\,\frac{d\zeta}{d\tau}=\frac{d\beta}{d\tau}\,\frac{d\zeta}{dt}-\frac{d\beta}{dt}\,\frac{d\zeta}{d\tau}-\Gamma_{,}(\zeta,\,t)\frac{d\zeta}{dt},$$

or, what is the same thing,

$$\frac{\dfrac{d\zeta}{dt}}{\dfrac{d\zeta}{d\tau}}=\frac{\dfrac{d\lambda}{dt}}{\dfrac{d\lambda}{d\tau}}-\Pi_{,}(\zeta,\,t)\frac{1}{\dfrac{d\lambda}{d\tau}},$$

and substituting in these equations the value in

$$\frac{d\beta}{dt}-\frac{d\beta}{d\tau}\,\frac{\dfrac{d\zeta}{dt}}{\dfrac{d\zeta}{d\tau}}=\frac{dl\rho}{dt}-\frac{\dfrac{d\lambda}{dt}}{\dfrac{d\lambda}{d\tau}}\,\frac{dl\rho}{d\tau}+\Pi_{,}(\zeta,\,t)\frac{\dfrac{dl\rho}{d\tau}}{\dfrac{d\lambda}{d\tau}}-\Gamma_{,}(\zeta,\,t),$$

or writing

$$T=\frac{d}{d\tau}\,\frac{\dfrac{d\zeta}{dt}}{\dfrac{d\zeta}{d\tau}},\quad R=\frac{d\beta}{dt}-\frac{d\beta}{d\tau}\,\frac{\dfrac{d\zeta}{dt}}{\dfrac{d\zeta}{d\tau}},$$

we have

$$T=\frac{1}{\dfrac{d\lambda}{d\tau}}\,\frac{d^2\lambda}{dt\,d\tau}-\frac{\dfrac{d^2\lambda}{d\tau^2}}{\left(\dfrac{d\lambda}{d\tau}\right)^2}\,\frac{d\lambda}{dt}+\Pi_{,}(\zeta,\,t)\frac{\dfrac{d^2\lambda}{d\tau^2}}{\left(\dfrac{d}{d\tau}\right)^2}-\Pi_{,}(\zeta,\,t)\frac{\dfrac{d\zeta}{d\tau}}{\dfrac{d\lambda}{d\tau}},$$

$$R=\frac{dl\rho}{dt}-\frac{\dfrac{dl\rho}{d\tau}}{\dfrac{d\lambda}{d\tau}}\,\frac{d\lambda}{d\tau}+\Pi_{,}(\zeta,\,t)\,\frac{\dfrac{dl\rho}{d\tau}}{\dfrac{d\lambda}{d\tau}}-\Gamma_{,}(\zeta,\,t).$$

Now from the equation $\lambda=\phi+\chi$ where χ is independent of τ, the differential coefficients of λ and ρ with respect to τ are at once deduced from those of f, r with respect to t, and we have,

$$\frac{d^2\lambda}{d\tau^2}=-\frac{2n^2a^3}{\rho^3}\,e\sin\phi,$$

$$\frac{d\lambda}{d\tau}=\frac{na^2\sqrt{(1-e^2)}}{\rho^2},$$

$$\frac{dl\rho}{d\tau}=\frac{nae\sin\phi}{\rho\sqrt{(1-e^2)}}.$$

Consequently

$$T = \frac{\rho^2}{na^2\sqrt{(1-e^2)}}\ \frac{d}{dt}\ \frac{na^2\sqrt{(1-e^2)}}{\rho^2} + \frac{2\rho e\sin\phi}{a(1-e^2)}\frac{d\lambda}{dt}$$

$$- \frac{2\rho e\sin\phi}{a(1-e^2)}\Pi_{,}(\zeta,\ t) - \frac{\rho^2}{na^2\sqrt{(1-e^2)}}\Pi'_{,}(\zeta,\ t)\frac{d\zeta}{dt},$$

$$= -2\frac{dl\rho}{dt} + \frac{2\rho e\sin\phi}{a(1-e^2)}\frac{d\lambda}{dt} + \frac{1}{na^2\sqrt{(1-e^2)}}\frac{d}{dt}na^2\sqrt{(1-e^2)}$$

$$- \frac{2\rho e\sin\phi}{a(1-e^2)}\Pi_{,}(\zeta,\ t) - \frac{\rho^2}{na^2\sqrt{(1-e^2)}}\Pi'_{,}(\zeta,\ t)\frac{d\zeta}{dt},$$

$$R = \frac{dl\rho}{dt} - \frac{\rho e\sin\phi}{a(1-e^2)}\frac{d\lambda}{dt} + \frac{\rho e\sin\phi}{a(1-e^2)}\Pi_{,}(\zeta,\ t) - \Gamma_{,}(\zeta,\ t),$$

and substituting in these equations the values of

$$\frac{dl\rho}{dt} - \frac{\rho e\sin\phi}{a(1-e^2)}\frac{d\lambda}{dt}\ \text{ and }\ \frac{d}{dt}na^2\sqrt{(1-e^2)}$$

we find

$$T = \left\{2\frac{\rho}{r}\cos(v_{,}-\lambda) - 1 + \frac{2\rho}{a(1-e^2)}\left(\cos(v_{,}-\lambda) - 1\right)\right\}\frac{na}{\sqrt{(1-e^2)}}\frac{d\Omega}{dv_{,}} + 2\frac{\rho}{r}\sin(v_{,}-\lambda)\frac{na}{\sqrt{(1-e^2)}}\,r\frac{d\Omega}{dr}$$

$$-2\Pi_{,}(\zeta,\ t)\ \frac{\rho e\sin\phi}{a(1-e^2)} - \Pi'_{,}(\zeta,\ t)\ \frac{\rho^2}{na^2\sqrt{(1-e^2)}}\frac{d\zeta}{dt}$$

$$R = -\left\{\frac{\rho}{r}\cos(v_{,}-\lambda) - 1 + \frac{\rho}{a(1-e^2)}\left(\cos(v_{,}-\lambda) - 1\right)\right\}\frac{na}{\sqrt{(1-e^2)}}\frac{d\Omega}{dv_{,}} - \frac{\rho}{r}\sin(v_{,}-\lambda)\frac{na}{\sqrt{(1-e^2)}}\,r\frac{d\Omega}{dr}$$

$$+\Pi_{,}(\zeta,\ t)\frac{\rho e\sin\phi}{a(1-e^2)} - \Gamma_{,}(\zeta,\ t),$$

which are Hansen's values, except that $\frac{d\lambda}{d\tau}$ in the coefficient of $\Pi'_{,}(\zeta,\ t)$ has been replaced by its value.

2, *Stone Buildings*, 31*st March*, 1855.