The Theory of Constructive Types.

(Principles of Logic and Mathematics).

Part II.

Cardinal Arithmetic.

By

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V. Complements of Part I.

A. Extension and Intension.

The Theory of Types, as explained in Part I, may be called the Pure Theory of Types, as it is based on the most general idea of logical types, and as it does not assume any other propositions, than the axioms of the Logical Calculus. This method enables us to get a system of Mathematics which appears to be a part of Logic, and as such may be called Pan-Mathematics. This system is more general than Classical Mathematics, as it does not enable us to prove that there is a class of inductive numbers other than the null-class, which does not contain the greatest element Nevertheless, if we assume the axiom of infinity as a hypothesis, we get a special system, which is as a matter of fact the same thing as what is called Classical Mathematics,— Cantor's theory appears then as a hypothetical system that we can get, if we assume the existence of alephs. Conformably to the hypotheses which we assume, we can get many special systems of Mathematics. As the Pure Theory of Types does not assume any existence-axiom and does not lead to Richard's paradox, it is a natural base for rational Semeiotics, a science whose importance can scarcely be denied. Note that the simplified theory of types, as expounded on p. 12 of Part I, may be used in Mathematics without any risk of getting a contradiction. To avoid such paradoxes as Richard's or König's, it is quite sufficient to assume a direction excluding from the scope of the system any function which is not constructed with the symbols of the system itself. An analogous method is used by mathematicians dealing with the system of axioms of Zermelo[1]). Such a method, though very convevenient, is nevertheless inconsistent with certain fundamental problems of Logic and Semeiotics. Moreover the simplified theory of types implies the existence of functions which cannot be built up, unless we assume that all functions are extensional functions (the Axiom of Extension).

Now, a purely formal system of Logic ought never to imply the existence of such functions; otherwise it might be asked why the axiom of infinity, or other existence-axioms are not to be assumed as primitive propositions.

The practical elimination of the Leibnizian idea of identity is an essential simplification of Logic, this idea being of no use in Mathematics, as we have no means to prove with it the identity of objects given by two different expressions.

Now, here is a most interesting metaphysical problem: Can an object be denoted by two different expressions?

This problem cannot be discussed in a system of formal Logic, as such a system does not contain the primitive idea „expression". On the other hand it is easy to see that such a problem cannot be solved at all, as we always get two contradictory solutions.

If you suppose that two different expressions denote two different objects, you cannot prove that two equivalent classes are identical. To prove that any equivalent classes are identical, we ought to suppose that there are objects denoted by two different expressions. The first hypothesis may form the base of a *Nominalistic* system of Metaphysics (Ontology), the other of a *Realistic* system. The Realistic Hypothesis, i. e the *axiom of extension* would be formulated as follows:

$$.(\bar{x}) . \alpha\{\bar{x}\} \equiv \beta\{\bar{x}\} . \supset . f\{\alpha\} \equiv f\{\beta\}:$$

This axiom seems to have had great success in recent years. I never should care to discuss its truth. I am convinced we never get a contradiction from using this axiom, but I am also convinced

[1]) Cf. Fraenkel: Der Begriff »definit« und die Unabhängigkeit des Auswahl-axioms. Sitzungsberichte der preußischen Akademie der Wissenschaften, Berlin 1922.

that its negation is consistent with the primitive propositions of the Logical Calculus and with the directions of the Pure Theory of Types.

Moreover, there are other general hypotheses which imply the negation of the axiom of extension, yet are at the same time very fruitful.

To see this, let us assume the following definition of the α-order Leibnizian identity, the definition of complete Leibnizian identity being in the Pure Theory of Types impossible.

We have:

$$13{\cdot}001 \quad (x \underset{L}{=} y)_\alpha \underset{d'}{=} (\bar{u}) : \bar{u}\{x\} \equiv \bar{u}\{y\} : \mathfrak{C}\{\bar{u}, \alpha\}.$$

With this definition we build up the theory of the α-order Leibnizian identity, just as in Principia. E. g. we have the following propositions:

$$13{\cdot}11 \quad \vdash . (x \underset{L}{=} y)_\alpha \equiv (\bar{u}) : \bar{u}\{x\} \supset \bar{u}\{y\} : \mathfrak{C}\{\bar{u}, \alpha\} :$$

$$13{\cdot}13 \quad \vdash .: \beta\{x\} . (x \underset{L}{=} y)_\alpha : \mathfrak{C}\{\beta, \alpha\} . \supset \beta\{y\}.$$

$$13{\cdot}15 \quad \vdash (x \underset{L}{=} x)$$

$$13{\cdot}16 \quad \vdash . (x \underset{L}{=} y) \equiv (y \underset{L}{=} x)_\alpha .$$

$$13{\cdot}17 \quad \vdash : (x \underset{L}{=} y)_\alpha . (y \underset{L}{=} z)_\alpha . \supset (x \underset{L}{=} z)_\alpha .$$

For the $\omega''''_{(\nu)}$-order Leibnizian identity of classes we have the following definition:

$$13{\cdot}0011 \quad (\alpha \underset{L}{=} \beta) \underset{d'}{=} (\bar{k}) : \bar{k}\{\alpha\} \supset \bar{k}\{\beta\} : \mathfrak{C}\{\bar{k}, \omega''''_{(\nu)}\}.$$

We have now the following propositions:

$$20{\cdot}14 \quad \vdash . (\alpha \underset{L}{=} \beta) \supset . \alpha = \beta :$$

$$\text{Dem.} \quad \vdash 12{\cdot}312 . (0{\cdot}271) . \supset$$

$$\vdash \mathfrak{C}\{\omega''''_{(\nu)}, \hat{v}[. \hat{v}\{x\} : \omega''''_{(\nu)}\{\hat{v}\} \vee \sim \omega''''_{(\nu)}\{\hat{v}\} :]\} \quad (1)$$

$$\vdash . (1) . (13{\cdot}0011) \supset \vdash Hp \supset$$

$$. \hat{v}[. \hat{v}\{x\} : \omega''''_{(\nu)}\{\hat{v}\} \vee \sim \omega''''_{(\nu)}\{\hat{v}\} :]\{\alpha\} \equiv$$

$$\hat{v}[. \hat{v}\{x\} : \omega''''_{(\nu)}\{\hat{v}\} \vee \sim \omega''''_{(\nu)}\{\hat{v}\} :]\{\beta\}.$$

$$[(0{\cdot}16)] \supset : \alpha\{x\} : \omega''''_{(\nu)}\{\alpha\} \vee \sim \omega''''_{(\nu)}\{\alpha\} : \equiv$$

$$. \beta\{x\} : \omega''''_{(\nu)}\{\beta\} \vee \sim \omega''''_{(\nu)}\{\beta\} : .$$

$$\supset \alpha\{x\} \equiv \beta\{x\}.$$

$$\supset \vdash Prop$$

We see that the $\omega''''_{(V)}$-order Leibnizian identity of two classes implies their identity (equivalence).

I shall use the following abbreviations:

$$13{\cdot}0012 \quad \Psi \underset{df}{=} \hat{u}\hat{z}[.:\hat{u} = V : \hat{u}\{\hat{x}\} : \mathfrak{C}\{\hat{u}, V\}.]$$

$$13{\cdot}00121 \quad W \underset{df}{=} \hat{R}\,[.\,\sigma'''\{\hat{R}\}.\,\mathfrak{C}\{\hat{R}, \Psi\}.]$$

$$13{\cdot}0013 \quad (R \underset{L}{=} P) \underset{df}{=} (\overline{\tau}) : \overline{\tau}\{R\} \supset \overline{\tau}\{P\}.\,\mathfrak{C}\{\overline{\tau}, W\}.$$

The definition of the Axiom of Intension, i. e. Intax, is as follows:

$$13{\cdot}002 \quad \text{Intax} \underset{df}{=} .\,(\hat{x}[.\beta\{\hat{x}\}|\gamma\{\hat{x}\}.] \underset{L}{=} \hat{x}[.\beta'\{\hat{x}\}|\gamma\{\hat{x}\}.]) \supset (\beta \underset{L}{=} \beta').$$

We have now the proposition:

$$13{\cdot}4 \quad \vdash .\,\text{Intax} \supset (\exists \overline{u}, \overline{v}) .: \sim (\overline{u} \underset{L}{=} \overline{v}) : \overline{u} = \overline{v} : \mathfrak{C}\{\overline{u}, V\} : \mathfrak{C}\{\overline{v}, V\}.$$

$$\text{Dem.} \left[13{\cdot}002 \; \frac{-\beta, V}{\beta', \gamma} .\, 10{\cdot}24 \right]$$

Thus it is obvious that the Intax is not consistent with the Axiom of Extension. Now, it is possible to prove, as we shall see below, that Intax implies Infinax (i. e. the Axiom of Infinity)[1].

On the contrary there seems to be no real simplification of Arithmetic, if we assume the axiom of extension — The problem of the Leibnizian identity of two equivalent classes or relations can be eliminated by simply dealing with extensional classes or relations, as we have seen in Part I. The axiom of extension would be needed only in the simplified theory of types, to avoid the proof of the existence of classes, which can never be explicitly given. This proof is as follows:

In the simplified theory of types we have the complete Leibnizian identity, which is to be defined as follows:

$$(x \underset{L}{=} y) \underset{df}{=} (\overline{\psi}) .\, \overline{\psi}\{x\} \supset \overline{\psi}\{y\}.$$

Now let us assume the following definition, using α, β as variable class-letters:

$$G \underset{df}{=} \hat{\alpha}\hat{x}[(\exists \overline{f}) .\, (\hat{z}[(\exists \overline{\beta})\, \overline{f}\{\overline{\beta}, \hat{z}\}] \underset{L}{=} \alpha) .\sim \overline{f}\{\hat{\alpha}, \hat{x}\}.]$$

[1]) The possibility of such a proof was suggested to me by Mr Greniewski.

We have the following proposition:

$$(A)\quad (\exists \overline{f}).(\hat{z}[\exists \overline{\beta})\,\overline{f}\{\overline{\beta}, \hat{z}\}] \underset{L}{=} \hat{z}[(\exists \overline{\beta}) G\{\overline{\beta}, z\}]).\sim \overline{f}\{\hat{z}[(\exists \overline{\beta})\, G\{\overline{\beta}, \hat{z}\}], x\}.$$

To see this, suppose we have: $\sim G\{\hat{z}[(\exists \overline{\beta})\, G\{\overline{\beta}, \hat{z}\}], x\}$, i. e.

$$(\overline{f}).(\hat{z}[(\exists \overline{\beta})\,\overline{f}\{\overline{\beta}, \hat{z}\}] \underset{L}{=} \hat{z}[(\exists \overline{\beta})\, G\{\overline{\beta}, \hat{z}\}]) \supset \overline{f}\{\hat{z}[(\exists \overline{\beta})\, G\{\overline{\beta}, \hat{z}\}], x\}.$$

This proposition being true for every f, it is true for G. Therefore we have:

$$.(\hat{z}[(\exists \overline{\beta})\, G\{\overline{\beta}, \hat{z}\}] = \hat{z}[\exists \overline{\beta})\, G\{\overline{\beta}, \hat{z}\}]) \supset G\{\hat{z}[(\exists \overline{\beta})\, G\{\overline{\beta}, \hat{z}\}], x\}.$$

Here, the hypothesis being true by 13·15, we have:

$G\{\hat{z}[(\exists \overline{\beta})\, G\{\overline{\beta}, \hat{z}\}], x\}$, i e. the proposition (A).

Now, it is obvious that the function f, whose existence is proved by (A), cannot be equivalent to G. Therefore we never shall have such a function, unless we assume the axiom of extension. As a system containing such an axiom in no longer one of pure logic, we see that there is no system of pure logic to be based on the simplified theory of types.

I do not know whether an adequate definition of the hypothesis of Nominalism is to be found in a system of Ontology using no other primitive ideas than purely logical ones. At any rate the Intax is a part of the hypothesis of Nominalism. Another constituent of this hypothesis would be the following axiom, which, as is at once to be seen, is inconsistent with Proposition (A):

$$(\hat{x}[(\exists \overline{v})\, f_{\nu}\{\overline{v}, \hat{x}\}] \underset{L}{=} \hat{x}[(\exists \overline{v})\, G_{\nu}\{\overline{v}, \hat{x}\}]) \supset$$
$$(\hat{u}\hat{x}[f_{\nu}\{\hat{u}, \hat{x}\}] \underset{L}{=} \hat{u}\hat{x}[G_{\nu}\{\hat{u}, \hat{x}\}])$$

This axiom enables us to prove that the class of functions of the type W is similar to the class of classes $\hat{x}[(\exists \overline{v})\, f_{\nu}\{\overline{v}, \hat{x}\}]$, where $\hat{u}\hat{x}[f_{\nu}\{\hat{u}, \hat{x}\}]$ is a function of the type W. The proof of this proposition is a trivial application of our axiom. We should then have in any type the same cardinal numbers in spite of Cantor's theory. Nevertheless the axiom in question seems unfruitful, if used in a system of Mathematics. To obtain a satisfactory one, we ought to suppose that any type is similar to a class of inductive numbers. We shall call this hypothesis the Axiom of Nominalism[1]). Note

[1]) Cf. Zasady czystej teorji typów, Przegląd fil. 1922 p. 28.

that this hypothesis, not less inconsistent with Cantor's theory than with the simplified Theory of Types, is nevertheless very natural and quite simple. It conforms to Poincaré's postulate, stating that there are no other mathematical objects than those we can build up into a given system. It is interesting to note that with the Axiom of Nominalism, we can prove the axiom of Zermelo [1]), and we have nevertheless to do with a continuum conceived as an ambiguous symbol (Cf. Part I, p. 19).

The researches concerning this subject seem to be very important, many interesting theorems of modern Mathematical Analysis being based on Zermelo's axiom. Note that with the axiom of Nominalism we prove, e. g. that a limit point of a class of points is a limit point of a progression of points, contained in the given class. As the Intax enables us to prove the Axiom of Infinity, it is obvious that a system based on the Axiom of Nominalism and on Intax should embody modern Mathematical Analysis.

B. Types.

It is to be remarked that the use of primitive letters is very limited. As a matter of fact, they are only used to build up the expression $\mathfrak{C}\{x, y\}$. Now, there is another method of obtaining an equivalent expression. Let us expunge the primitive letters from our system and assume the following definitions:

12·001 $\mathfrak{C}'_\alpha(x) \underset{df}{=} . \, \alpha\{x\} \equiv \alpha\{x\}.$

12·002 $\mathfrak{C}_\alpha(x, y) \underset{df}{=} . \, \mathfrak{C}'_\alpha(x) \, . \, \mathfrak{C}'_\alpha(y).$

It is obvious that the symbol $\mathfrak{C}_\alpha(x, y)$ denotes the proposition „x is of the same type as y" as much as the symbol $\mathfrak{C}\{x, y\}$. The elimination of primitive letters would be an essential simplification of the Pure Theory of Types. If we omit the primitive letters, we can have a very simple direction for the construction of functions of the same type, i. e:

D Two functional expressions, containing no primitive letters, denote functions of the same type,

1° if they denote at the same time functions with I variable (or with II, or with III, or with IV), their corresponding variables

[1]) Cf. Trzy odczyty odnoszące się do pojęcia istnienia. Przegląd fil. 1917.

being determined by the same functional expressions, or being individual letters; and

2° if they contain the same elementary letters, and the same undetermined variables occurring in both as constituents of the same functional or propositional expression.

With this direction we can write significantly $\mathfrak{T}_\alpha (E, G)$, without using the directions 0·2, by simply looking on the letters occurring in the expressions E, G.

In consequence of our direction, an intuitive use of the pure Theory of Types appears to be possible. Nevertheless I still keep to directions 0·2, as being more convenient in symbolic practice.

Note that by the direction D, we have:

$$\mathfrak{T}_\sigma \{\hat{u} [. \mathfrak{T}_\omega \{\hat{u}, a\} \,|\, (\overline{v})\, \mathfrak{T}_\omega \{\overline{v}, a\}.], \hat{u} [\mathfrak{T}_\omega \{\hat{u}, a\}]\},$$

a formula impossible to attain by the directions 0·2. We see at once that this difference is not essential. The first method is most in harmony with practice; the second with the primitive idea of a logical type. In Principia we have an analogous difference between first-order matrices and first-order functions.

The pure Theory of Types does not enable us to prove the existence of functions of a given property, without having an instance of such a function. Nevertheless, it enables us to prove the existence of individuals, without having any instance of them. Now, Prof. Wilkosz has remarked that a purely formal system of Logic ought not to be of any use in proving the existence of objects which are not explicitly given in the system. To have such a system, it is sufficient to deal with individuals in conformity with the method we have applied to classes. We begin with introducing the letters l, m, n, which shall be called i n d i v i d u a l c o n s t a n t s. We suppose that these letters denote individuals; and we agree that these letters can never be used as noted or apparent variables. As there are metaphysical reasons to admit the existence of individuals of different types, we shall never use such expressions as $\mathfrak{T}_\alpha \{l, m\}$; or similarly as $\hat{x} [f\{\hat{x}\}]$ or $(\overline{x}) f \{\overline{x}\}$. To have noted or apparent variables, we shall be obliged to begin with such expression as $. f\{x\} . \mathfrak{T}_\alpha \{x, m\} .$[1]), where the real variable x is determined

[1]) Mr Skarżeński, has remarked that we get a serious simplification of the system, if we use $f_{(m)} \{x\}$ as a fundamental idea. I see that this method would be most conformable to the real meaning of the idea of a propositional function.

by the constant m. Then the fundamental Principles of the Calculus of Functions will be:

I. The Principle of Deduction:

10·01 $\vdash .(\bar{x})\, f_m\{\bar{x}\} \supset f_m\{y\}.$

II. The Principle of Disjunction:

10·02 $\vdash .\, (\bar{x}) . p \vee f_m\{\bar{x}\}. \supset . p \vee (\bar{x})\, f_m\{\bar{x}\}:$

With these Principles we can prove that there is an individual of the same type as m, we having the proposition:

$$(\exists\, \bar{x})\ \mathcal{C}_\alpha\{\bar{x}, m\}$$

but we cannot prove that there is an individual.

Such a modification of the Pure Theory of Types will be made in the complete system of Logic and Mathematics which I intend to publish later.

The theory of functions of the same type, as expounded in Part I, is far from being complete. To have a full theory of functions of the same type, occurring in a system, it would be necessary to write a very big book, — as a matter of fact a commonplace one. It is therefore more reasonable to prove no other propositions than those which are to be used immediately. In this paper we shall use the following propositions, which do not occur in Part I:

12·31 $\vdash \mathcal{C}\{\hat{x}\,\hat{y}\,[.\, f\{\hat{x}\} \mid g\{\hat{y}\}.],\ \hat{x}\,\hat{y}\,[.\, f\{\hat{x}\} \vee g\{\hat{y}\}.]\}$

[12·2421 . (0·34)]

12·3121 $\vdash \mathcal{C}\{\hat{x}\,\hat{y}\,[.\, f\{\hat{x}\} \mid g\{\hat{y}\}.],\ \hat{x}\,\hat{y}\,[.\, f\{\hat{x}\} \supset g\{\hat{y}\}.]\}$

[Constr. as in 12·312]

12·3122 $\vdash \mathcal{C}\{\hat{x}\,\hat{y}\,[.\, f\{\hat{x}\} \mid g\{\hat{y}\}.],\ \hat{x}\,\hat{y}\,[.\, f\{\hat{x}\} . g\{\hat{y}\}.]\}$

Constr. $\vdash$ 12·22 . 3·01. $\supset$

$\vdash \mathcal{C}\{\hat{x}\,\hat{y}\,[.\, f\{\hat{x}\} . g\{\hat{y}\}.],\ \hat{x}\,\hat{y}\,[.\sim f\{\hat{x}\} \vee \sim g\{\hat{y}\}.]\}$

[0·43·44 14·15] $\supset \vdash \mathcal{C}\{\hat{x}\,\hat{y}\,[.\, f\{\hat{x}\} . g\{\hat{y}\}.],\ \hat{x}\,\hat{y}\,[.\sim f\{\hat{x}\} \vee g\{\hat{y}\}.]\}$

[12·3121 . 1·01] $\supset \vdash$ Prop.

12·4 $\vdash \mathcal{C}\{\hat{x}\,[(\bar{y}) . p .\, f\{\bar{y}, \hat{x}\}.],\ \hat{x}\,[.\, p .\, (\bar{y})\, f\{\bar{y}, \hat{x}\}.]\}$

[0·27 . 12·3122.]

12·41 $\vdash \mathcal{C}\{\hat{x}\,[(\bar{y}) . p \supset f\{\bar{y}, \hat{x}\}.],\ \hat{x}[.\, p \supset (\bar{y})\, f\{\bar{y}, \hat{x}\}.]\}$

[0·27 . 12·3121.]

12·42 $\vdash \mathcal{C}\{\hat{x}\,[(\bar{y}) . f\{\bar{y}, \hat{x}\} \mid p.],\ \hat{x}\,[.\, (\bar{y})\, f\{\bar{y}, \hat{x}\} \mid p.]\}$ [0·27·24]

12·421 $\vdash \mathcal{C}\{\hat{x}\,[(\bar{u}) . f_a\{\bar{u}, \hat{x}\} \mid p.],\ \hat{x}\,[.\, (\bar{u})\, f_a\{\bar{u}, \hat{x}\} \mid p.]\}$ [0·27·24]

12·43 $\vdash \mathfrak{C}\{\hat{x}\ [(y).f\{\bar{y}, \hat{x}\} \supset p.], \hat{x}\ [.(\bar{y})\ f\ \{\bar{y}, \hat{x}\} \supset p.]\}$
[12·42·3121]

12·5 $\vdash \mathfrak{C}\{z\ [(\bar{x}, \bar{y})\ f\{\hat{x}, \bar{y}, \hat{z}\}], \hat{z}[(\bar{x})\ f\{\bar{x}, y', \hat{z}\}]\}$

Constr. $\vdash .12{\cdot}1.0{\cdot}441. \supset \vdash \mathfrak{C}\{\hat{z}\ [f\{x, y', \hat{z}\}], \hat{z}\ [f\{x', x, \hat{z}\}]\}$

[0·25·4] $\supset \vdash \mathfrak{C}\{\hat{z}\ [(\bar{x})\ f\{\bar{x}, y', \hat{z}\}], \hat{z}\ [(\bar{y})\ f\{x', \bar{y}, \hat{z}\}]\}$

[0·252] $\supset \mathfrak{C}\{\hat{z}\ [(\bar{x})\ f\{\bar{x}, y', \hat{z}\}], \hat{z}\ [.(\bar{x}) f\{\bar{x}, y', \hat{z}\} \mid (\bar{y})\ f\{x', \bar{y}, \hat{z}\}.]\}$ (1)

$\vdash .12{\cdot}1.0{\cdot}252{\cdot}412. \supset \vdash \mathfrak{C}\{\hat{z}\ [f\{x, y, \hat{z}\}], \hat{z}\ [.f\{x, y', \hat{z}\} \mid f\{x'\ y, \hat{z}\}.]\}$

[0·252] $\supset \vdash \mathfrak{C}\{\hat{x}\ [(\bar{x}, \bar{y}) f\{\bar{x}, \bar{y}, \hat{z}\}], \hat{z}\ [(\bar{x}, \bar{y}).f\{\bar{x}, y', \hat{z}\} \mid f\{x'. \bar{y}, \hat{z}\}.]\}$ (2)

$\vdash .(1).(2). \supset \vdash$ Prop.

12·51 $\vdash \mathfrak{C}\{\hat{y}\ [(\bar{x}, \bar{u}) f_a\{\bar{u}, \bar{x}, \hat{y}\}].\ \hat{y}\ [(\bar{u}, \bar{x})\ f_a\ \{\bar{u}, \bar{x}, \hat{y}\}]\}$

Constr. $\vdash 12{\cdot}1.0{\cdot}242{\cdot}261{\cdot}26{\cdot}4 \supset$

$\vdash \mathfrak{C}\{\hat{y}\ (\bar{x}, \bar{u}) f_a\{\bar{u}, \bar{x}. \hat{y}\}], \hat{y}\ [(\bar{x}, \bar{u}).f_a\{\bar{u}, x', y'\} \mid f_a\{v, \bar{x}, \hat{y}\}.]\}$

[12·421.0 26·261.] $\supset$

$\vdash \mathfrak{C}\{\hat{y}\ [(\bar{x}, \bar{u}) f_a\{\bar{u}. \bar{x}. \hat{y}\}], \hat{y}\ [(\bar{x}).(\bar{u})\ f_a\ \{\bar{u}, x', y'\} \mid f_a\{\bar{v}, \bar{x}, \bar{y}\}.]\}$

[0·27] $\supset \vdash \mathfrak{C}\{\hat{y}\ [(\bar{x}, \bar{u})\ f_a\ \{\bar{u}, \bar{x}, \hat{y}\}], \hat{y}\ [.(\bar{u}) f_a\ \{\bar{u}, x', y'\} \mid (\bar{x})\ f_a\ \{v, \bar{x}, \hat{y}\}.]\}$

[12·421.0·26·261.] $\supset$

$\vdash \mathfrak{C}\{\hat{y}\ [(\bar{x}, \bar{u})\ f_a\ \{\bar{u}, \bar{x}. \hat{y}\}], \hat{y}\ [(\bar{u}).f_a\ \{\bar{u}, x', y'\} \mid (\bar{x}) f_a\ \{v, \bar{x}, \hat{y}\}]\}$

[0·27] $\supset \vdash \mathfrak{C}\{\hat{y}\ [(\bar{x}, \bar{u})\ f_a\ \{\bar{u}, \bar{x}, \hat{y}\}].\ \hat{y}\ [(\bar{u}, \bar{x}).f_a\ \{\bar{u}, x', y'\} \mid f_a\ \{v, \bar{x}, \hat{y}\}.]\}$

[0·252] $\supset \vdash$ Prop.

We shall use the following abbreviations:

12·011 $\mathfrak{C}\{x, y, z\} = .\ \mathfrak{C}\{x, z\}.\ \mathfrak{C}\{y, z\}.$

12·012 $\mathfrak{C}\{x, y, z. z'\} = :\ \mathfrak{C}\{x, z'\}.\ \mathfrak{C}\{y. z'\}:\ \mathfrak{C}\{z, z'\}.$

Note that, if we take $f_a\{v\}$ for $\hat{u}\ [f_a\{\hat{u}\}]\ \{v\}$, this is by no means effected by the simple application of 0·16 :but we first take by 12·01·02

$u\ [.f\{\hat{u}\}.(\bar{\varphi}): \bar{\varphi}\{\hat{u}\} \supset \bar{\varphi}\ \{\hat{u}\}. \supset . \bar{\varphi}\ \{a\} \supset \bar{\varphi}\ \{a\}:.]$ for $\hat{u}\ [f_a\ \{\hat{u}\}]$
and then we apply the direction 0·16 to this function.

. Automatical construction of assertions.

It is to be remarked that the number of definitions needed in practice is very great. In this paper we shall not give all definitions explicitly in cases from which any ambiguity is excluded. We shall also use some simplifications in our construction of expressions which are by no means an essential modification of our directions, and may without any difficulty be omitted in a complete system. E. g. we shall omit one external dot on both sides of our assertions; likewise the brackets in defined symbols, in cases excluding any ambiguity; we shall also omit the letter a in defined symbols, by a proceeding to be explained later.

VI. Prolegomena to Cardinal Arithmetic.

This chapter contains certain definitions and propositions to be explicitly used in Cardinal Arithmetic. In spite of the general method expounded in Part I., we shall have to deal with definitions built up for special types.

a. Complements of the Theory of Deduction:

3·02 $\quad .p.q.r. \underset{df}{=} :p.q:r.$

3·021 $\quad .p.q.r.s. \underset{df}{=} :p.q.r:s.$

3·022 $\quad .p.q.r.s.p'. \underset{df}{=} :p.q.r.s:p'.$

3·44 $\quad \vdash :q \supset p :. r \supset p : \supset : q \vee r . \supset p.$

3·47 $\quad \vdash :p \supset r :. q \supset s : \supset :p.q. \supset .r.s:$

4·1 $\quad \vdash .p \supset q . \equiv . \sim q \supset \sim p.$ [to be called: Transp]

5·5 $\quad \vdash p \supset :p \supset q . \equiv q.$

5·75 $\quad \vdash :r \supset \sim q :. p \equiv . q \vee r :. \supset :p . \sim q . \equiv r.$

10·28 $\quad \vdash (\overline{x}) . f\{\overline{x}\} \supset g\{\overline{x}\} . \supset . (\exists \overline{x}) f\{\overline{x}\} \supset (\exists \overline{x}) g\{\overline{x}\}.$

10·34 $\quad \vdash (\overline{x}) . g\{\overline{x}\} \supset p . \equiv . (\exists \overline{x}) g\{\overline{x}\} \supset p.$

b. Classes and Relations.

We shall use the following abbreviations:

20·04 $\quad \text{extens}_a\ (\varkappa) \underset{df}{=} (\overline{u}. \overline{v}) : \overline{u} \underset{a,\, a}{=} \overline{v} . \supset . \varkappa\{u\} \equiv \varkappa\{\overline{v}\}:$

20·041 $\quad \text{extens}\ (\varkappa) \underset{df}{=} \text{extens}_a\ (\varkappa)$

21·04 $\quad \text{extens}_a (R) \underset{df}{=} (\overline{u}, \overline{v}, \overline{w}, \overline{t}) :. \overline{u} \underset{a,\, a}{=} \overline{v} :. \overline{w} \underset{a,\, a}{=} \overline{t} : \supset . R\{\overline{u}, \overline{w}\} \equiv R\{\overline{v}. \overline{t}\}:$

21·041 $\quad \text{extens}\ (R) \underset{df}{=} \text{extens}_a (R)$

The difference between the use of $\text{extens}_a(\varkappa)$ and extens $(\varkappa)$, (or $\text{extens}_a(R)$ and $\text{extens}(R)$), is that the first symbol can be automatically applied to higher types, so as to have e. g. extens$_{P'_{(a)}}(\varkappa)$ for $(\overline{P}\,\overline{Q}) : \overline{P} \underset{P'_{(a)}, P'_{(a)}}{=} \overline{Q} . \supset . \varkappa_{(P'_{(a)})}\{\overline{P}\} \equiv \varkappa_{(P'_{(a)})}\{\overline{Q}\}$.

The second symbol i. e. extens $(\varkappa)$ stands simply for extens$_a(\varkappa)$. Such simplified symbols as extens $(\varkappa)$ will be used below without being expressly defined, in conformity with the remark on p. 14.

I pass now to the following list of definitions, which are built up for special types to be used in Cardiual Arithmetic.

Definitions:

20·042 $u\, \varepsilon_a\, \varkappa \underset{df}{=} . \varkappa\{u\} . \text{extens}_a(\varkappa) . \mathfrak{C}\{u, a\}$.

20·0421 $u, v\, \varepsilon_a\, \varkappa \underset{df}{=} . u\, \varepsilon_a\, \varkappa . v\, \varepsilon_a\, \varkappa$.

20·0422 $u, v, w\, \varepsilon_a\, \varkappa \underset{df}{=} . u\, \varepsilon_a\, \varkappa\ \ v, w\, \varepsilon_a\, \varkappa$.

20·043 $\text{extens}_a(\varkappa, \omega) \underset{df}{=} . \text{extens}_a(\varkappa) . \text{extens}_a(\omega)$.

22·06 $. \varkappa \underset{a}{\subset} \omega . \underset{df}{=} (\overline{u}) . \varkappa_{(a)}\{\overline{u}\} \supset \omega_{(a)}\{\overline{u}\}$.

22·061 $. \varkappa \underset{a}{=} \omega . \underset{df}{=} . \varkappa_{(a)} = \omega_{(a)}$.

22·062 $. \varkappa \underset{a}{\neq} \omega . \underset{df}{=} . \varkappa_{(a)} \neq \omega_{(a)}$.

22·07 $(\varkappa \cap_a \omega) \underset{df}{=} \hat{u}[. \hat{u}\, \varepsilon_a\, \varkappa . \hat{u}\, \varepsilon_a\, \omega .]$

22·071 $(\varkappa \cup_a \omega) \underset{df}{=} \hat{u}[. \hat{u}\, \varepsilon_a\, \varkappa \vee \hat{u}\, \varepsilon_a\, \omega .]$

22·072 $(\varkappa \underset{a}{-} \omega) \underset{df}{=} (\varkappa \cap_a - \omega_{(a)})$

22·08 $(\underset{a}{-}\, \omega) \underset{df}{=} \hat{u}[\sim \hat{u}\, \varepsilon_a\, \omega]$

51·01 $\iota^{\mathfrak{c}}{}_a u \underset{df}{=} \hat{v}[. \hat{v} \underset{a}{=} u .]$

22·09 $K \underset{df}{=} (\varkappa''' \cup_a \iota^{\mathfrak{c}}{}_a u)$

21·042 $u[R]_a v \underset{df}{=} . R\{u, v\} . \text{extens}_a(R) . \mathfrak{C}\{u, v, a\}$.

21.06 $. P \underset{a}{\subset} Q . \underset{df}{=} (\overline{u}, \overline{v}) . P_{(a)}\{\overline{u}, \overline{v}\} \supset Q_{(a)}\{\overline{u}, \overline{v}\}$.

21·061 $. P \underset{a}{=} Q . \underset{df}{=} . P_{(a)} = Q_{(a)}$.

21·062 $. P \underset{a}{\neq} Q . \underset{df}{=} . P_{(a)} \neq Q_{(a)}$.

21·07 $(P \cap_a Q) \underset{df}{=} \hat{u}\hat{v}[. \hat{u}[P]_a \hat{v} . \hat{u}[Q]_a \hat{v} .]$

21·071 $(P \cup_a Q) \underset{df}{=} \hat{u}\hat{v}[. \hat{u}[P]_a \hat{v} \vee \hat{u}[Q]_a \hat{v} .]$

21·072 $(P \underset{a}{-} Q) \underset{df}{=} (P \bigcap_a - Q_{(a)})$

21·08 $(\underset{a}{-} P) \underset{df}{=} \hat{u}\,\hat{v}[\sim \hat{u}\,[P]_a\,\hat{v}]$

21·081 $A \underset{df}{=} \hat{u}\,\hat{v}[(\exists \overline{\varkappa}) . \hat{u}, \hat{v}\,\varepsilon_a \overline{\varkappa} . \mathfrak{C}\{\overline{\varkappa}, K\}.]$

21·09 $\text{extens}_a^A (R) \underset{df}{=} (\overline{u}, \overline{v})(\overline{P}, \overline{Q}) :. \overline{u} \underset{a,\,a}{=} \overline{v} :\ \overline{P} \underset{A,\,A}{=} \overline{Q} : \supset . R\{\overline{u}, \overline{P}\} \equiv$
$\equiv R\{\overline{v}, \overline{Q}\}:$

31·013 $Cnv_a R \underset{df}{=} Cnv^{\text{c}}\, R_{(a,\,a)}$

31·014 $Cnv_a^A R \underset{df}{=} \hat{P}\,\hat{u}\,[R_{(u,\,A)}\{\hat{u}, \hat{P}\}]$

31·015 $Cnv_A^a R \underset{df}{=} \hat{u}\,\hat{P}[R_{(A,\,a)}\{\hat{P}, \hat{u}\}]$

21·091 $u\,[R]_a^A\,Q \underset{df}{=} . R\,\{u, Q\} . \text{extens}_a^A (R) . \mathfrak{C}\{u, a\} . \mathfrak{C}\{Q, A\}.$

21·092 $Q\,[R]_a^A\,u \underset{df}{=} . R\{Q, u\} . \text{extens}_A^a (Cnv_A^a R) . \mathfrak{C}\{u, a\} . \mathfrak{C}\{Q, A\}.$

24·04 $\exists_a \varkappa \underset{df}{=} (\exists \overline{u})\,\overline{u}\,\varepsilon_a\,\varkappa$

25·04 $\exists_a R \underset{df}{=} (\exists \overline{u}, \overline{v})\,\overline{u}\,[R]_a \overline{v}$

25·041 $\exists_a^A R \underset{df}{=} (\exists \overline{u})(\exists\,\overline{P})\,\overline{u}\,[R]_a^A\,\overline{P}$

25·042 $\exists_A^a R \underset{df}{=} (\exists \overline{u})(\exists\,\overline{P})\,\overline{P}\,[R]_A^a\,\overline{u}$

33·05 $D_a R \underset{df}{=} \hat{u}\,[(\exists \overline{v})\,\hat{u}\,[R]_a \overline{v}]$

33·051 $D_a^A R \underset{df}{=} \hat{u}\,[(\exists \overline{P})\,\hat{u}\,[R]_a^A\,\overline{P}]$

33·052 $D_A^a R \underset{df}{=} \overline{P}[(\exists \hat{u})\,\overline{P}[R]_A^a\,\hat{u}]$

33·06 $\text{Ɑ}_a R \underset{df}{=} \hat{u}\,[(\exists \overline{v})\,\overline{v}\,[R]_a \overline{u}]$

33·061 $\text{Ɑ}_a^A R \underset{df}{=} \hat{P}[(\exists \overline{v})\,\overline{v}\,[R]_a^A\,\hat{P}]$

33·062 $\text{Ɑ}_A^a R \underset{df}{=} \hat{u}[(\exists \overline{P})\,\overline{P}\,[R]_A^a\,\hat{u}]$

37·001 $\vartheta \underset{df}{=} \sigma''''$

37·0011 $\vartheta' \underset{df}{=} \sigma'''''$

37·002 $\pi \underset{df}{=} \tau''''$

37·0021 $\pi' \underset{df}{=} \tau'''''$

37·01 $[R^{\text{cc}}]_a\,\varkappa \underset{df}{=} \hat{u}\,[(\exists \overline{v}) . \hat{u}\,[R]_a \overline{v} . \overline{v}\,\varepsilon_a\,\varkappa .]$

37·011 $[R^{\text{cc}}]_a^A\,\pi \underset{df}{=} \hat{u}\,[(\exists \overline{P}) . \hat{u}\,[R]_a^A\,\overline{P} . \overline{P}\,\varepsilon_A\,\pi .]$

37·012 $[R^{\text{cc}}]_A^a\,\varkappa \underset{df}{=} \hat{P}\,[(\exists \overline{u})\,\hat{P}\,[R]_A^a\,\overline{u} . \overline{u}\,\varepsilon_a\,\pi .]$

40·01 $\mathfrak{p}{}^{\text{c}}\sigma \underset{df}{=} \hat{u}[(\bar{\varkappa}).\bar{\varkappa}\,\varepsilon_K\,\sigma \supset \hat{u}\,\varepsilon_a\,\bar{\varkappa}.]$

50·01 $\mathrm{I} \underset{df}{=} \hat{u}\,v[.\hat{u} \underset{a,a}{=} \hat{v}.]$.

Propositions:

24·23 $\vdash (\varkappa \cap_a \Lambda_{(a)}) \underset{a}{=} \Lambda_{(a)}$

24·24 $\vdash \text{extens}(\varkappa) \supset .(\varkappa \cup_a \Lambda_{(a)}) \underset{a}{=} \varkappa.$

22·42 $\vdash \varkappa \underset{a}{\subset} \varkappa$

22·43 $\vdash (\varkappa \cap_a \omega) \underset{a}{\subset} \varkappa$

22·44 $\vdash : \varkappa \underset{a}{\subset} \omega :. \omega \underset{a}{\subset} \omega' : \supset . \varkappa \underset{a}{\subset} \omega'.$

22·621 $\vdash \text{extens}(\varkappa, \omega) \supset : \varkappa \underset{a}{\subset} \omega . \equiv . (\varkappa \cap_a \omega) \underset{a}{=} \varkappa:$

22·68 $\vdash ((\varkappa \cap_a \omega) \cup_a (\varkappa \cap_a \omega')) \underset{a}{=} (\varkappa \cap_a (\omega \cup \omega'))$

22·81 $\vdash \text{extens}(\varkappa, \omega) \supset : \varkappa \underset{a}{\subset} \omega . \equiv . \underset{a}{-} \omega \underset{a}{\subset} \underset{a}{-} \varkappa:$

22·87 $\vdash \text{extens}(\varkappa, \omega) \supset .(\underset{a}{-} \varkappa \cap_a \underset{a}{-} \omega) \equiv \underset{a}{-} (\varkappa \cup_a \omega).$

25·4 $. \text{extens}_a(P) . \text{extens}_a(Q) . \supset : (P \cap_a Q) \underset{a}{=} \dot{\Lambda}_{(a,a)} .$
$\equiv . ((P \cup_a Q) \underset{a}{-} P) \underset{a}{=} Q :$

37·2 $\vdash . \text{extens}(\varkappa, \omega) : \varkappa \underset{a}{\subset} \omega : \supset . [R^{\text{cc}}]_a \varkappa \underset{a}{\supset} [R^{\text{cc}}]_a \omega .$

37·51 $\vdash \text{extens}(\varkappa) \supset : \varkappa \underset{a}{\subset} \mathfrak{a}_a R . \equiv . \varkappa \underset{a}{\subset} [Cnv_a R^{\text{cc}}]_a [R^{\text{cc}}]_a \varkappa .$

40·12 $\vdash \varkappa\,\varepsilon_K\,\sigma \supset . \mathfrak{p}{}^{\text{c}}\sigma \underset{a}{\subset} \varkappa .$

51·36 $\vdash \text{extens}(\varkappa) \supset . \sim u\,\varepsilon_a\,\varkappa \equiv . \varkappa \underset{a}{\subset} \underset{a}{-} \iota{}^{\text{c}} u:$

C. Theory of Relations.

Relative products of two relations.

34·01 $(R \overset{a}{\underset{a}{|}} S) \underset{df}{=} \hat{u}\hat{v}[\exists\bar{w}) . \hat{u}[R]_a \bar{w} . \bar{w}[S]_a \hat{v} .]$

34·011 $(R \underset{a}{|} S) \underset{df}{=} (R \overset{a}{\underset{a}{|}} S)$

34·012 $(R \overset{\mathbf{A}}{\underset{a}{|}} S) \underset{df}{=} \hat{u}\hat{v}[(\exists\bar{P}) . \hat{u}[R]^{\mathbf{A}}_a \bar{P} . \bar{P}[S]^a_{\mathbf{A}} \hat{v} .]$

34·013 $(R \overset{a}{\underset{a,\mathbf{A}}{|}} S) \underset{df}{=} \hat{u}\hat{P}[\exists\bar{w}) . \hat{u}[R]_a \bar{w} . \bar{w}[S]^{\mathbf{A}}_a \hat{P} .]$

34·014 $(R \overset{a}{\underset{\mathbf{A}}{|}} S) \underset{df}{=} \hat{P}\hat{Q}[(\exists\bar{w}) . \hat{P}[R]^a_{\mathbf{A}} \bar{w} . \bar{w}[S]^{\mathbf{A}}_a \hat{Q} .]$

34·015 $(R \overset{\mathbf{A}}{\underset{a,\mathbf{A}}{|}} S) \underset{df}{=} \hat{u}\hat{P}[(\exists\bar{Q}) . \hat{u}[R]^{\mathbf{A}}_a \bar{Q} . \bar{Q}[S]_{\mathbf{A}} \hat{P} .]$

34·016 $(R \overset{a}{\underset{A,a}{|}} S) \underset{df}{=} \hat{P}\hat{u}[(\exists \bar{w}) . \hat{P}[R]^a_A \bar{w} . \bar{w}[S]_a \hat{u}.]$

34·017 $(R \overset{A}{\underset{A,a}{|}} S) \underset{df}{=} \hat{P}\hat{u}[(\exists \bar{Q}) . \hat{P}[R]_A \bar{Q} . \bar{Q}[S]^a_A \hat{u}.]$

34·04 $(R \overset{a}{\underset{a}{|}} S \overset{a}{\underset{a}{|}} T) \underset{df}{=} ((R \overset{a}{\underset{a}{|}} S) \overset{a}{\underset{a}{|}} T)$

34·041 $(R | S | T) \underset{df}{=} (R \overset{a}{\underset{a}{|}} S \overset{a}{\underset{a}{|}} T)$

Analogous abbreviations will be used for relative products of relations of any type.

34·21 ⊢ $(R \overset{a}{\underset{a}{|}} S \overset{a}{\underset{a}{|}} T) \underset{a}{=} (R \overset{a}{\underset{a}{|}} (S \overset{a}{\underset{a}{|}} T))$

Analogous propositions for other types are here tacitly assumed.

Limited domains and converse domains.

35·01 $(\varkappa \overset{a}{\underset{a}{\upharpoonleft}} R) \underset{df}{=} \hat{u}\hat{v}[.\hat{u}\,\varepsilon_a \varkappa . \hat{u}[R]_a \hat{v}.]$

35·011 $(\varkappa \overset{A}{\underset{a}{\upharpoonleft}} R) \underset{df}{=} \hat{u}\hat{P}[.\hat{u}\,\varepsilon_a \varkappa . \hat{u}[R]^A_a \hat{P}.]$

35·012 $(\pi \overset{a}{\underset{A}{\upharpoonleft}} R) \underset{df}{=} \hat{P}\hat{u}[.\hat{P}\varepsilon_A \pi . \hat{P}[R]^a_A \hat{u}.]$

35·02 $(R \overset{a}{\underset{a}{\upharpoonright}} \varkappa) \underset{df}{=} \hat{u}\hat{v}[.\hat{u}[R]_a \hat{v} . \hat{v}\,\varepsilon_a \varkappa.]$

35·021 $(R \overset{a}{\underset{A}{\upharpoonright}} \varkappa) \underset{df}{=} \hat{P}\hat{u}[.\hat{P}[R]^a_A \hat{u} . \hat{u}\,\varepsilon_a \varkappa.]$

35·022 $(R \overset{A}{\underset{a}{\upharpoonright}} \pi) \underset{df}{=} \hat{u}\hat{P}[.\hat{u}[R]^A_a \hat{P} . \hat{P}\varepsilon_A \pi.]$

35·412 ⊢ $(R \overset{a}{\underset{a}{\upharpoonright}} (\varkappa \cup_a \omega)) \equiv ((R \overset{a}{\underset{a}{\upharpoonright}} \varkappa) \cup_a (R \overset{a}{\underset{a}{\upharpoonright}} \omega))$

35·431 ⊢ extens $(\varkappa, \omega) \supset : \varkappa \underset{a}{\subset} \omega . \supset . (R \overset{a}{\underset{a}{\upharpoonright}} \varkappa) \underset{a}{\subset} (R \overset{a}{\underset{a}{\upharpoonright}} \omega):$

35·65 ⊢ $: \varkappa \underset{a}{\subset} \sigma_a R :$ extens $(\varkappa) . \supset . D_a(\varkappa \overset{a}{\underset{a}{\upharpoonleft}} R) \underset{a}{=} \varkappa .$

Ordinal couples:

55·01 $(u \overset{a}{\downarrow} v) \underset{df}{=} \hat{w}\hat{t}[: \hat{w} \underset{a,a}{=} u :. \hat{t} \underset{a,a}{=} v :]$

55·03 $(u \underset{A}{\downarrow}) \underset{df}{=} \hat{P}\hat{w}[: \hat{P} \underset{a}{=} (u \overset{a}{\downarrow} \hat{w}) : \mathfrak{C}\{\hat{w}, a\} . \mathfrak{C}\{\hat{P}, A\}.]$

55·04 $(\underset{A}{\downarrow} u) \underset{df}{=} \hat{P}\hat{w}[: \hat{P} \underset{a}{=} (\hat{w} \overset{a}{\downarrow} u) : \mathfrak{C}\{\hat{w}, a\} . \mathfrak{C}\{\hat{P}, A\}.]$

55·2 ⊢ $. (u \overset{a}{\downarrow} v) \underset{a}{=} (u' \overset{a}{\downarrow} v') . \equiv : u \underset{a,a}{=} u' :. v \underset{a,a}{=} v' :.$

One-many, many-one and one-one relations.

71·01 $R\,\varepsilon\,1 \underset{a}{\rightarrow} Cls \underset{df}{=} .(\bar{u},\bar{v},\bar{w}) : \bar{u}\,[R]_a\,\bar{w}\,.\,\bar{v}\,[R]_a\,\bar{w}\,.\supset \bar{u} \underset{a,a}{=} \bar{v}$: extens$_a(R)$.

71·011 $R\,\varepsilon\,1 \underset{a,A}{\rightarrow} Cls \underset{df}{=} .(\bar{u},\bar{v})(\bar{P}) : \bar{u}\,[R]^A_a\,\bar{P}\,.\,\bar{v}\,[R]^A_a\,\bar{P}\,.\supset .\,\bar{u} \underset{a,a}{=} \bar{v}$:
extens$^A_a(R)$.

77·012 $R\,\varepsilon\,1 \underset{A,a}{\rightarrow} Cls \underset{df}{=} .(\bar{P},\bar{Q})(\bar{u}) : \bar{P}\,[R]^a_A\,\bar{u}\,.\,\bar{Q}\,[R]^a_A\,\bar{u}\,.\supset .\,\bar{P} \underset{a}{=} \bar{Q}$:
extens$^A_a(Cnv^a_A R)$.

71·02 $R\,\varepsilon\,Cls \underset{a}{\rightarrow} 1 \underset{df}{=} (\bar{u},\bar{v},\bar{w})\,.\,\bar{w}\,[R]_a\,\bar{u}\,.\,\bar{w}\,[R]_a\,\bar{v}\,.\supset .\,\bar{u} \underset{a,a}{=} \bar{v}$: extens$_a(R)$.

71·03 $R\,\varepsilon\,1 \underset{a}{\rightarrow} 1 \underset{df}{=} : R\,\varepsilon \underset{a}{\rightarrow} Cls\,.\,R\,\varepsilon\,Cls \underset{a}{\rightarrow} 1.$

I omit the definitions of $R\,\varepsilon\,Cls \underset{a,A}{\rightarrow} 1$, $R\,\varepsilon\,Cls \underset{A,a}{\rightarrow} 1$, $R\,\varepsilon\,1 \underset{a,A}{\rightarrow} 1$, $R\,\varepsilon\,1 \underset{A,a}{\rightarrow} 1$, which are to be got by the same method.

71·19 $\vdash R\,\varepsilon\,1 \underset{a}{\rightarrow} Cls \equiv .(R \underset{a}{|} Cnv_a R) \underset{a}{=} (I \upharpoonright_a D_a R).$

71·192 $\vdash R\,\varepsilon\,1 \underset{a}{\rightarrow} 1 \equiv : (R \,|\, Cnv_a R) \underset{a}{=} (I \upharpoonright_a D_a R) :. (Cnv_a R \,|\, R) \underset{a}{=} (I \upharpoonright \mathsf{G}_a R)$:

71·24 $\vdash .R\,\varepsilon\,1 \underset{a}{\rightarrow} Cls\,.\,S\,\varepsilon\,1 \underset{a}{\rightarrow} Cls : (\mathsf{G}_a R \cap_a \mathsf{G}_a S) \underset{a}{=} \wedge_{(a)} : \supset$
$(R \cup_a S)\,\varepsilon\,1 \underset{a}{\rightarrow} Cls$

71·25 $\vdash .R\,\varepsilon\,1 \underset{a}{\rightarrow} Cls\,.\,S\,\varepsilon\,1 \underset{a}{\rightarrow} Cls\,.\supset (R \,|\, S)\,\varepsilon\,1 \underset{a}{\rightarrow} Cls$

71·252 $\vdash .R\,\varepsilon\,1 \underset{a}{\rightarrow} 1\,.\,S\,\varepsilon\,1 \underset{a}{\rightarrow} 1\,.\supset (R \,|\, S)\,\varepsilon\,1 \underset{a}{\rightarrow} 1$

71·26 $\vdash R\,\varepsilon\,1 \underset{a}{\rightarrow} Cls \supset (R \upharpoonright_a \varkappa)\,\varepsilon\,1 \underset{a}{\rightarrow} Cls$

71·36 $\vdash R\,\varepsilon\,1 \underset{a}{\rightarrow} Cls \supset : u = R^{\mathfrak{c}}{}_{(a,a)}\,v\,. \equiv u[R]_a v.$

72·184 $\vdash .(u \underset{A}{\downarrow})\,\varepsilon\,1 \underset{A,a}{\rightarrow} 1\,.\,(\underset{A}{\downarrow} u)\,\varepsilon\,1 \underset{A,a}{\rightarrow} 1.$

72·503 $\vdash R\,\varepsilon\,1 \underset{A,a}{\rightarrow} 1 : \omega \underset{a}{\subset} \mathsf{G}^a_A R : \supset [Cnv^a_A R^{\mathfrak{cc}}]^A_a [R^{\mathfrak{cc}}]^a_A\,\omega \underset{a}{=} \omega$

D. Similarity of Classes.

We shall use the following classes and relations to determine the types:

73·002 $B \underset{df}{=} \hat{P}\hat{Q}[\hat{P},\hat{Q}\,\varepsilon_A\,D^a_A(a \underset{A}{\downarrow})]$

73·003 $C \underset{df}{=} \hat{u}\hat{v}[\hat{u},\hat{v}\,\varepsilon_a\,\mathsf{G}^a_A(a \underset{A}{\downarrow})]$

Our propositions, 12·31—12·51 and 12·1, imply the following propositions concerning the types:

73·004 $\vdash \mathfrak{C}\{C, ((Cnv^a_A(\underset{A}{a\downarrow}) \overset{A}{\underset{a,A}{|}} B) \overset{A}{\underset{a}{|}} (a\downarrow))\}$

73·005 $\vdash \mathfrak{C}\{B, (((\underset{A}{a\downarrow}) \overset{a}{\underset{A,a}{|}} C) \overset{a}{\underset{A}{|}} Cnv^a_A(\underset{A}{a\downarrow}))\}$

73·006 $\vdash \mathfrak{C}\{D_a C, [Cnv^a_A(\underset{A}{a\downarrow})^{cc}]^A_a D_A B\}$

73·007 $\vdash \mathfrak{C}\{D_A B, [(\underset{A}{a\downarrow})^{cc}]^A_a D_a C\}$

73·01 $\varkappa \leftarrow (R)_a \rightarrow \omega \underset{df}{=} . R\varepsilon 1 \underset{a}{\rightarrow} 1 : \varkappa \underset{a}{=} D_a R :. \omega \underset{a}{=} Q_a R :$

73·011 $\varkappa \leftarrow (R) \rightarrow \omega = \varkappa \leftarrow (R)_a \rightarrow \omega$

Note that, if $\varkappa$ or ω is not extensional, we have $\sim \varkappa \leftarrow (R)_a \rightarrow \omega$.

The proposition „$\varkappa\, sm\, \omega$“ is defined as follows:

73·02 $\varkappa\, sm\, \omega \underset{df}{=} (\exists \overline{R}) . \varkappa \leftarrow (\overline{R}) \rightarrow \omega . \mathfrak{C}\{\overline{R}, C\}$.

For classes of relations of the type A, we have the definition:

73·03 $\vartheta\, sm'\, \pi \underset{df}{=} (\exists \overline{\overline{R}}) . \vartheta \leftarrow (\overline{\overline{R}})_A \rightarrow \pi . \mathfrak{C}\{\overline{\overline{R}}, B\}$.

This definition is to be used only in a small number of propositions. The full theory of similarity will be given for $\varkappa\, sm\, \omega$. — By the method of Principia we get the following propositions:

73·11 $\vdash \mathfrak{C}\{\varkappa, D_a C\} \supset . \varkappa sm \omega \equiv (\exists \overline{R}) . \varepsilon 1 \underset{a}{\rightarrow} 1 : \varkappa \underset{a}{\subset} D_a \overline{R} :. \omega = [Cnv R^{cc}]_a \varkappa :$
extens $[D_a C_{(a)}]\{\varkappa\} . \mathfrak{C}\{\overline{R}, C\}:$

73·12 $\vdash \mathfrak{C}\{\omega, D_a C\} \supset . \varkappa\, sm\, \omega \equiv (\exists \overline{R}) . \overline{R} \varepsilon 1 \underset{a}{\rightarrow} 1 : \omega \subset Q_a \overline{R} : \varkappa = [\overline{R}^{cc}]_a \omega :$
extens $[D_a C_{(a)}]\{\omega\} . \mathfrak{C}\{\overline{R}, C\}:$

73·2 $\vdash . R \varepsilon 1 \underset{a}{\rightarrow} 1 . \mathfrak{C}\{R, C\} . \supset . D_a R\, sm\, Q_a R . Q_a R\, sm\, D_a R$.

73·21 $\vdash . R\varepsilon 1 \underset{a}{\rightarrow} 1 : \omega \underset{a}{\subset} D_a R : \text{extens}(\omega) . \supset \omega \leftarrow ((\omega \overset{a}{\underset{a}{\urcorner}} R)) \rightarrow [Cnv R^{cc}]_a \omega$

73·22 $\vdash . R \varepsilon 1 \underset{a}{\rightarrow} 1 : \omega \underset{a}{\subset} Q_a R : \text{extens}(\omega) . \supset [R^{cc}]_a \omega \leftarrow ((R \overset{a}{\underset{a}{\upharpoonright}} \omega)) \rightarrow \omega$

73·3 $\vdash$ extens $[(D_a C)_{(a)}]\{\varkappa\} \supset . \varkappa\, sm\, \varkappa . \varkappa \leftarrow ((I \overset{a}{\underset{a}{\upharpoonright}} \varkappa)) \rightarrow \varkappa$.

73·32 $\vdash . \mathfrak{C}\{\varkappa, \omega, \varkappa', D_a C\} . \varkappa\, sm\, \omega . \omega\, sm\, \varkappa' . \supset \varkappa\, sm\, \varkappa'$

73·37 $\vdash . \mathfrak{C}\{\varkappa, \omega, \varkappa', D_a C\} . \varkappa\, sm\, \varkappa' . \supset . \omega\, sm\, \varkappa \equiv \omega\, sm\, \varkappa'$.

73·611 $[(\underset{A}{\downarrow u})^{cc}]^a_A \varkappa\, sm'\, [(\underset{A}{a\downarrow})^{cc}]^a_A \varkappa$

73·69 $\vdash . \varkappa \leftarrow (R) \rightarrow \omega . \text{extens}(\varkappa') : (\varkappa \cap_a \varkappa') \underset{a}{=} \wedge_{(a)} :. (\omega \cap_a \varkappa') \underset{a}{=} \wedge_{(a)} :$
$\supset (\varkappa \cup_a \varkappa') \leftarrow ((R \cup_a (I \overset{a}{\underset{a}{\upharpoonright}} \varkappa'))) \rightarrow (\omega \cup_a \varkappa')$

73·7 $\vdash .$ exstens $[D_a C_{(a)}]\{\varkappa'\} . \varkappa sm\, \omega : (\varkappa \cap_a \varkappa') = \wedge_{(a)} :. (\omega \cap_a \varkappa') \underset{a}{=} \wedge_{(a)} :$
$\supset (\varkappa \cup_a \varkappa')\, sm\, (\omega \cup_a \varkappa')$

73·71 $\vdash . \varkappa \, sm \, \omega . \varkappa' \, sm \, \omega' : (\varkappa \cap_a \varkappa') \underset{a}{=} \wedge_{(a)} :. (\omega \cap_a \omega') \underset{a}{=} \wedge_{(a)} : \supset$
$(\varkappa \cup_a \varkappa') \, sm \, (\omega \cup_a \omega')$

73·72 $\vdash .$ extens $(\varkappa, \omega) . (\varkappa \cup \iota^{\epsilon}{}_a u) \, sm \, (\omega \cup \iota^{\epsilon}{}_a v) : (\varkappa \cap \iota^{\epsilon}{}_a u) \underset{a}{=} \wedge_{(a)} :$
$. (\omega \cap \iota^{\epsilon}{}_a v) \underset{a}{=} \wedge_{(a)} : \supset \varkappa \, sm \, \omega$

73·73 $\vdash .$ extens $(\varkappa, \omega) . [a \underset{A}{\downarrow})^{\epsilon\epsilon}]^a_A \varkappa \, sm' \, \vartheta . [(a \underset{A}{\downarrow})^{\epsilon\epsilon}]^a_A \omega \, sm' \, \vartheta . \supset \varkappa \, sm \, \omega .$

Dem. $\vdash$ 72·184 . 71·252 . 73·004 . $\supset$

$\vdash \mathrm{Hp} \supset :. [(a \underset{A}{\downarrow})^{\epsilon\epsilon}]^a_A \varkappa \leftarrow (R)_A \rightarrow \vartheta . [(a \underset{A}{\downarrow})^{\epsilon\epsilon}]^a_A \omega \leftarrow (S)_A \rightarrow \vartheta . \mathfrak{T}\{R, S, B\} . \supset$

$\varkappa \leftarrow (((Cnv^a_A (a \underset{A}{\downarrow}) \overset{A}{\underset{a,A}{|}} R) | (S \overset{A}{\underset{A,a}{|}} (a \underset{A}{\downarrow})))) \rightarrow \omega . \mathfrak{T}\{((Cnv^a_A (a \underset{A}{\downarrow}) \overset{A}{\underset{a,A}{|}} R) | (S \overset{A}{\underset{,a}{|}} (a \underset{A}{\downarrow}))), C\} .$

[10·28] $\supset \vdash$ Prop

73·31 $\vdash . \omega \, sm \, \varkappa . [(a \underset{A}{\downarrow})^{\epsilon\epsilon}]^a_A \varkappa \, sm' \, \vartheta . \supset [(a \underset{A}{\downarrow})^{\epsilon\epsilon}]^a_A \omega \, sm' \, \vartheta$

Dem. $\vdash$. 72·184 . 71·252 . $\supset$

$\vdash . \mathfrak{T}\{R, C\} . \mathfrak{T}\{S . B\} . \omega \leftarrow (R)_A \rightarrow \varkappa . [(a \underset{A}{\downarrow})^{\epsilon\epsilon}]^a_A \varkappa \leftarrow (S) \underset{A}{\rightarrow} \vartheta .$

$\supset . [(a \underset{A}{\downarrow})^{\epsilon\epsilon}]^a_A \omega \leftarrow ((((a \underset{A}{\downarrow}) \overset{a}{\underset{A\,a}{|}} R) \overset{a}{\underset{A}{|}} Cnv^a_A (a \underset{A}{\downarrow}))) \underset{A}{\rightarrow} [(a \underset{A}{\downarrow})^{\epsilon\epsilon}]^a_A \varkappa .$

$[(a \underset{A}{\downarrow})^{\epsilon\epsilon}]^a_A \varkappa \leftarrow (S)_A \rightarrow \vartheta .$

[71·252] $\supset [(a \underset{A}{\downarrow})^{\epsilon\epsilon}]^a_A \omega \leftarrow ((((a \underset{A}{\downarrow}) \overset{a}{\underset{A,a}{|}} R) \overset{a}{\underset{A}{|}} (Cnv^a_A (a \underset{A}{\downarrow}) \overset{A}{\underset{a,A}{|}} S)))_A \rightarrow \vartheta$

[10·28] $\supset \vdash$ Prop.

31·01 $Cnv \underset{df}{=} \hat{Q} \hat{P} [: \hat{Q} \underset{a}{=} Cnv_A \hat{P} . \mathfrak{T}\{\hat{Q}, \hat{P}, A\} .]$

72·11 $\vdash Cnv \, \varepsilon \, 1 \underset{A}{\rightarrow} 1$

73·4 $\vdash . [Cnv^{\epsilon\epsilon}]_A \vartheta \, sm' \, \vartheta . [Cnv^{\epsilon\epsilon}]_A \vartheta \leftarrow ((Cnv \overset{A}{\underset{A}{\lceil}} \vartheta))_A \rightarrow \vartheta .$ [72·11]

73·71 $\vdash . \vartheta \, sm' \, \pi . \vartheta' \, sm' \, \pi : (\vartheta \cap_A \vartheta') \underset{A}{=} \wedge_{(A)} :. (\pi \cap_A \pi') \underset{A}{=} \wedge_{(A)} . \supset$
$(\vartheta \cup_A \vartheta') \, sm' \, (\pi \cup_A \pi')$

I now pass to the Schröder-Bernstein theorem and its lemmas, assuming the following definitions:

73·79 $l(\omega) \underset{df}{=} : \omega \underset{a}{\subset} D_a R : (\omega'' - G_a R) \underset{a}{\subset} \omega :. [Cnv_a R^{\epsilon\epsilon}]_a \omega \underset{a}{\subset} \omega :$
extens (ω'').

73·791 $l_0 \underset{df}{=} \hat{\varkappa} [. l(\hat{\varkappa}) .$ extens $[K_{(a)}]\{\varkappa\} .]$

Note that $l(\omega)$ is ambiguous as to the order of ω, therefore $l(\mathfrak{p}^{\epsilon} l_0)$ is a proposition, the expression $l_0\{\mathfrak{p}^{\epsilon} l_0\}$ being meaningless. We now have the following proposition to be got by the method of Principia:

73·81 $\vdash . R \,\varepsilon\, 1 \underset{a}{\to} Cls : \mathfrak{C}_a R \underset{a}{\subset} \omega'' :. \omega'' \underset{a}{\subset} D_a R : \mathrm{extens}(\omega'') .$

$(\exists \bar{\varkappa}) : \bar{\varkappa} \underset{a}{=} D_a R : \mathfrak{C}\{\bar{\varkappa}, K\} . \supset l(\mathfrak{p}\text{‘}l_0)$

The hypothesis $R \,\varepsilon\, 1 \to Cls$ is here irrelevant, but it is necessary in the following lemmas. It is no serious limitation of our theory, as we have to apply our lemmas to one-one relations. Following Principia, I write „Hp 73·81“ for „the hypothesis of 73·81“.

We have:

73·811 $\vdash$ Hp 73·81 $\supset . [Cnv_a R^{\text{‘‘}}]_a \mathfrak{p}\text{‘}l_0 \underset{a}{\subset} (\mathfrak{p}\text{‘}l_0 \underset{a}{-} (\omega'' \underset{a}{-} \mathfrak{C}_a R)) .$

73·812 $\vdash$. Hp 73·81 . $\sim u \,\varepsilon_a ((\omega'' \underset{a}{-} \mathfrak{C}_a R) \cup_a [Cnv_a R^{\text{‘‘}}]_a \mathfrak{p}\text{‘}l_0) . \supset$

$(\exists \bar{\varkappa}) . [Cnv_a R^{\text{‘‘}}]_a (\bar{\varkappa} \underset{a}{-} \iota\text{‘}_a u) \underset{a}{\subset} (\bar{\varkappa} \underset{a}{-} \iota\text{‘}_a u) : \bar{\varkappa} \,\varepsilon_K\, l_0 .$

Dem. $\vdash$ 22.81 $\supset \vdash$ Hp $\supset \sim u \,\varepsilon_a [Cnv_a R^{\text{‘‘}}]_a \mathfrak{p}\text{‘}l_0$

[. Hp . 71·36 .] $\supset \sim (R\text{‘}_{(a,a)} u) \,\varepsilon_a\, \mathfrak{p}\text{‘}l_0$

[(40·01)] $\supset (\exists \bar{\varkappa}) . \bar{\varkappa} \,\varepsilon_K\, l_0 : [Cnv R^{\text{‘‘}}]_a \bar{\varkappa} \underset{a}{\subset} \underset{a}{-} \iota\text{‘}_a u : (1)$

$\vdash$. (1) . (73·791) . $\supset \vdash$ Hp $\supset (\exists \bar{\varkappa}) : [Cnv R^{\text{‘‘}}]_a \bar{\varkappa} \underset{a}{\subset} (\bar{\varkappa} \underset{a}{-} \iota\text{‘}_a u) : \bar{\varkappa} \,\varepsilon_K\, l_0 :$

[37·2] $\supset \vdash$ Prop

73·82 $\vdash$ Hp 73·812 $\supset : (\mathfrak{p}\text{‘}l_0 \underset{a}{-} \iota\text{‘}_a u) \underset{a}{=} \mathfrak{p}\text{‘}l_0 : \sim u \,\varepsilon_a\, \mathfrak{p}\text{‘}l_0 .$

Dem. $\vdash$ 22·87 . 51·36 . $\supset \vdash$ Hp $\supset . (\omega'' \underset{a}{-} \mathfrak{C}_a R) \underset{a}{\subset} \underset{a}{-} \iota\text{‘}_a u$. (1)

[73·81] $\supset \vdash$. Hp . $\varkappa \,\varepsilon_K\, l_0 . \supset . (\omega'' \underset{a}{-} \mathfrak{C}_a R) \underset{a}{\subset} (\varkappa \underset{a}{-} \iota\text{‘}_a u) .$

[73·812 . (73·79)] $\supset \vdash$ Hp $\supset (\exists \bar{\varkappa}) . (\bar{\varkappa} \underset{a}{-} \iota\text{‘}_a u) \,\varepsilon_K\, l_0 . \mathfrak{C}\{\varkappa, K\} .$

[40·12] $\supset . p\text{‘}l_0 \underset{a}{\subset} (p\text{‘}l_0 \underset{a}{-} \iota\text{‘}_a u) .$

[51·36 . 22·43] $\supset \vdash$ Prop.

73·821 $\vdash$. Hp 73·81 . $u \,\varepsilon_a (\mathfrak{p}\text{‘}l_0 \underset{a}{-} (\omega'' \underset{a}{-} \mathfrak{C}_a R)) . \supset$

$u \,\varepsilon_a [Cnv_a R^{\text{‘‘}}]_a \mathfrak{p}\text{‘}l_0$ [73·82]

73·83 $\vdash$ Hp 73·81 $\supset . (\mathfrak{p}\text{‘}l_0 \underset{a}{-} (\omega'' \underset{a}{-} \mathfrak{C}_a R)) \underset{a}{=} [Cnv_a R^{\text{‘‘}}]_a \mathfrak{p}\text{‘}l_0 :$

$. \mathfrak{p}\text{‘}l_0 \underset{a}{=} ((\omega'' \underset{a}{-} \mathfrak{C}_a R) \cup_a [Cnv_a R^{\text{‘‘}}]_a \mathfrak{p}\text{‘}l_0) :$

[73·821 . 811·81 .]

73·84 $\vdash$ Hp 73·81 $\supset . \omega'' \underset{a}{=} (\mathfrak{p}\text{‘}l_0 \cup_a (\mathfrak{C}_a R \underset{a}{-} [Cnv_a R^{\text{‘‘}}]_a \mathfrak{p}\text{‘}l_0))$ [73·83]

73·8401 $\vdash \mathfrak{C}\{R, C\} . \mathfrak{C}\{\omega'', DC\} . \supset$

$\mathfrak{C}\{((\mathfrak{p}\text{‘}l_0 \underset{a}{\upharpoonleft} R) \cup_a (I \underset{a}{\upharpoonright} (D_a R \underset{a}{-} [Cnv_a R^{\text{‘‘}}]_a \mathfrak{p}\text{‘}l_0))), C\}$

[(0·252) . 12·3121 · 3122 · 4 · 5]

73·841 $\vdash . \mathrm{Hp}\ 73{\cdot}81 . \mathfrak{T}\{\omega'', DC\} . \mathfrak{T}\{R, C\} . R \varepsilon 1 \to 1$

$\supset . \omega''\ sm\ D_a R . \omega''\ sm_a\ \text{Ɑ}_a R .$

$[73 \cdot 8 \cdot 21 \cdot 83 \cdot 84]$

73·86 $\vdash : \text{Ɑ}_a R \underset{a}{\subset} D_a S :. \text{Ɑ}_a S \underset{a}{\subset} D_a R . \mathrm{extens}(R) . \mathrm{extens}(S) . \supset$

$: \text{Ɑ}_a S \underset{a}{\subset} D_a(R/S) :. D_a(R/S) \underset{a}{=} D_a R :. \text{Ɑ}_a(R/S) \underset{a}{\subset} \text{Ɑ}_a S :$

73·87 $\vdash . R \varepsilon 1 \underset{a}{\to} 1 . S \varepsilon 1 \underset{a}{\to} 1 : \text{Ɑ}_a R \underset{a}{\subset} D_a S :. \text{Ɑ}_a S \underset{a}{\subset} D_a R : \mathfrak{T}\{R, S, C\}.$

$\supset . (\exists \bar{\varkappa}) : D_a R \underset{a}{=} \bar{\varkappa} : \mathfrak{T}\{\bar{\varkappa}, K\} . \supset D_a R\ sm\ D_a S$

$[.3 \cdot 86 \cdot 841 \cdot 2 . (0{\cdot}252)]$

73·88 $\vdash . \varkappa\ sm\ \varkappa' . \omega\ sm\ \omega' : \varkappa' \underset{a}{\subset} \omega :. \omega' \underset{a}{\subset} \varkappa : \mathfrak{T}\{\varkappa, K\} . \mathfrak{T}\{\varkappa', \omega, \omega', D_a C\}.$

$. \supset \varkappa\ sm\ \omega$

This is the Schröder-Bernstein theorem. Note that this theorem is true, when $\varkappa$, $\varkappa'$, ω, ω' are of the type K, but it is not proved for $\varkappa$, $\varkappa'$ ω, ω' being of the type $D_a C$.

VII. Cardinal numbers.

The Theory of Cardinal numbers, as given în Principia, is based on certain conventions enabling us to deal with numbers of ambiguous types. These conventions are far from being general directions of meaning, as they concern arithmetical operations. These conventions being required in proofs of propositions, can hardly be omitted, therefore it may be doubted whether we can build up Arithmetic without supplementary directions. Now the use of ascending cardinals seems to be scarcely possible without these. Moreover it would be quite useless in our system, as we can prove nothing concerning cardinal number of the Universum of a given type. There is this essential difference between the Pure Theory of Types and a simplified one, that the simplified Theory enables us to prove that the cardinal number of the classes of classes contained in a given class is greater than the cardinal number of this class [1]). With this theorem, we can prove that if σ is a cardinal number other than the null-class, there is a cardinal

[1]) Cf. Principia * 102·1.

number $(\sigma + 1)$ other than the null-class. Therefore in the simplified theory it is useful to deal with ascending cardinals; but, as I think, we ought at any rate to set aside the special conventions of Principia. In the Pure Theory of Types we have to do simply with homogeneous cardinals. This limitation enables us to get the theory of cardinals without any supplementary convention.

A. Homogeneous Cardinals.

Let us assume the following abbreviations:

101·0001 $\exists' \underset{df}{=} \exists_{DC}$

101·0002 $\varepsilon' \underset{df}{=} \varepsilon_{DC}$

103·001 $Red_1(\omega) \underset{df}{=} (\overline{\varkappa'}) :. \overline{\varkappa'} \underset{a}{\subset} \omega : \text{extens}\,[D_a C_{(a)}]\,\{\overline{\varkappa'}\}.$
$\supset (\exists \overline{\varkappa}) : \overline{\varkappa} \underset{a}{=} \overline{\varkappa'} : \mathfrak{T}\{\overline{\varkappa'}, K\} :. \text{extens}\,[D_a C_{(a)}]\,\{\omega\}.$

103·002 $Red_2(\omega) \underset{df}{=} (\overline{\varkappa'})\,(\overline{P}) :: \overline{\varkappa'} \underset{a}{\subset} \omega : \text{extens}\,[D_a C_{(a)}]\,\{\overline{\varkappa'}\}.\ \overline{P}\varepsilon 1 \rightarrow Cls :$
$\mathfrak{C}_a \overline{P} \underset{a}{=} \overline{\varkappa'} : \mathfrak{T}\{\overline{P}, C\}. \supset (\exists \overline{Q})\ \overline{Q} \underset{a}{=} \overline{P} : \mathfrak{T}\{\overline{Q}, A\} :. \text{extens}\,[D_a C_{(a)}]\,\{\omega\}.$

103·003 $Red(\omega) \underset{df}{=} .\ Red_1(\omega)\,.\ Red_2(\omega)\,.\ (\exists \overline{S})\ \mathfrak{T}\{\overline{S}, B\}.$

103·004 $Red(\varkappa, \omega) \underset{df}{=} .\ Red(\varkappa)\,.\ Red(\omega)\,.$

We see that $Red(\omega)$ is the hypothesis of reducibility of subclasses of ω, as much as of one-many relations, whose converse domain is a sub-class of ω. We shall deal with reducible classes, i. e. classes which satisfy $Red(\omega)$, using a method analogous to that which we have applied to the problem of extension. As we cannot prove that there are in the type $D_a C$ classes which are not identical with $\iota^{\epsilon}{}_a a$ or $\iota^{\epsilon}{}_a - a$, the existence of cardinals other than 0,1 and 2 is not assured by any means in this type. As we can prove that the null-class, as well as the classes containing elements identical with a unit element, or with one of two given elements, are reducible classes our dealing with reducible classes is no serious limitation of the theory of homogeneous cardinals.

I assume the following definition of the relation of similarity between reducible classes:

103·004 $sm\,r \underset{df}{=} \hat{\varkappa}\,\hat{\omega}\,[.\ Red(\hat{\varkappa}, \hat{\omega})\,.\ \varkappa\ sm\ \omega.]$

This relation enables us to have the following definition of a homogeneous cardinal number of the class $\varkappa$ having the type $D_a C$. —

103·01 $\quad Nc^c \varkappa \underset{df}{=} \hat{\omega}[\hat{\omega}\, smr\, \varkappa]$

We see that $Nc^c \varkappa$ is the class of all reducible classes of the type $D_a C$, which are similar to $\varkappa$.

We shall use the abbreviation:

101·001 $\quad \Omega \underset{df}{=} D_a C$

101·002 $\quad \wedge' \underset{df}{=} (\wedge_{(a)} \cap_a \Omega)$

The numbers 0, 1, 2 are defined as follows:

101·01 $\quad 0 \underset{df}{=} Nc^c \wedge'$

101·02 $\quad 1 \underset{df}{=} Nc^c (\iota^c_a a \cup_a \wedge')$

101·03 $\quad 2 \underset{df}{=} Nc^c ((\iota^c_a a \cup_a \iota^c_a - a) \cup_a \wedge')$

101·003 $\quad \wedge'' \underset{df}{=} (\wedge_{(\Omega)} \cap_\Omega 1)$

The class of cardinals other than $\wedge''$ being denoted by NC, we have the following definition:

103·02 $\quad NC \underset{df}{=} \hat{\sigma}[(\exists \bar{\varkappa}) : \hat{\sigma} \underset{\Omega}{=} Nc^c \bar{\varkappa} : \exists ! \hat{\sigma} . \mathfrak{C}\{\hat{\sigma}, Nc^c \Omega\}.]$

We now have the following propositions concerning the use of $Red(\omega)$.

101·1001 $\vdash : \varkappa \underset{a}{\subset} \omega : \text{extens}\,[\Omega_{(a)}]\{\varkappa\}. Red(\omega). \supset Red(\varkappa)$

Dem $\vdash$ 22·44 $\supset \vdash . Hp : \omega' \underset{a}{\subset} \varkappa : \supset \omega' \underset{a}{\subset} \omega : Red(\omega).$

$\quad [10·33;1] \supset \vdash$ Prop.

101·1002 $\vdash Red(\varkappa, \omega) \supset Red((\varkappa \cup_a \omega))$

Dem. $\vdash$ 22·43·68·621 $\underset{a}{\supset} \vdash : \varkappa' \underset{a}{\subset} (\varkappa \cup_a \omega) : \text{extens}\,[\Omega_{(a)}]\{\varkappa'\}. \supset$

$\quad : (\varkappa' \cap_a \varkappa) \underset{a}{\subset} \varkappa :. (\varkappa' \cap_a \omega) \underset{a}{\subset} \omega :. \varkappa' \underset{a}{=} ((\varkappa' \cap_a \varkappa) \cup_a (\varkappa' \cap_a \omega)) :$ (1)

$\vdash$ 35·431·65·412 $\supset \vdash : \varkappa' \underset{a}{\subset} (\varkappa \cup_a \omega) : \text{extens}\,[\Omega_{(a)}]\{\varkappa'\}. P \varepsilon 1 \underset{a}{\rightarrow} Cls :$

$\quad \mathrm{G}_a P \underset{a}{=} \varkappa' : \mathfrak{C}\{P, C\}.$

$\supset . (P \overset{a}{\underset{a}{\upharpoonright}} (\varkappa' \cap_a \varkappa)) \varepsilon 1 \underset{a}{\rightarrow} Cls . (P \overset{a}{\underset{a}{\upharpoonright}} (\varkappa' \cap_a \omega)) \varepsilon 1 \underset{a}{\rightarrow} Cls : \mathrm{G}_a (P \overset{a}{\underset{a}{\upharpoonright}} (\varkappa' \cap_a \varkappa)) \underset{a}{\subset} \varkappa :$

$\quad . \mathrm{G}_a (P \overset{a}{\underset{a}{\upharpoonright}} (\varkappa' \cap_a \omega)) \underset{a}{\subset} \omega :. P \underset{a}{=} ((P \overset{a}{\underset{a}{\upharpoonright}} (\varkappa' \cap_a \varkappa)) \cup_a (P \overset{a}{\underset{a}{\upharpoonright}} (\varkappa' \cap_a \omega))).$ (2)

$[(1)] \supset :. \varkappa'' \underset{a}{=} (\varkappa' \cap_a \varkappa) :. \omega'' \underset{a}{=} (\varkappa' \cap_a \omega) : \mathcal{T}\{\varkappa'', \omega'', K\} : Q \underset{a}{=} (P \overset{a}{\underset{a}{\upharpoonright}} (\varkappa' \cap_a \varkappa)) :$

$Q' \underset{a}{=} (P \overset{a}{\underset{a}{\upharpoonright}} (\varkappa' \cap_a \omega)) : \mathcal{T}\{Q, Q', A\}.$

$\supset : \varkappa' \underset{a}{=} (\varkappa'' \cup_a \omega'') :. P \underset{a}{=} (Q \cup_a Q') : \mathcal{T}\{(\varkappa'' \cup_a \omega''), K\} . \mathcal{T}\{(Q \cup_a Q'), A\} :$

$[10{\cdot}24] \qquad \supset (\exists \overline{\varkappa}''') (\exists \overline{Q}'') : \overline{\varkappa}' \underset{a}{=} \overline{\varkappa}''' :. P \underset{a}{=} \overline{Q}'' : \mathcal{T}\{\varkappa''', K\} . \mathcal{T}\{\overline{Q}'', A\}. \quad (3)$

$\vdash (1)(2)(3) . 10{\cdot}34 \supset \vdash$ Prop.

101·1003 $\vdash$ $\mathit{Red}(\wedge')$

101·1004 $\vdash$ $\mathit{Red}((\iota^{c} u \cup_a \wedge'))$

101·1005 $\vdash$ $\mathit{Red}(\varkappa) \supset \mathit{Red}((\varkappa \cup_a \iota^{c}_{a} u))$ [101·1002·1004]

101·1006 $\vdash$ $\mathit{Red}(((\iota^{c}_{a} u \cup_a \iota^{c}_{a} u) \cup_a \wedge'))$ [101·1004·1005]

With these lemmas we prove now without any difficulty the following propositions, by the method of Principia.

103·11 $\vdash$ $. \varkappa \varepsilon' Nc^{c} \omega \equiv \varkappa \, smr \, \omega :. \varkappa \varepsilon' Nc^{c} \omega \equiv \omega \varepsilon' Nc^{c} \varkappa .$

103·12 $\vdash$ $\exists' Nc^{c} \varkappa \supset \varkappa \varepsilon' Nc^{c} \varkappa$

103·13 $\vdash$ $\exists' Nc^{c} \varkappa \equiv \mathit{Red}(\varkappa)$

103·14 $\vdash$ $\exists' Nc^{c} \varkappa \supset : Nc^{c} \varkappa \underset{\Omega}{=} Nc^{c} \omega . \equiv \varkappa \, smr \, \omega .$

101·11 $\vdash$ $0 \varepsilon_1 NC$ [101·1003]

101·21 $\vdash$ $1 \varepsilon_1 NC$ [101·1004]

101·31 $\vdash$ $2 \varepsilon_1 NC$ [101·1005]

101·14 $\vdash$ $. Nc^{c} \varkappa \underset{\Omega}{=} 0 . \equiv . \varkappa \underset{a}{=} \wedge' .$ [(101·01) . 103·11]

Here ε_1 is no special sign; we simply use the class 1 to denote the types, according to the definition 20·042.

101·22 $\vdash$ $1 \underset{\Omega}{\neq} 0$

101·33 $\vdash$ $\varkappa, \omega \varepsilon' 1 : (\varkappa \cap_a \omega) \underset{a}{=} \wedge_{(a)} : \supset (\varkappa \cup_a \omega) \varepsilon' 2$

101·34 $\vdash$ $. 2 \underset{\Omega}{\neq} 0 :. 2 \underset{\Omega}{\neq} 1 .$

103·2 $\vdash$ $\sigma \varepsilon_1 NC \equiv (\exists \overline{\varkappa}) : \sigma \underset{\Omega}{=} Nc^{c} \overline{\varkappa} : \exists' \sigma .$

103·27 $\vdash$ $. \exists' \sigma : \sigma \underset{\Omega}{=} Nc^{c} \varkappa : \equiv . \sigma \varepsilon_1 NC . \varkappa \varepsilon' \sigma .$

B. Greater and less.

Our dealing with homogenous cardinals enables us to have the following simple definition of „greater or equal“.

117·05 $. \sigma \geqslant \tau . \underset{df}{=} . \sigma, \tau \varepsilon_1 NC . (\exists \overline{\varkappa}, \overline{\omega}) . \overline{\varkappa} \varepsilon' \sigma . \overline{\omega} \varepsilon' \tau : \overline{\omega} \underset{a}{\subset} \overline{\varkappa} :.$

We then have:

117·06 $\quad . \tau \leqslant \sigma . \underset{df}{=} . \sigma \geqslant \tau .$

117·01 $\quad . \sigma > \tau . \underset{df}{=} : \sigma \geqslant \tau :. \sigma \underset{\Omega}{\neq} \tau :$

117·04 $\quad . \tau < \sigma . \underset{df}{=} . \sigma > \tau .$

117·23 $\quad \vdash : \sigma \geqslant \tau :. \tau \geqslant \sigma . \supset . \tau \underset{\Omega}{=} \sigma .$ [103·14·26 73·88]

This important proposition is got by means of the Schröder-Bernstein theorem. Note that our definition of NC enables us to use this theorem, as we are dealing with no other classes but reducible and extensional ones.

117·1031 $\quad \vdash : \sigma \underset{\Omega}{=} \tau : \sigma, \tau \varepsilon_1 NC . \supset . \sigma \geqslant \tau .$ [(117·05) . 22·42]

117·104 $\quad \vdash . \sigma \geqslant \tau . \equiv :. \sigma > \tau . \vee :. \sigma \underset{\Omega}{=} \tau : \sigma . \tau \varepsilon_1 NC :.$ [(117·01) . 5·75.]

117·281 $\quad \vdash . \sigma > \tau . \equiv : \sigma \geqslant \tau : \sim . \tau \geqslant \sigma :$ [117·1031 23]

117·4 $\quad \vdash : \sigma \geqslant \tau' :. \tau' \geqslant \tau : \supset . \sigma \geqslant \tau .$ [73·23]

117·42 $\quad \vdash \sim . \mu > \mu .$

117·45 $\quad \vdash : \sigma \geqslant \tau' :. \tau' > \tau : \supset . \sigma > \tau .$

Dem $\vdash$ 117·4 $\supset \vdash$ Hp $\supset . \sigma \geqslant \tau .$

[117·104] $\quad \supset :. \sigma > \tau . \vee :. \sigma \underset{\Omega}{=} \tau : \sigma, \tau \varepsilon_1 NC :.$ (1)

$\left[\text{Hp} \frac{\sigma}{\tau}\right] \vdash \supset . \text{Hp} : \sigma \underset{\Omega}{=} \tau : \sigma, \tau \varepsilon_1 NC . \supset : \sigma \geqslant \tau' : \tau' \geqslant \sigma :. \sigma \underset{\Omega}{\neq} \tau' :$

[117.23] $\quad \supset : \sigma \underset{\Omega}{=} \tau' :. \sigma \underset{\Omega}{\neq} \tau' :$

[Transp. (1) .] $\supset \vdash$ Prop.

117·46 $\quad \vdash : \sigma > \tau' :. \tau' \geqslant \tau : \supset . \sigma > \tau .$ [Proof as in 117·45]

117·47 $\quad \vdash : \sigma > \tau' :. \tau' > \tau : \supset . \sigma > \tau .$ [117·45 (117·01)]

117·5 $\quad \vdash \sigma \varepsilon_1 NC \supset . \sigma \geqslant 0 .$

117·501 $\quad \vdash \sigma \varepsilon_1 NC \equiv . \sigma \geqslant 0 .$

117·511 $\quad \vdash \sigma \varepsilon_1 (NC \underset{1}{-} \iota'_1 0) \equiv . \sigma > 0 .$ [117·501 . (127·01)]

117·531 $\quad \vdash \sigma \varepsilon_1 (NC \underset{1}{-} \iota'_1 0) \equiv . \sigma \geqslant 1 .$ [117·53·501·23 . 101·22.]

117·54 $\quad \vdash . 1 \geqslant \sigma . \equiv :. \sigma \underset{\Omega}{=} 0 . \vee \sigma \underset{\Omega}{=} 1 :$ [117 531·501·104 . Transp.]

117·551 $\quad \vdash \sigma \varepsilon_1 ((NC \underset{1}{-} \iota'_1 0) \underset{1}{-} \iota'_1 1)) \equiv . \sigma \geqslant 2 .$

[(101·03) . (117·05) . 101·34.]

C. Addition.

The sum of two homogeneous cardinals is to be defined as follows:

110·02 $\quad (\sigma+\tau) \underset{df}{=} \hat{\varkappa}'[.\exists'\sigma.\exists'\tau.(\exists\bar{\varkappa},\bar{\omega}):\sigma=Nc'\bar{\varkappa}:.\tau=Nc'\bar{\omega}.(\bar{\varkappa}\cap_a\bar{\omega}) \underset{a}{=}$

$\Lambda_{(a)}: \bar{\varkappa}'\, smr\,(\bar{\varkappa}\cup_a\bar{\omega}).]$.

The dealing with homogeneous cardinals makes the more general definition, as given in Principia, irrelevant to our purposes. Now we have the following propositions:

110·1001 $\quad \vdash\ \mathfrak{C}\{\sigma,\tau,1\} \supset \mathfrak{C}\{(\sigma+\tau),1\}$

[12·2421·42·5]

110·14 $\quad \vdash : (\varkappa\cap_a\omega) \underset{a}{=} \Lambda_{(a)}: \varkappa\,\varepsilon'\sigma.\,\omega\,\varepsilon'\tau.\,\sigma.\tau\,\varepsilon_1\,NC. \supset (\varkappa\cup_a\omega)\,\varepsilon'(\sigma+\tau)$

[110·03.101·1002]

110·4 $\quad \vdash\ \exists'(\sigma+\tau) \supset \sigma,\tau\,\varepsilon_1\,NC$

110·42 $\quad \vdash\ (\sigma+\tau)\,\varepsilon_1\,(NC\cup_1\iota'_1\Lambda'')$

110·51 $\quad \vdash\ (\sigma+\tau) \underset{\Omega}{=} (\tau+\sigma)$

110·552 $\quad \vdash : (\varkappa\cap_a\omega) \underset{a}{=} \Lambda_{(a)}: \varkappa''smr(\varkappa\cup_a\omega):\varkappa,\omega\,\varepsilon'\,\text{extens}[\Omega_{(a)}].$

$\supset(\exists\bar{\varkappa}',\bar{\omega}').\,\bar{\varkappa}'smr\varkappa.\,\omega'smr\,\omega:\varkappa'' \underset{a}{=} (\bar{\varkappa}'\cup_a\bar{\omega}'):.(\bar{\varkappa}'\cap_a\bar{\omega}') \underset{a}{=} \Lambda_{(a)}:$

Dem $\vdash$ 101·1001.73·22. $\supset$

$\vdash : (\varkappa\cap_a\omega) \underset{a}{=} \Lambda_{(a)}: Red(\varkappa'',(\varkappa\cup_a\omega)).\,\varkappa'',(\varkappa\cup_a\omega)\varepsilon'\,\text{extens}[\Omega_{(a)}].\,\text{Hp}. \supset$

$: \mathfrak{C}\{S,C\}.\,\varkappa''\leftarrow(S)_a\rightarrow(\varkappa\cap_a\omega). \supset .[S^{\epsilon\epsilon}]_a\varkappa\, smr\,\varkappa.\,[S^{\epsilon\epsilon}]_a\omega\, smr\,\omega:\varkappa'' \underset{a}{=}$

$([S^{\epsilon\epsilon}]_a\varkappa\cup_a[S^{\epsilon\epsilon}]_a\omega):([S^{\epsilon\epsilon}]_a\varkappa\cap_a[S^{\epsilon\epsilon}]_a\omega) \underset{a}{=} \Lambda_{(a)}:.$

[10·34] $\supset \vdash$ Prop.

110·56 $\quad \vdash\ ((\sigma+\tau)+\sigma') \underset{\Omega}{=} (\sigma+(\tau+\sigma'))$ [110·552]

110·6 $\quad \vdash\ \sigma\varepsilon_1(NC\cup_1\iota'_1\Lambda'') \supset .(\sigma+0) \underset{\Omega}{=} \sigma.$

110·62 $\quad \vdash .(\sigma+\tau) \underset{\Omega}{=} 0 . \equiv : \sigma \underset{\Omega}{=} 0 :. \tau \underset{\Omega}{=} 0:$ [103·26]

110·631 $\quad \vdash\ \sigma\varepsilon_1(NC\cup_1\iota'_1\Lambda'') \supset .(\sigma+\tau) \underset{\Omega}{=} \hat{\varkappa}[.\,\sigma\varepsilon_1 NC.(\exists\bar{\omega})(\exists\bar{u}).$

$\bar{\omega}\,\varepsilon'\sigma.\sim\bar{u}\,\varepsilon_a\,\bar{\omega}.\,\hat{\varkappa}\, smr\,(\bar{\omega}\cup_a\iota'\bar{u})].$

110·632 $\quad \vdash\ \sigma\varepsilon_1(NC\cup_1\iota'_1\Lambda'' \supset$

$.(\sigma+1) \underset{\Omega}{=} \hat{\varkappa}[.\,\sigma\varepsilon_1 NC.(\exists\bar{u}).\,\bar{u}\,\varepsilon_a\hat{\varkappa}.(\hat{\varkappa} \underset{a}{-} \iota'\bar{u})\varepsilon'\sigma.\,\text{extens}[\Omega_{(a)}]\{\hat{\varkappa}\}.]$

110·64 $\quad \vdash\ (0+0) \underset{\Omega}{=} 0$ [110·62]

110·641 $\quad \vdash\ (1+0) \underset{\Omega}{=} 1$ [110·6]

110·643 $\quad \vdash\ \underset{\Omega}{=} (1+1)$ [110·641]

117·31 $\quad \vdash .\ \sigma \geqslant \tau . \equiv . \sigma,\tau\,\varepsilon_1\,NC.(\exists\bar{\sigma}').\,\sigma \underset{\Omega}{=} (\tau+\bar{\sigma}'):$

117·561 $\quad \vdash : \sigma \geqslant \tau : \exists'(\sigma+\tau'). \supset .(\sigma+\tau') \geqslant (\tau+\tau').$

Dem $\vdash$ 101·1001 . 73·22 $\supset$

$\vdash . \mathrm{Hp}: \varkappa'' \underset{a}{=} (\varkappa' \cup_a \omega') : \varkappa', \varkappa \varepsilon' \sigma . \omega' \varepsilon' \tau' : (\varkappa' \cap_a \omega') \underset{a}{=} \wedge_{(a)} : \omega \varepsilon' \tau : \omega \underset{a}{\subset} \varkappa :$

$\varkappa' \leftarrow (R)_{(a)} \rightarrow \varkappa . \supset$

$: [R^{cc}]_a \omega \underset{a}{\subset} \varkappa' :. ([R^{cc}]_a \omega \cap_a \omega') \underset{a}{=} \wedge_{(a)} : [R^{cc}]_a \omega \, sm \, \omega .$ (1)

[110·14] $\supset . \varkappa'' \varepsilon' (\sigma + \tau) : ([R^{cc}]_a \omega \cup_a \omega') \underset{a}{\subset} \varkappa'' :$

[101·1001 . 10·34] $\supset \vdash$ Prop.

117·6 $\vdash (\sigma + \tau) \varepsilon_1 NC \supset . (\sigma + \tau) \geqslant \sigma .$ [(110·02) . 110·14.]

D. Multiplication.

I take from Principia the following definition of the arithmetical product of two classes:

113·01 $(\varkappa \times_A \omega) \underset{df}{=} \hat{P} [(\exists \bar{u} . \bar{v}) . \bar{u} \varepsilon_a \varkappa . \bar{v} \varepsilon_a \omega : \hat{P} \underset{a}{=} (\bar{u} \overset{a}{\downarrow} \bar{v}) : \mathfrak{C}\{\hat{P}, A\}.]$

The definition of the product of two cardinals is now:

113·02 $(\sigma \times \tau) \underset{df}{=} \hat{\varkappa}' [. \exists' \sigma . \exists' \tau . (\exists \bar{\varkappa}, \bar{\omega}) : \sigma \underset{\Omega}{=} Nc^{\varepsilon} \bar{\varkappa} :. \tau \underset{\Omega}{=} Nc^{\varepsilon} \bar{\omega} :$

$[a \underset{A}{\downarrow})^{cc}]^a_A \hat{\varkappa}' sm' (\bar{\varkappa} \times_a \bar{\omega}) . Red(\hat{\varkappa}') .]$

We have now the following propositions:

113·001 $\vdash \mathfrak{C}\{\sigma, \tau, 1\} \supset \mathfrak{C}(\sigma \times \tau, 1\}$ [12·2421·42·5.]

113·13 $\vdash . \varkappa \, smr \, \varkappa' \;\; \omega \, smr \, \omega' . \supset (\varkappa \times_A \omega) sm' (\varkappa' \times_A \omega')$

Dem $\vdash$ 72·184 . 71·252 . $\supset$

$\vdash . \varkappa \leftarrow (R)_a \rightarrow \varkappa' . \omega \leftarrow (S)_a \rightarrow \omega' : T \underset{A}{=} \hat{P} \hat{Q} [(\exists \bar{u}, \bar{w}) .$

$\hat{P} [(((\bar{u} \underset{A}{\downarrow}) \overset{a}{\underset{A,a}{|}} S) \overset{a}{\underset{A}{|}} Cnv^a_A (\bar{w} \underset{A}{\downarrow})]_A \hat{Q} . \bar{u} [R]_a \bar{w} .]$

$\supset . T \varepsilon 1 \underset{A}{\rightarrow} 1 : D_a T \underset{A}{=} (\varkappa \times_A \omega) :. Q_A T \underset{A}{=} (\varkappa' \times_A \omega') :$

[10·28] $\supset \vdash$ Prop.

103·203 $\vdash (\sigma \times \tau) \varepsilon_1 (\wedge C \cup_1 \iota^{\varepsilon}_1 \wedge'')$ [73·73·731]

113·27 $\vdash (\sigma \times \tau) = (\tau \times \sigma)$ [73·4]

117·571 $\vdash : \sigma \geqslant \tau : \exists' (\sigma \times \tau') . \supset . (\sigma \times \tau') \geqslant (\tau \times \tau') .$

Dem $\vdash$ 37·2 $\supset \vdash : \omega \underset{a}{\subset} \varkappa : \mathrm{extens}(\omega, \varkappa) . \supset . (\omega \times_A \omega') \underset{A}{\subset} (\varkappa \times_A \omega') .$ (1)

$\vdash$ 113·13 $\supset \vdash . \varkappa \varepsilon' \sigma . \omega \varepsilon' \tau : \omega \underset{a}{\subset} \varkappa : \varkappa' \varepsilon' \sigma . \omega' \varepsilon' \tau' . [(a \underset{A}{\downarrow})^{cc}]^a_A \varkappa'' sm' (\varkappa' \times_A \omega') .$

$Red(\varkappa'') . \supset [(a \underset{A}{\downarrow})^{cc}]^a_A \varkappa'' \, sm' (\varkappa \times_A \omega')$ (12)

[(1) . 73·22 . 37·2·51 . 72·502] $\supset [(a \underset{A}{\downarrow})^a_A \varkappa'' \leftarrow (R)_A \rightarrow (\varkappa \times_A \omega') \supset$

$: [R^{cc}]_A (\omega \times_A \omega') \subset [(a \underset{A}{\downarrow})^{cc}]^a_A \varkappa'' :. [Cnv^a_A (a \underset{A}{\downarrow})^{cc}]^A_a [R^{cc}]_A (\omega \times_A \omega') \subset \varkappa'' :$

$[Cnv^a_A (a \underset{A}{\downarrow})^{cc}]^A [R^{cc}]_A (\omega \times_A \omega') \varepsilon' (\tau \times \tau') .$

$\supset \vdash$ Prop.

113·621 $\vdash \sigma \varepsilon_1 (NC \cup_1 \iota\text{‘}_1 \wedge'') \supset . (\sigma \times 1) \underset{\Omega}{=} \sigma . [73·611]$

117·62 $\vdash (\sigma \times \tau) \varepsilon_1 NC . \sigma, \tau \varepsilon_1 \underset{1}{-} \iota\text{‘}_1 0 . \supset . (\sigma \times \tau) \geqslant \sigma .$

[117·571·53 . 113·621 .]

113·601 $\vdash \sigma \varepsilon_1 NC \supset . (\sigma \times 0) \underset{\Omega}{=} 0 .$

113·602 $\vdash . (\sigma \times \tau) = 0 . \equiv :. \sigma \underset{\Omega}{=} 0 . \vee . \tau \underset{\Omega}{=} 0 :.$

113·4 $\vdash ((\varkappa \cup_a \omega) \times_A \omega') \underset{A}{=} ((\varkappa \times_A \omega') \cup_A (\omega \times_A \omega'))$ [40·31]

113·43 $\vdash . \exists'(\sigma + \tau) \vee . \tau' \underset{\Omega}{\neq} 0 : \supset . ((\sigma + \tau) \times \tau') \underset{\Omega}{=} ((\sigma \times \tau') + (\tau \times \tau')) .$

Dem $\vdash$ 113·601 . 117·561·4 . 110·62 $\supset$

$\vdash : \tau' \underset{\Omega}{\neq} 0 : \exists'((\sigma \times \tau') + (\tau \times \tau')) . \supset \exists'(\sigma + \tau)$ (1)

$\vdash$ 101·1001 . 113·4 . 73·71 . $\supset \vdash \exists'(\sigma + \tau) \supset . ((\sigma + \tau) \times \tau') \underset{\Omega}{=}$

$((\sigma \times \tau') + (\tau \times \tau'))$. (2)

$\vdash$ (1) . (2) . $\supset \vdash$ Prop.

I shall use the following abbreviations:

113·501 $X(w) \underset{df}{=} (((\varkappa'' \overset{A}{\underset{a}{\uparrow}} Cnv^a_A(a \underset{A}{\downarrow}) \overset{A}{\underset{a.A}{|}} R) \overset{A}{\underset{a}{|}} ((w \underset{A}{\downarrow}) \overset{a}{\underset{A}{\upharpoonright}} \omega))$

113·502 $Y(t) \underset{df}{=} (((\omega \overset{A}{\underset{a}{\uparrow}} Cnv^a_A(t \underset{A}{\downarrow})) \overset{A}{\underset{a,A}{|}} Cnv_A S) \overset{A}{\underset{a}{|}} ((a \underset{A}{\downarrow}) \overset{a}{\underset{A}{\upharpoonright}} \omega''))$

113·503 $Z_X \underset{df}{=}$

$\hat{P}\hat{Q}[(\exists \bar{u}, \bar{u}')\hat{P}[(((\underset{A}{\downarrow} \bar{u}') \overset{a}{\underset{A}{\upharpoonright}} \varkappa'') \overset{a}{\underset{A,a}{|}} X(\bar{u})) \overset{a}{\underset{A}{|}} (Y(\bar{u}') \overset{a}{\underset{a,A}{|}} (\omega'' \overset{A}{\underset{a}{\uparrow}} Cnv^a_A(\underset{A}{\downarrow} \bar{u}))))]Q]$

113·51 $\vdash . [a \underset{A}{\downarrow})^{cc}]^a_A \varkappa'' \leftarrow (R)_A \rightarrow (\varkappa \times_A \omega) . [(a \underset{A}{\downarrow})^{cc}]^a_A \omega'' \leftarrow (S)_A \rightarrow (\varkappa' \times_A \omega)$

$. \mathfrak{C}\{R, S, B\} . \supset (\varkappa'' \times_A \varkappa') \, sm' (\omega'' \times^A \varkappa)$

Dem $\vdash$ 72·184 . 71·252 . $\supset . Z_X \varepsilon 1 \underset{A}{\rightarrow} 1 : D_A Z_X \underset{A}{=} (\varkappa'' \times_A \varkappa') :. \mathfrak{C}_A Z_X \underset{A}{=}$

$(\omega'' \times_A \varkappa) : \mathfrak{C}\{Z_X, B\} .$

[10·24] $\supset \vdash$ Prop.

113·54 $\vdash . \exists'(\sigma \times \tau) . \exists'(\sigma \times \tau') . \supset ((\sigma \times \tau) \times \tau') \underset{\Omega}{=} (\sigma \times (\tau \times \tau'))$.

[113·51 . 105·6 . 113·27]

113·66 $\vdash (\sigma \times 2) = (\sigma + \sigma)$ [110·643 . 113·43·621 . 101·32]

113·771 $\vdash . \exists'(\sigma + 1) \vee . \tau \neq 0 \supset . (\tau \times (\sigma + 1)) \underset{\Omega}{=} ((\tau \times \sigma) + \tau))$

[113·43·621]

E. Exponentiation.

According to the method of Cantor, I start from the following definition:

116·01 $(\varkappa \exp \omega) \underset{df}{=} \hat{P}[.\, \hat{P}\varepsilon 1 \underset{a}{\rightarrow} Cls : D_a\hat{P} \underset{a}{\subseteq} \varkappa :.\, ꓷ_a\hat{P} \underset{a}{=} \omega : \mathfrak{T}\{\hat{P}, A\}\,.$ extens $(\varkappa, \omega).]$

We see that $(\varkappa \exp \omega)$ is a class of relations of the type A, like $(\varkappa \times_A \omega)$. By means of $(\varkappa \exp \omega)$ we define σ^τ as follows.

116·02 $\sigma^\tau \underset{df}{=} \hat{\varkappa}'[.\exists'\sigma\,.\,\exists'\tau\,.\,(\exists \bar{\varkappa}, \bar{\omega}) : \sigma \underset{\Omega}{=} Nc^{\epsilon}\bar{\varkappa} :.\, \tau \underset{\Omega}{=} Nc^{\epsilon}\bar{\omega} :$

$[(\underset{A}{a\downarrow})^{\epsilon\epsilon}]^a_A \hat{\varkappa}'\, sm'\, (\bar{\varkappa} \exp \omega)\,.\, Red\,(\hat{\varkappa}').]$

We have now the following proposition:

116·001 $\vdash\ \mathfrak{T}\{\sigma, \tau, 0\} \supset \mathfrak{T}\{\sigma^\tau, 0\}$ [12·2421·42·5]

116·19 $\vdash\ .\, \varkappa\, smr\, \varkappa'\,.\, \omega\, smr\, \omega'\,. \supset (\varkappa \exp \omega)\, sm'\, (\varkappa' \exp \omega')$

Dem $\vdash$ 71·25 $\supset$

$\vdash\ .\, \mathfrak{T}\{\varkappa, \varkappa', K\}\,.\, \mathfrak{T}\{\omega, \omega', K\}\,.\, \mathfrak{T}\{R, S, A\}\,.\, \varkappa \leftarrow (R)_a \rightarrow \varkappa'\,.\, \omega \leftarrow (S)_a \rightarrow \omega'\,.$

$\supset : P\varepsilon 1 \underset{a}{\rightarrow} Cls \supset .\, \mathfrak{T}\{(Cnv_a R \mid P \mid S), A\}\,.\, (Cnv_a R \mid P \mid S)\varepsilon 1 \underset{a}{\rightarrow} Cls :$ (1)

[71·192 . 34·21] $\supset$:

$T \underset{A}{=} \hat{P}\hat{Q}[: \hat{Q} \underset{a}{=} (Cnv_a R \mid P \mid S) : \hat{Q}\varepsilon_A(\varkappa' \exp \omega')\,.\, \hat{P}\varepsilon_A(\varkappa \exp \omega).]\,. \supset$

$.\, \mathfrak{T}\{T, B\}\,.\, T\varepsilon 1 \underset{A}{\rightarrow} 1 : D_A T \underset{A}{=} (\varkappa \exp \omega) :.\, ꓷ_A T \underset{A}{=} (\varkappa' \exp \omega') :.$ (2)

[10·28 . (103·004) .] $\supset \vdash$ Prop.

116·23 $\vdash\ \sigma^\tau \varepsilon_1 NC \vee .\, \sigma^\tau \underset{\Omega}{=} \wedge''\,.$ [73·73·731]

116·301 $\vdash\ \sigma\varepsilon_1 NC \supset .\, \sigma^0 \underset{\Omega}{=} 1.$

116·311 $\vdash\ \sigma\varepsilon_1 (NC \underset{1}{-} \iota^{\epsilon}{}_1 0) \supset .\, 0^\sigma \underset{\Omega}{=} 0\,.$

116·321 $\vdash\ \sigma\varepsilon_1 NC \supset .\, \sigma^1 \underset{\Omega}{=} \sigma.$

116·331 $\vdash\ \sigma\varepsilon_1 NC \supset .\, 1^\sigma \underset{\Omega}{=} 1.$

116·51 $\vdash\ \sigma^\tau \underset{\Omega}{=} 0 \ \equiv : \sigma \underset{\Omega}{=} 0 : \sigma\varepsilon_1 (NC \underset{1}{-} \iota^{\epsilon}{}_1 0)$

116·35 $\vdash\ .\, [(\underset{A}{a\downarrow})^{\epsilon\epsilon}]^a_A \varkappa'' \leftarrow (R)_A \rightarrow (\varkappa \exp \omega)\,.$

$[(\underset{A}{a\downarrow})^{\epsilon\epsilon}]^a_A \leftarrow (S)_A \rightarrow (\varkappa \exp \omega') : (\omega \cap_a \omega') \underset{a}{=} \wedge_{(a)} :$

$.\, \mathfrak{T}\{R, S, B\}\,.\, \mathfrak{T}\{\omega, \omega', K\}\,. \supset (\varkappa'' \times \omega'')\, sm'\, (\varkappa \exp (\omega \cup_a \omega'))$

Dem $\vdash$ 72·184 · 71·252 . $\supset$

$\vdash\ .\, \mathrm{Hp} : T \underset{A}{=} \hat{P}\hat{Q}[(\exists \bar{u}, \bar{v}) : \hat{P} \underset{a}{=} (\hat{u} \overset{a}{\downarrow} \hat{v}) : \bar{u}[((\varkappa'' \overset{A}{\underset{a}{\rceil}} Cnv^a_A(a\downarrow)) \overset{A}{\underset{A}{\mid}} R)] \underset{a,A}{(}\hat{Q} \overset{A}{\underset{a}{\upharpoonright}} \omega\underset{a}{)}.$

$\bar{v}[((\omega'' \underset{a}{\overset{A}{\rceil}} Cnv^a_A(a\downarrow)) \underset{A}{\overset{A}{|}} S)]^A_{\cdot\, a,A}(Q \overset{a}{\upharpoonright} \omega') : \hat{Q}\varepsilon_A(\varkappa \exp(\omega \cup_a \omega')) . \hat{P}\varepsilon_A(\varkappa'' \times_A \omega'').]:$

$\supset . \mathfrak{C}\{T, B\} . T\varepsilon 1 \underset{A}{\rightarrow} 1 : D_A T \underset{A}{=} (\varkappa'' \times \omega'') :. \text{ᗡ}_A T \underset{A}{=} (\varkappa \exp(\omega \cup_a \omega')) :$

$[10{\cdot}24] \supset \vdash$ Prop.

117·581 $\vdash : \sigma \geqslant \tau : \tau' \varepsilon_1 NC . \supset . \sigma^{\tau'} \geqslant \tau^{\tau'}$.

Dem $\vdash$ $22{\cdot}44 \supset \vdash . \omega \underset{a}{\subset} \varkappa . \supset . (\omega \exp \omega') \underset{A}{\subset} (\varkappa \exp \omega')$.

$\supset \vdash$ Prop.

117·591 $\vdash : \sigma \geqslant \tau : \tau' \varepsilon_1 (NC \underset{1}{-} \iota'_1 0) . \supset . \tau'^{\sigma} \geqslant \tau'^{\tau}$.

Dem $\vdash . 71{\cdot}24 . 25{\cdot}4 . \supset$

$\vdash . \omega, \varkappa \varepsilon_K \text{extens}[K_{(a)}] : \omega \underset{a}{\subset} \varkappa : u \varepsilon_a \varkappa' : T \underset{a}{=} \hat{P}\hat{Q}[\hat{P}\varepsilon_A(\varkappa' \exp \omega) :$

$\hat{Q} \underset{a}{=} (\hat{P} \cup_a \hat{v}\hat{w}[: \hat{v} \underset{a,a}{=} u : \hat{w}\varepsilon_a(\varkappa \underset{a}{-} \omega).]) : \mathfrak{C}\{\hat{Q}, A\}.]:$

$\supset . \mathfrak{C}\{T, B\} . T\varepsilon 1 \underset{A}{\rightarrow} 1 : D_A T \underset{A}{=} (\varkappa' \exp \omega) :. \text{ᗡ}_A T \underset{A}{\subset} (\varkappa' \exp \varkappa) :.$ (1)

$[(1) . 72{\cdot}184 . 71{\cdot}252] \supset : [(a \underset{A}{\downarrow})^{\prime\prime}]\varkappa'' \leftarrow (R)_A \rightarrow (\varkappa' \exp \omega)$.

$[(a \underset{A}{\downarrow})^{\prime\prime}]\omega'' \leftarrow (S)_A \rightarrow (\varkappa' \exp \varkappa) : T' \underset{a}{=} (((Cnv^A_a(a\downarrow)) \underset{A}{\overset{A}{|}} (R \underset{a,A}{\overset{A}{|}} T \underset{A}{\overset{A}{|}} Cnv_A S)) \underset{a}{\overset{A}{|}} (a \underset{A}{\downarrow})) :$

$\supset . T' \varepsilon 1 \underset{a}{\rightarrow} 1 : D_a T' \underset{a}{=} \varkappa'' :. \text{ᗡ}_a T' \underset{a}{\subset} \omega'' :$ (2)

$\vdash (1) . (2) . \supset$ Prop.

117·592 $\vdash : \sigma^{\tau} \underset{\Omega}{=} 1 :. \sigma \underset{\Omega}{\neq} 0 :. \sigma \underset{\Omega}{\neq} 1 : \supset . \tau \underset{\Omega}{=} 0$.

$[117{\cdot}551{\cdot}53{\cdot}581{\cdot}591 . 116{\cdot}321 . 110{\cdot}643 . 117{\cdot}6 .]$

116·52 $\vdash \exists'(\sigma + \tau) \supset . (\tau'^{\sigma} \times \tau'^{\tau}) \underset{\Omega}{=} \tau'^{(\sigma+\tau)} . [116{\cdot}51 . 117{\cdot}591{\cdot}6]$

To prove the remaining laws of exponentiation, I shall use the following temporary abbreviations:

116·521 $Z_1 \underset{df}{=} ((\varkappa'' \underset{a}{\overset{A}{\rceil}} Cnv^a_A(a\downarrow)) \underset{A}{\overset{A}{|}}_{a,A} S)$

116·5211 $Z_2 \underset{df}{=} ((\omega'' \underset{a}{\overset{A}{\rceil}} Cnv^a_A(a\downarrow)) \underset{A}{\overset{A}{|}}_{a,A} R)$

116·5212 $Z_3 \underset{df}{=} ((\omega' \underset{a}{\overset{A}{\rceil}} Cnv^a_A(a\downarrow)) \underset{A}{\overset{A}{|}}_{a,A} T)$

116·522 $L_0 \underset{df}{=} \hat{P}\hat{Q}[. \hat{P}\varepsilon_A(\varkappa'' \times_A \omega'') . \hat{Q}\varepsilon_A(\varkappa' \exp \omega') . (\exists \bar{u}'', \bar{v}'') . \bar{u}''[\hat{P}]_a \bar{v}''$.

$(\exists \bar{P}', \bar{Q}') . \bar{u}''[Z_1]^A_a \bar{P}' . \bar{v}''[Z_2]^A_a \bar{Q}' : \hat{Q} \underset{a}{=} \hat{u}'\hat{v}'[(\exists \bar{v}) . \bar{v}[\bar{Q}']_a \hat{v}'$.

$\hat{u}'[((Z_3 \underset{a}{\overset{A}{|}} ((\downarrow \bar{v}) \overset{a}{\underset{A}{\upharpoonright}} \varkappa)) \underset{a}{\overset{a}{|}} \bar{P}')]_a \hat{v}'.]:]$

116·523 $L_1 \underset{df}{=} \hat{P}\hat{Q}[. \hat{P}\varepsilon_A(\varkappa \exp \omega'') . \hat{Q}\varepsilon_A(\varkappa'' \exp \omega') : \hat{Q} \underset{a}{=}$

$\hat{u}''\hat{v}'[\hat{u}''[Z_1]^A_a((P \underset{a,A}{\overset{a}{|}} Z_2) \underset{a}{\overset{A}{|}} ((\downarrow \hat{v}') \underset{A}{\overset{a}{\upharpoonright}} \omega))]:]$

116·524 $H_1 \underset{df}{=} \hat{u}\,\hat{v}''[(\exists\bar{v}')(\exists\bar{P}')\,.\,\bar{P}'[(Cnv^A_a Z_1 \mid Q)]^a_A\,\bar{v}'\,.$

$\hat{u}[((\bar{P}' \underset{a,A}{\overset{a}{|}} (\omega \underset{a}{\overset{A}{\urcorner}} Cnv^a_A(\underset{A}{\downarrow}\bar{v}')) \underset{a}{\overset{A}{|}} Cnv^A_a Z_2)]_a \hat{v}''.]$

We have the following propositions concerning these relations:

116·525 $\vdash : D_A S \underset{A}{=} [(a\underset{A}{\downarrow})^{cc}]^a_A\,\varkappa'' : S \varepsilon 1 \underset{A}{\rightarrow} 1\,.\supset$

$.\,Z_1 \varepsilon 1 \underset{a,A}{\rightarrow} 1 : D^A_a Z_1 \underset{a}{=} \varkappa'' :.\, \mathfrak{a}^A_a Z_1 \underset{A}{=} \mathfrak{a}_A S$: [72·184 . 71·252]

116·526 $\vdash : D_A R \underset{A}{=} [(a\underset{A}{\downarrow})^{cc}]^a_A\,\omega'' : R \varepsilon 1 \underset{A}{\rightarrow} 1\,.\supset$

$.\,Z_2 \varepsilon 1 \underset{a.A}{\rightarrow} 1 : D_A Z_2 \underset{a}{=} \omega'' :.\, \mathfrak{a}^A_a Z_2 \underset{A}{=} \mathfrak{a}_A R$:

116·527 $\vdash : D_A T \underset{A}{=} [(a\underset{A}{\downarrow})^{cc}]^a_A\,\varkappa' : T \varepsilon 1 \underset{A}{\rightarrow} 1\,.\supset$

$.\,Z_3 \varepsilon 1 \underset{a,A}{\rightarrow} 1 : D^A_a Z_3 \underset{a}{=} \omega' :.\, \mathfrak{a}^A_a Z_3 \underset{A}{=} \mathfrak{a}_A T$:

116·5401 $\vdash [(a\underset{A}{\downarrow})^{cc}]^a_A\varkappa'' \leftarrow (S)_A \rightarrow (\varkappa \exp \omega').[(a\underset{A}{\downarrow})^{cc}]^a_A\omega'' \leftarrow (R)_A \rightarrow (\omega \exp \omega')$

$.\,[(a\underset{A}{\downarrow})^{cc}]^a_A\,\varkappa' \leftarrow (T)_A \rightarrow (\varkappa \times_A \omega)\,.$

$Red(\varkappa)\,.\,Red(\varkappa', \omega')\,.\,Red(\varkappa'', \omega'')\,.\,\mathfrak{C}\{R, S, T, B\}\,.\,\exists_a\,\omega''\,.\,\exists_a\varkappa''\,.$

$Q \varepsilon_A (\varkappa \exp \omega')\,.\supset Q \varepsilon_A \mathfrak{a}_A L_0$

Dem $\vdash$. 71·192·25·252·36 . 116·527 . $\supset$

$\vdash$ Hp . $Q' \varepsilon_A (\omega \exp \omega') : P' \underset{a}{=} \hat{u}\,\hat{v}'[(\exists\bar{v})\,.\,\bar{v}[Q']_a \hat{v}'\,.$

$\hat{u}[(((\varkappa \underset{a}{\overset{A}{\urcorner}} Cnv^a_A(\underset{A}{\downarrow}\bar{v})) \underset{a}{\overset{A}{|}} Cnv^A_a Z_3) \underset{a}{\overset{a}{|}} Q)]_a \hat{v}'\,.]$:

$u''[Z_1]^A_a P'\,.\,v''[Z_2]^A_a Q'\,.\,P \underset{a}{=} (u'' \overset{a}{\downarrow} v'') : \supset.$

$Q \underset{a}{=} \hat{u}'\,\hat{v}'[(\exists\bar{v})\,.\,\bar{v}[Q']_a \hat{v}'\,.\,\hat{u}'[((Z_3 \underset{a}{\overset{A}{|}} ((\underset{A}{\downarrow}v) \overset{a}{\underset{A}{\upharpoonright}} \varkappa)) \underset{a}{\overset{a}{|}} P')]_a \hat{v}\,.]$.

[10·24 . (115·522)] $\supset P[L_0]_A Q$

[10·34 . Hp] $\supset$ Prop.

116·5402 $\vdash$ Hp 116·541 $\supset L_0 \varepsilon 1 \underset{A}{\rightarrow} 1$ [116·525·526]

116·54 $\vdash .\,[(a\underset{A}{\downarrow})^{cc}]^a_A\,\varkappa''\,sm'(\varkappa \exp \omega')\,.\,[(a\underset{A}{\downarrow})^{cc}]^a_A\,\omega''\,sm'(\omega \exp \omega')\,.$

$[(a\underset{A}{\downarrow})^{cc}]^a_A\,\varkappa'\,sm'(\varkappa \times_A \omega)\,.$

$Red(\varkappa'', \omega'')\,.\,Red(\varkappa)\,.\,Red(\varkappa', \omega')\,.\supset (\varkappa'' \times_A \omega'')\,sm'(\varkappa \exp \omega')$

[116·5401·5402·35 . 113·601]

116·55 $\vdash \exists'(\sigma \times \tau) \supset .\,(\sigma^{\tau'} \times \tau^{\tau'}) \underset{\Omega}{=} (\sigma \times \tau)^{\tau'}$

[116·54 . 117·581·62 . 113·601·311·621 . 116·301]

116·615 $\vdash . [(\underset{A}{a\downarrow})^{\epsilon\epsilon}]^a_A \varkappa'' \leftarrow (S)_A \rightarrow (\varkappa \exp \omega)$.

$$[(\underset{A}{a\downarrow})^{\epsilon\epsilon}]^a_A \omega'' \leftarrow (R)_A \rightarrow (\omega \times \omega') . \mathfrak{C}\{R, S, B\} .$$

$$Red(\varkappa'', \omega'') . Red(\varkappa, \omega) . Red(\omega') . \exists_a \varkappa'' . \exists_a \omega'' . \exists_a \omega . Q \varepsilon_A (\varkappa'' \exp \omega') :$$

$$P \underset{a}{=} H_1 : \supset P[L_1]_A Q$$

Dem $\vdash$. 116·525 . 71·25 . $\supset \vdash$ Hp $\supset . (Cnv^A_a Z_1 \overset{a}{\underset{A,a}{|}} Q) \varepsilon 1 \underset{A,a}{\rightarrow} Cls$:

$$D^a_A(Cnv^A_a Z_1 \overset{a}{\underset{A,a}{|}} Q) \underset{A}{\subset} (\varkappa \exp \omega) :. \, \mathfrak{a}^a_A(Cnv^A_a Z_1 \overset{a}{\underset{A,a}{|}} Q) \underset{a}{=} \varkappa' :$$

[71·36] $\supset \vdash$ Hp $\supset . P''[(Cnv^A_a Z_1 \overset{a}{\underset{A,a}{|}} Q)]^a_A v' \equiv . P'' \underset{a}{=} \hat{u}\hat{v}[(\exists \overline{P}') .$

$$\overline{P}'[(Cnv^A_a Z_1 \overset{a}{\underset{a}{|}} Q)]^a_A v' . \hat{u}[\overline{P}']_a \hat{v} .] :$$

$$\supset : u''[Z_1]^A_a P'' . P''[(Cnv^A_a Z_1 \overset{a}{\underset{A,a}{|}} Q)]v' . \equiv . u''[Z_1]^A_a P'' :$$

$$P'' \underset{a}{=} \hat{u}\hat{v}[(\exists \overline{P}') . \overline{P}'[(Cnv^A_a Z_1 \overset{a}{\underset{A,a}{|}} Q)]^a_A v' . \hat{u}[\overline{P}']_a \hat{v} .] :.$$

[71·192] $\supset . u''[Q]_a v' \equiv (\exists \overline{Q}') . u''[Z_1]^A_a \overline{Q}' : \overline{Q}' \underset{a}{=} \hat{u}\hat{v}[(\exists \overline{P}') .$

$$\overline{P}'[(Cnv^A_a Z_1 \overset{a}{\underset{A,a}{|}} Q)]^a_A v' . \hat{u}[\overline{P}']_a \hat{v} .] :$$

71·192] $\supset \vdash$ Hp $\supset . u''[Q]_a v' \equiv (\exists \overline{Q}') . u''[Z_1]^A_a \overline{Q}' : \overline{Q}' \underset{a}{=} \hat{u}\hat{v}[(\exists \overline{P}') .$

$\overline{P}'[(Cnv^A_a Z_1 \overset{a}{\underset{A,a}{|}} Q)]^a_A v' . \hat{u}[(((\overline{P}' \overset{a}{\underset{a,A}{|}} (\omega \overset{A}{\underset{a}{\rceil}} Cnv(\downarrow v'))) \overset{A}{\underset{a,A}{|}} (Cnv^A_a Z_2 \overset{a}{\underset{A}{|}} Z_2)) \overset{A}{\underset{a}{|}} ((\downarrow v') \overset{a}{\underset{A}{\restriction}} \omega))]_a \hat{v} .]$:

[71·192.55·2] $\equiv (\exists \overline{Q}') . u''[Z_1]^A_a \overline{Q}' : \overline{Q}' \underset{a}{=} \hat{u}\hat{v}[(\exists \overline{v}'', \overline{w}')(\exists \overline{P}') .$

$$\overline{P}'[(Cnv^A_a Z_1 \overset{a}{\underset{A,a}{|}} Q)]^a_A \overline{w}' . \hat{u}[((\overline{P}' \overset{a}{\underset{a,A}{|}} (\omega \overset{A}{\underset{a}{\rceil}} Cnv^a_A(\downarrow \overline{w}'))) \overset{A}{\underset{a}{|}} Cnv Z_2)]_a \overline{v}'' .$$

$$\overline{v}''[(Z_2 \,|\, ((\underset{A}{\downarrow} v') \overset{a}{\underset{A}{\restriction}} \omega))]_a \hat{v} .] :$$

[(116 524)) $\equiv (\exists \overline{Q}') . u''[Z_1]^A_a \overline{Q}' : \overline{Q}' \underset{a}{=} ((P \overset{a}{\underset{a,A}{|}} Z_2) \overset{A}{\underset{a}{|}} ((\underset{A}{\downarrow} v') \overset{a}{\underset{a}{\restriction}} \omega))$:

[(116·523)] $\supset \vdash$ Prop.

116·616 $\vdash$ Hp 116·65 $\supset L_1 \varepsilon 1 \underset{A}{\rightarrow} 1$

Dem $\vdash$. 71·192·25·252 . 116·525 . $\supset \vdash P[L_1]_A Q \supset$

$$. u''[Q]_a v' \equiv (\exists \overline{P}') . u''[Z_1]^A_a \overline{P}' : P \underset{a}{=} ((P' \overset{a}{\underset{a,A}{|}} (\omega \rceil Cnv^a_A(\underset{A}{\downarrow} v'))) \overset{A}{|} Cnv^A_a Z_2) :.$$

[71·192] $\supset P''[(Cnv^A_a Z_1 \overset{a}{\underset{A,a}{|}} Q)]^a_A v' \equiv . P'' \varepsilon (\varkappa \exp \omega) : P \underset{a}{=}$

$$((P'' \overset{a}{\underset{a,A}{|}} (\omega \rceil Cnv^a_A(\underset{A}{\downarrow} v'))) \overset{A}{\underset{a}{|}} Cnv^A_a Z_2) :.$$

[(116·525)] $\supset$ $\vdash$ Prop.

116·62 $\vdash . [a\underset{A}{\downarrow})^{cc}]^{a}_{A}\varkappa'' sm'(\varkappa \exp \omega) . [(a\underset{A}{\downarrow})^{cc}]^{a}_{A}\omega'' sm'(\omega \times_{A} \omega') .$
$Red(\varkappa'', \omega'') . Red(\omega) .$
$\supset (\varkappa \exp \omega'') sm' (\varkappa'' \exp \omega')$
[116·615·616·311·301·35·113·602]

116·63 $\vdash \exists'(\tau \times \tau') \supset . \sigma^{(\tau \times \tau')} \underset{\Omega}{=} (\sigma^{\tau})^{\tau'}$ [116·62.117·62.116·301·351]

This proposition is the third law of exponentiation. We have got the laws of exponentiation by another method than that used in Principia. The proofs are here very much shortened; newerthless, as the fundamental relations we have to deal with are explicitly given, there is no further difficulty to obtain full demonstrations.

117·66 $\vdash \sigma \varepsilon_1 NC \supset . 2^{\sigma} \geqslant \sigma .$

Dem $\vdash$ 72·184 $\supset$

$\vdash : \varkappa \underset{a}{=} (\iota^{c}{}_{a} u \bigcup_{a} \iota^{c}{}_{a} v) : \sim . u \underset{a,a}{=} v : T \underset{A}{=} \hat{P}\hat{Q}[.\hat{P}\varepsilon_{A}(\varkappa \exp \omega) . (\exists \overline{w}) :$
$\overline{w} \underset{a}{=} (Cnv_{a}\hat{P})^{c} u : \hat{Q}[(\underset{A}{\downarrow} a)]^{a}_{A} w .] : Red(\omega) .$
$\supset . T \varepsilon 1 \underset{A}{\rightarrow} 1 : D_{A} T \underset{A}{\subset} (\varkappa \exp \omega) :. \text{Ꭰ}_{A} T \underset{A}{=} [(\underset{A}{\downarrow} a)^{cc}]^{a}_{A} \omega :$

[10·24] $\supset (\exists \overline{\vartheta}) : \overline{\vartheta} \underset{A}{\subset} (\varkappa \exp \omega) : \overline{\vartheta} sm' [(\underset{A}{\downarrow} a)^{cc}]_{A} \omega .$

[73·22] $\supset . [(a\underset{A}{\downarrow})^{cc}]^{a}_{A} \varkappa' \leftarrow (R)_{A} \rightarrow (\varkappa \exp \omega) \supset : [R^{cc}]_{A} \vartheta \underset{a}{\subset} \varkappa' : [R^{cc}]_{A} \vartheta sm \omega .$

[101·1001] $\supset . \varkappa' \varepsilon' 2^{\sigma} . \omega \varepsilon' \sigma . \supset (\exists \overline{\omega}') : \overline{\omega}' \underset{a}{\subset} \varkappa' : \omega' \varepsilon' \sigma .$

[10·28] $\supset$ $\vdash$ Prop.

F. Subtraction.

119·01 $(\sigma - \tau) \underset{df}{=} \hat{\varkappa}[:(\tau + Nc^{c}\hat{\varkappa}) \underset{\Omega}{=} \sigma : NC\{\sigma\} . NC\{\tau\} .]$

119·11 $\vdash \exists'(\sigma - \tau) \supset \sigma, \tau \varepsilon_1 NC$

119·14 $\vdash \varkappa \varepsilon'(\sigma - \tau) \supset Nc^{c}\varkappa \underset{\Omega}{\subset} (\sigma - \tau)$

119·26 $\vdash \exists'(\sigma - \tau) \supset . \sigma \geqslant \tau .$

119·27 $\vdash . \sigma \geqslant \tau . \supset \exists'(\sigma - \tau)$

119·64 $\vdash . \sigma \geqslant \tau . \equiv \exists'(\sigma - \tau)$

119·32 $\vdash ((\sigma + \tau) - \tau) \varepsilon_1 NC \supset . \sigma \underset{\Omega}{=} ((\sigma + \tau) - \tau) .$

119·34 $\vdash (\sigma - \tau) \varepsilon_1 NC \supset . ((\sigma - \tau) + \tau) \underset{\Omega}{=} \sigma .$

119·35 $\vdash . \exists'(\sigma' + \tau) . (\sigma - \tau) \varepsilon_1 NC . \supset . (\sigma' + \sigma) \underset{\Omega}{=} ((\sigma' + \tau) + (\sigma - \tau)) .$

119·45 $\vdash . ((\sigma + \tau) - \sigma')\varepsilon_1 NC . (\tau - \sigma')\varepsilon_1 NC . \supset$
$. ((\sigma + \tau) - \sigma') \underset{\Omega}{=} (\sigma + (\tau - \sigma'))$.

The proof of these propositions is to be got by a direct application of the method of Principia and of lemmas 101·1001 1002.

VIII. Inductive numbers.

The Theory of iductive numbers, as based on the pure theory of Types, is quite complete and seems very simple. Moreover, it can be exposed in a quite popular way, without any serious difficulty.

I begin with the definition of an hereditary class.

120·001 $H(g, \tau) \underset{df}{=} . \tau\varepsilon_1 g . (\bar{\sigma}) : \bar{\sigma}\varepsilon_1 g \supset (\bar{\sigma} + 1)\varepsilon_1 g : \mathfrak{C}\{\bar{\sigma}, 1\}$:

We have now following proposition:

120·101 $\vdash H(\hat{\sigma}'[. H(g, \tau) \supset \hat{\sigma}'\varepsilon_1 g .], \tau)$

Dem $\vdash$ 5·5 $\supset \vdash H(g, \tau) \supset : H(g, \tau) \supset \sigma'\varepsilon_1 g . \equiv \sigma'\varepsilon_1 g$.

[(120·001)] $\supset H(\hat{\sigma}'[. H(g, \tau) \supset \hat{\sigma}'\varepsilon_1 g .], \tau)$ (1)

$\vdash$ 2·21 $\supset \vdash \sim H(g, \tau) \supset : H(g, \tau) \supset \sigma'\varepsilon_1 g . \equiv V_{(1)}\{\tau\}$.

[(120·001)] $\supset H(\hat{\sigma}'[. H(g, \tau) \supset \hat{\sigma}'\varepsilon_1 g .], \tau)$ (2)

$\vdash$ (1) . (2) . $\supset \vdash$ Prop.

Note that $H(g, \tau)$ is ambiguous in respect to the type of g. I proceed now to the definition of inductive numbers. The class of inductive numbers ought to be the logical product of all hereditary classes, i. e. all classes g such that $H(g, 0)$. Now, we cannot speak about „all hereditary classes“, these classes not having the same type. Therefore we cannot speak simply of inductive numbers, as we have different orders of them.

Let us now lay down the definition:

120·002 $\Phi \underset{df}{=} \hat{\tau}[(\exists\bar{\sigma}) . \bar{\sigma}\varepsilon_1 NC . \hat{\tau}\varepsilon_1 NC .]$

Our definition of the Φ-order inductive numbers will be:

120·01 $NC(\Phi)$ induct $\underset{df}{=} \hat{\tau}[(\bar{g}) : H(\bar{g}, 0) \supset \hat{\tau}\varepsilon_1 g : \mathfrak{C}\{\bar{g}, \Phi\} .]$

This definition is a pattern for definitions of inductive numbers in any order. We shall here use the definition.

120·001 $N_0C \text{ induct} \underset{df}{=} NC(\Phi) \text{ induct}$

It is obvious that the Φ-order inductive numbers do not necessarily belong to all hereditary classes of another order. Now, it is useful to speak about N_0C induct-order inductive numbers. We shall see below that the inductive numbers of this order have all fundametal group-properties of the natural numbers. Therefore we shall call these numbers simply i n d u c t i v e n u m b e r s, and we shall use the following definition:

120·013 $NC \text{ induct} \underset{df}{=} NC(N_0C \text{ induct}) \text{ induct}$

It is easy to prove that all „inductive nnmbers" are Φ-order inductive numbers. We have the following proposition:

120·1011 $\vdash \tau\varepsilon_1 NC \text{ induct} \supset \tau\varepsilon_1 N_0C \text{ induct}$

Dem $\vdash 24{\cdot}23{\cdot}24 \supset$

$\vdash . \text{Hp} . H(g,0) . \mathfrak{C}\{g,\Phi\} . \supset H((g \cup_1 (N_0C \text{ induct} \cap_1 \wedge_{(1)})), 0)$

[Hp] $\supset \tau\varepsilon_1(g \cup_1 (N_0C \text{ induct} \cap_1 \wedge_{(1)}))$

[24·23·24] $\supset \tau\varepsilon_1 g$

$\supset \vdash$ Prop.

The following propositions concerning N_0Cinduct can be proved by the same method for NCinduct.

120·1 $\vdash \sigma\varepsilon_1 N_0C \text{induct} \equiv (\overline{g}) : H(\overline{g},0) \supset \sigma\varepsilon_1 \overline{g} . \mathfrak{C}\{\overline{g},\Phi\}.$

To prove this proposition we use 120·001 and we show in an easy manner that N_0C induct is an extensional class.

120·11 $\vdash . H(g,0) . \mathfrak{C}\{g,\Phi\} . \sigma\varepsilon_1 N_0C \text{ induct} . \supset \sigma\varepsilon_1 g$

120·12 $\vdash H(g,0) \supset 0\varepsilon_1 g$

120·121 $\vdash . H(g,0) \supset \sigma\varepsilon_1 g . \supset . H(g,0) \supset (\sigma \quad 1)\varepsilon_1 g.$

120·122 $\vdash H(g,0) \supset 1\varepsilon_1 g$ [120·12·121 . 110·641 .]

120·123 $\vdash H(g,0) \supset 2\varepsilon_1 g$ [120·122·121 . 110·643 .]

Note that, if any natural number, e. g. 1918, is defined in our system, we can prove the proposition 1918 ε_1 N_0Cinduct, using 1918 times the method of the proof of 110·122.

120·124 $\vdash (\sigma+1) \underset{\Omega}{\neq} 0$ [110·4 . 101·11 . 110·632.]

120·14 $\vdash N_0C \text{ induct} \underset{1}{\subset} (NC \cup_1 \iota'_1 \wedge'')$]120·121·12·11 .]

120·02 $Cls(\Phi) \text{ induct} \underset{df}{=} \hat{\varkappa}[(\exists\overline{\sigma}) . \overline{\sigma}\varepsilon_1 NC(\Phi) \text{ induct} . \hat{\varkappa}\varepsilon_1 \overline{\sigma}.]$

120·021 $Cls_0 \text{ induct} \underset{df}{=} Cls(\Phi) \text{ induct}$

120·023 $J(h) \underset{df}{=} (\overline{\varkappa})(\overline{u}) : \text{extens}(\overline{\varkappa}) . \overline{\varkappa}\varepsilon' h . \supset (\overline{\varkappa} \cup_a \iota' \overline{u})\varepsilon' h .$

120·212 $\vdash H(g,0) \supset Nc^{\epsilon}\Lambda'\varepsilon_1 g$ [120·12.(101·01)]

120·2121 $\vdash \wedge'\varepsilon' Cls_0$ induct [120·212.(120·02·021)]

120·213 $\vdash H(g,0) \supset Nc^{\epsilon}(\iota^{\epsilon}{}_a u \cup_a \wedge')\varepsilon_1 g$ [120·122.(101·02)]

120·2131 $\vdash (\iota^{\epsilon}{}_a u \cup_a \wedge')\varepsilon' Cls_0$ induct [120·213(120·0 ·021)]

120·21 $\vdash \varkappa\varepsilon' Cls_0 \text{ induct} \equiv Nc^{\epsilon}\varkappa\varepsilon_1(N_0 C \text{ induct} \underset{1}{-} \iota^{\epsilon}\wedge'')$

[120·14.(120·02) 103·27]

120·201 $\vdash \varkappa smr\varkappa' \supset . Nc^{\epsilon}\varkappa\varepsilon_1(N_0 C \text{induct} \underset{1}{-} \iota^{\epsilon}\wedge'') \equiv$

$Nc^{\epsilon}\varkappa'\varepsilon_1(N_0 C \text{induct} - \iota^{\epsilon}\wedge'')$. [103·14]

120·214 $\vdash \varkappa smr\varkappa' \supset . \varkappa\varepsilon' Cls_0 \text{ induct} \equiv \varkappa'\varepsilon' Cls_0$ induct. [120·201]

120·215 $\vdash \text{extens}(\omega) \supset : Nc^{\epsilon}(\omega \cup_a \iota^{\epsilon}{}_a u) \underset{\Omega}{=} Nc^{\epsilon}\omega . \vee . Nc^{\epsilon}(\omega \cup_a \iota^{\epsilon}{}_a u) \underset{\Omega}{=}$

$(Nc^{\epsilon}\omega + 1)$: [110·63]

120·22 $\vdash . \mathfrak{C}\{g, \Phi\} . H(g,0) . (\bar{h}) :. \wedge'\varepsilon'\bar{h} . J(\bar{h}) . \supset \omega'\varepsilon_1 \bar{h}$:

$\mathfrak{C}\{\bar{h}, \hat{\omega}[\Phi\{Nc^{\epsilon}\hat{\omega}\}]\} : \supset Nc^{\epsilon}\omega'\varepsilon_1 g$

Dem $\vdash$ 120·215 $\supset \vdash H(g,0) \supset J(\hat{\omega}[g\{Nc^{\epsilon}\hat{\omega}\}])$ (1)

$\vdash$ (1). 120·212 $\supset \vdash \text{Hp} \supset \omega'\varepsilon_1\hat{\omega}[g\{Nc^{\epsilon}\hat{\omega}\}]$

$\supset \vdash$ Prop.

120·221 $\vdash . J(h) : Nc^{\epsilon}\omega \underset{\Omega}{\subset} h : \supset . (Nc^{\epsilon}\omega + 1) \underset{\Omega}{\subset} h$. [103·11 . 110·63]

120·222 $\vdash . J(h) . \sigma\varepsilon_1 NC : \sigma \underset{\Omega}{\subset} h : \supset . (\sigma + 1) \underset{\Omega}{\subset} h$.

. [103·2 . 120·211 . 110·4]

120·251 $\vdash . H(g,0) . \sigma'\varepsilon_1 g . \omega\varepsilon_1\sigma' . \supset (\exists\bar{\sigma}) . \bar{\sigma}\varepsilon_1 g . (\omega \cup_a \iota^{\epsilon}{}_a u)\varepsilon_1\bar{\sigma}$.

Dem $\vdash$ 120·121 . 101·1005 . $\supset \vdash \text{Hp} \supset . Nc^{\epsilon}\omega\varepsilon_1 g . Nc^{\epsilon}(\omega \cup_a \iota^{\epsilon}{}_a u)\varepsilon_1 g$.

[120·215) $\supset \vdash$ Prop.

120·302 $\vdash . \exists_1 Nc^{\epsilon}\varkappa : (Nc^{\epsilon}\varkappa + 1) \underset{\Omega}{=} \wedge'' : \equiv : \varkappa \underset{a}{=} V_{(a)} : Red(\varkappa)$.

Dem $\vdash$ 110·631 $\supset \vdash \omega\varepsilon'(Nc^{\epsilon}\varkappa + 1) \equiv$

$(\exists\bar{u}) . \sim \bar{u}\varepsilon_a\varkappa . \omega\varepsilon' Nc^{\epsilon}(\varkappa \cup_a \iota^{\epsilon}{}_a \bar{u}) . \text{extens}[\Omega_{(a)}]\{\varkappa\}$. (1)

$\supset \vdash \sim \omega\varepsilon'(Nc^{\epsilon}\varkappa + 1) \equiv (\bar{u}) : \omega\varepsilon' Nc^{\epsilon}(\varkappa \cup_a \iota^{\epsilon}{}_a \bar{u}) . \text{extens}[\Omega_{(a)}]\{\varkappa\} . \supset \bar{u}\varepsilon_a\varkappa$.

$\vdash$. 10·271 . 10·1 .

$\supset \vdash : (Nc^{\epsilon}\varkappa + 1) \underset{\Omega}{=} \wedge'' : \exists' Nc^{\epsilon}\varkappa$.

$\supset (\bar{u}) . (\varkappa \cup_a \iota^{\epsilon}{}_a \bar{u})\varepsilon' Nc^{\epsilon}(\varkappa \cup_a \iota^{\epsilon}{}_a \bar{u}) \supset \bar{u}\varepsilon_a\varkappa$.

[103·13·12 . 101·1005.] $\supset (\bar{u}) \bar{u}\varepsilon_a\varkappa$

$\supset : \varkappa \underset{a}{=} V_{(a)} : Red(\varkappa)$. (2)

$\vdash$ (1) $\supset \vdash : \varkappa \underset{a}{=} V_{(a)} : Red(\varkappa) . \supset \sim \omega\varepsilon'(Nc^{\epsilon}\varkappa + 1)$

$\supset . (Nc^{\epsilon}\varkappa + 1) \underset{\Omega}{=} \wedge''$. (3)

[103·13] $\supset \exists' Nc'\varkappa$ (4)

$\vdash$ (2) . (3) . (4) $\supset \vdash$ Prop.

120·303 $\vdash \Lambda'' \varepsilon_1 N_0 C$ induct $\supset Nc'(\underset{a}{-}\Lambda')\, \varepsilon_1\, NC$ induct

Dem $\vdash$ 120·12 $\supset \vdash . H(g, 0) . \sim Nc'(-\Lambda')\varepsilon_1 g . \supset (g \underset{1}{-} \iota'_1\Lambda'')\{0\}$ (1)

[120·121·302] $\supset . (g - \iota'_1\Lambda'')\{\sigma\} \supset (g \underset{1}{-} \iota'_1\Lambda'')\{(\sigma+1)\}$. (2)

[.(1).(2).] $\supset H(\hat{\sigma}[(g \underset{1}{-} \iota'_1\Lambda'')\{\hat{\sigma}\}], 0)$

$\supset \sim \Lambda'' \varepsilon_1 NC$ induct (3)

$\vdash$. Transp. (33) . $\supset \vdash$ Hp $\supset . H(g, 0) \supset Nc'(\underset{a}{-}\Lambda')\, \varepsilon_1\, g$.

[120·01] $\supset \vdash$ Prop.

120·31 $\vdash . \exists'(Nc'\varkappa + 1) : (Nc'\varkappa + 1) \underset{\Omega}{=} (Nc'\omega + 1) : \supset . Nc'\varkappa \underset{\Omega}{=} Nc'\omega$.
[110·63 . 73·72·3·32 . 103·14]

120·311 $\vdash . \exists'(\sigma + 1) : (\sigma + 1) \underset{\Omega}{=} (\tau + 1) : \supset . \sigma \underset{\Omega}{=} \tau$.
[120·31 . 110·4 . 103·2]

120·32 $\vdash \sigma\varepsilon_1(NC$ induct $\underset{1}{-} \iota'_1\Lambda'') . \supset . \sigma \underset{\Omega}{\neq} (\sigma + 1)$.
[101·22 . 110·64 . 120·311·11]

120·41 $\vdash . \exists'(\sigma + \tau) . \tau\varepsilon_1 NC$ induct $: (\sigma + \tau) \underset{\Omega}{=} (\sigma' + \tau) : \supset . \sigma \underset{\Omega}{=} \sigma'$.

Dem $\vdash$. 120·311 . 110·61 . 120·11 . $\supset \vdash$ Prop.

Remark: As we deal with numbers of the same type, the complicatad proof of Principia is here supplied by the simple application of 120·11 to the class:

$$\hat{\tau}[:. \exists'(\sigma + \hat{\tau}) : (\sigma + \hat{\tau}) \underset{\Omega}{=} (\sigma' + \hat{\tau}) : \supset . \sigma \underset{\Omega}{=} \sigma' : \vee (\Phi \cap_1 \Lambda_{(1)})\{\hat{\tau}\} .]$$

120·411 $\vdash \sigma'\varepsilon_1 N_0 C$ induct $\supset : \exists'(\sigma - \sigma') \supset (\sigma - \sigma')\, \varepsilon_1 NC :$
$: \sigma \geqslant \sigma' . \equiv (\sigma - \sigma')\, \varepsilon_1 NC :$
[(119·01) . 120·41 . 119·14·27 . 103·2 .]

120·412 $\vdash . \sigma'\varepsilon_1 N_0 C$ induct $: \tau' \geqslant \sigma' : \supset . (\bar{\tau}, \bar{\sigma}) :. (\tau' - \sigma') \underset{\Omega}{=} \bar{\tau} :.$
$(\tau' - \sigma') \underset{\Omega}{=} \bar{\sigma} : \supset . \bar{\tau} \underset{\Omega}{=} \bar{\sigma} :. (\bar{\tau}, \bar{\sigma}) :. (\sigma' + \bar{\tau}) \underset{\Omega}{=} \tau' :. (\sigma' + \bar{\sigma}) \underset{\Omega}{=} \tau' : \supset . \bar{\tau} \underset{\Omega}{=} \bar{\sigma} :.$
[120·41·411 . 119·34]

120·413 $\vdash \sigma\varepsilon_1 NC \supset . (\sigma - 0) \underset{\Omega}{=} \sigma$. [(119·01) . 110·6]

120·414 $\vdash \sigma\varepsilon_1(NC \underset{1}{-} \iota'_1 0) \supset (\sigma - 1)\varepsilon_1 NC$ [117·53 . 120·411]

120·416 $\vdash . \sigma'\varepsilon_1 NC$ induct $. \exists'(\sigma - \sigma') . \supset . ((\sigma - \sigma') + \sigma') \underset{\Omega}{=} \sigma$.
[120·411 . 119·34]

120·417 $\vdash \sigma\varepsilon_1(NC \underset{1}{-} \iota^{c}_1 0) \supset . (\sigma + \sigma') \underset{\Omega}{=} ((\sigma' + 1) + (\sigma - 1))$.

[120·414 . 119·35 .]

120·418 $\vdash . \sigma\varepsilon_1 NC$ induct $: \tau \geqslant \sigma : \supset . (\sigma' + \tau) \underset{\Omega}{\doteq} ((\sigma' + \sigma) + (\tau + \sigma))$.

[120·411 . 119·35]

120·42 $\vdash . \sigma\varepsilon_1 NC$ induct $. \exists'\sigma : \sigma' \underset{\Omega}{\neq} 0 : \supset . \sigma \underset{\Omega}{\neq} (\sigma + \sigma')$.

[120·41·14]

120·422 $\vdash (\sigma + 1)\varepsilon_1(N_0 C \text{induct} \underset{1}{-} \iota^{c}_1 \Lambda'') \supset \sigma\varepsilon_1(N_0 C \text{induct} \underset{1}{-} \iota^{c}_1 \Lambda'')$

Dem $\vdash . \exists'(\sigma + 1) . H(g, 0) . \sim \sigma\varepsilon_1 g . \supset . (g \underset{1}{-} \iota^{c}_1(\sigma + 1))\{0\}$ (1)

[120·41] $\supset (g \underset{1}{-} \iota^{c}_1(\sigma + 1))\{\tau\} \supset (g \underset{1}{-} \iota^{c}_1(\sigma + 1))\{(\tau + 1)\}$. (2)

[(1) . (2)] $\supset . H((g \underset{1}{-} \iota^{c}_1(\sigma + 1)), 0) . \sim (g \underset{1}{-} \iota^{c}_1(\sigma + 1))\{(\sigma + 1)\}$. (3)

$\vdash$. 120·1 . (3) $\supset \vdash . \exists'(\sigma + 1) . H(g, 0) . \sim \sigma\varepsilon_1 g . \supset$

$\sim (\sigma + 1)\varepsilon_1(NC \text{ induct} \underset{1}{-} \iota^{c}_1 \Lambda'')$

[Transp] $\supset \vdash (\sigma + 1)\varepsilon_1(NC \text{induct} \underset{1}{-} \iota^{c}_1 \Lambda'') \supset . H(g, 0) \supset \sigma\varepsilon_1 g$. (4)

$\vdash$ 110·4 $\supset \vdash (\sigma + 1)\varepsilon_1(\underset{1}{-} \iota^{c}_1 \Lambda'') \supset \exists'\sigma$ (5)

$\vdash$ (4) . (5) $\supset \vdash$ Prop.

120·423 $\vdash \sigma\varepsilon_1(N_0 C \text{induct} \underset{1}{-} \iota^{c}_1 0) \equiv (\exists\bar{\tau}) . \bar{\tau}\varepsilon_1 N_0 C \text{induct} : \sigma \underset{\Omega}{=} (\bar{\tau} + 1) :$

Dem $\vdash$. 120·14·414 . $\supset \vdash . \sigma\varepsilon_1(N_0 C \text{induct} \underset{1}{-} \iota^{c}_1 0) . \exists'\sigma . \supset (\sigma - 1)\varepsilon_1 NC$

[120·422] $\supset (\sigma - 1)\varepsilon_1(NC \text{ induct} \underset{1}{-} \iota^{c}_1 \Lambda'')$

[120·416] $\supset (\exists\bar{\tau}) . \bar{\tau}\varepsilon_1 N_0 C \text{ induct} : \sigma \underset{\Omega}{=} (\bar{\tau} + 1) :$ (1)

$\vdash$. Transp . 119·11 $\supset \vdash . \sigma\varepsilon_1(N_0 C \text{ induct} \underset{1}{-} \iota^{c}_1 0) : \sigma \underset{\Omega}{=} \Lambda'' . \supset$

$: (\sigma - 1) \underset{\Omega}{=} \Lambda'' : (\sigma - 1)\varepsilon_1 N_0 C$ induct.

[Hp . 110·202] $\supset (\exists\bar{\tau}) . \bar{\tau}\varepsilon_1 N_0 C \text{ induct} : \sigma \underset{\Omega}{=} (\bar{\tau} + 1) :$ (2)

$\vdash$ (1) . (2) $\supset \vdash \sigma\varepsilon_1(N_0 C \text{induct} \underset{1}{-} \iota^{c}_1 0) \supset (\exists\bar{\tau}) . \bar{\tau}\varepsilon_1 N_0 C \text{induct} :$

$\sigma \underset{\Omega}{=} (\bar{\tau} + 1) :$ (3)

$\vdash$ 120·422·124 $\supset \vdash (\exists\bar{\tau}) . \bar{\tau}\varepsilon_1 N_0 C \text{induct} : \sigma \underset{\Omega}{=} (\bar{\tau} + 1) : \supset$

$\sigma\varepsilon_1(N_0 C \text{ induct} \underset{1}{-} \iota^{c}_1 0)$ (4)

$\vdash$ (3) · (4) $\supset \vdash$ Prop.

120·4231 $\vdash \sigma \varepsilon_1 N_0 C$ induct $\supset (\exists \bar{\tau}) . \bar{\tau} \varepsilon_1 (N_0 C \text{ induct} \underset{1}{-} \iota^{\text{c}}{}_1 \wedge'') :.$
$(\sigma + 1) \underset{\Omega}{=} (\bar{\tau} + 1) :$

Dem $\vdash . \text{Hp} . \exists'(\sigma + 1) \supset (\exists \bar{\tau}) . \bar{\tau} \varepsilon_1 (NC \text{ induct} \underset{1}{-} \iota^{\text{c}}{}_1 \wedge'') :.$
$(\sigma + 1) \underset{\Omega}{=} (\bar{\tau} + 1) :$ (1)

$\vdash$ 120·302·303 . $\supset \vdash . \text{Hp} : (\sigma + 1) \underset{\Omega}{=} \wedge'' : \supset$
$. Nc^{\text{c}}(\underset{a}{-} \wedge') \varepsilon_1 (NC \text{ induct} \underset{1}{-} \iota^{\text{c}}{}_1 \wedge'') :. (\sigma + 1) \underset{\Omega}{=} (Nc^{\text{c}}(\underset{a}{-} \wedge') + 1) :$
$\supset (\exists \bar{\tau}) . \bar{\tau} \varepsilon_1 (N_0 C \text{ induct} \underset{1}{-} \iota^{\text{c}}{}_1 \wedge'') :. (\sigma + 1) \underset{\Omega}{=} (\bar{\tau} + 1) :$ (2)
$\vdash$ (1) . (2) . $\supset \vdash$ Prop.

120·4232 $\vdash \sigma \varepsilon_1 (N_0 C \text{ induct} \underset{1}{-} \iota^{\text{c}}{}_1 0) \equiv (\exists \bar{\tau}) . \bar{\tau} \varepsilon_1 (N_0 C \text{ induct} \underset{1}{-} \iota^{\text{c}}{}_1 \wedge'') :$
$\sigma \underset{\Omega}{=} (\bar{\tau} + 1) :$ [120·423·4231]

120·424 $\vdash : \sigma \underset{\Omega}{\neq} 0 : \exists'(\tau + \sigma) . \supset . ((\tau + \sigma) - 1) \underset{\Omega}{=} (\tau + (\sigma - 1)) .$
[110·42·62 . 120·414 . 110·4 . 120·416 . 110·56 . 120·311 .]

120·425 $\vdash (\sigma + \tau) \varepsilon_1 (NC \underset{1}{-} \iota^{\text{c}}{}_1 0) \supset : ((\sigma + \tau) - 1) \underset{\Omega}{=} (\sigma + (\tau - 1)) . \vee .$
$. ((\sigma + \tau) - 1) \underset{\Omega}{=} ((\sigma - 1) + \tau) :$ [110·62 . 120·424 . 103·2 .]

120·426 $\vdash . \varkappa \varepsilon_1 Cls_0 \text{ induct} : \varkappa \underset{a}{\subset} \omega : \exists_a (\omega \cap_a - \varkappa) . \exists' Nc^{\text{c}} \omega .$
$\supset : Nc^{\text{c}} \varkappa < Nc^{\text{c}} \omega : \sim \omega \, sm \, \varkappa .$
[110·14 . 101·14 . 120·42 . (117·05) .]

120·427 $\vdash . R \varepsilon 1 \underset{a}{\rightarrow} 1 : G_a R \underset{a}{\subset} D_a R : \exists_a (D_a R - G_a R) . \mathfrak{C}\{R, C\} .$
$Red (D_a R) . \supset \sim D_a R \varepsilon' Cls_0$ induct
[120·426 . Transp .]

120·428 $\vdash . \sigma \varepsilon_1 N_0 C \text{ induct} . \exists'(\sigma + \tau) : \tau \underset{\Omega}{\neq} 0 : \supset . (\sigma + \tau) > \sigma .$
[117·511 . 110·4 . 117·561 . 120·42 .]

120·429 $\vdash \sigma \varepsilon_1 N_0 C \text{ induct} \supset : \tau > \sigma . \equiv . \tau \geqslant (\sigma + 1) .$
[120·428 . 117·47·31·531]

120·43 $. \sigma \text{ spec } \tau . \underset{df}{=} : \sigma < \tau . \vee . \sigma \geqslant \tau :$

120·432 $\vdash . \sigma \text{ spec } \tau \equiv : \sigma \leqslant \tau . \vee . \sigma \geqslant \tau :$ [117·281]

120·436 $\vdash . \sigma \text{ spec } \sigma' . \equiv . \sigma, \sigma' \varepsilon_1 NC . (\exists \bar{\tau}) :. (\sigma + \bar{\tau}) \underset{\Omega}{=} \sigma' . \vee . (\sigma' + \bar{\tau}) \underset{\Omega}{=} \sigma :$
$. \mathfrak{C}\{\bar{\tau}, 1\} :$ [120·432 . 117·31 .]

120·437 $\vdash \sigma \varepsilon_1 NC \supset . 0 \text{ spec } \sigma .$ [120·432 . 117·281]

120·438 $\vdash . \sigma \text{ spec } \tau . \exists'(\sigma + 1) . \supset . (\sigma + 1) \text{ spec } \tau .$
[120·436·417 . 110·4·61 .]

120·44 $\vdash . \sigma \varepsilon_1 NC . \tau \varepsilon_1 (N_0 C \text{induct} \underset{1}{-} \iota'_1 \Lambda'') . \supset \tau \,\text{spec}\, \sigma$.

[120·37·438 . 110·4 .]

120·442 $\vdash . \sigma \varepsilon_1 (NC \text{induct} \underset{1}{-} \iota'_1 \Lambda'') . \tau \varepsilon_1 NC . \supset$

$: . \sigma < \tau . \equiv \sim . \sigma \geqslant \tau :: . \sigma > \tau . \equiv \sim . \sigma \leqslant \tau :.$

[120·44 . 117·281 .]

120·443 $\vdash \mathfrak{C}\{ \Phi, \hat{\sigma}[. \Phi\{(\tau' + \hat{\sigma})\} . \mathfrak{C}\{\tau', \hat{\sigma}, 1\} .]\}$

Constr.

$\vdash . 12·1 . 0·261 . \supset \vdash \mathfrak{C}\{ \hat{\tau}[. \Phi\{\sigma'\} | \Phi\{\hat{\tau}\} .], \hat{\tau}[. \Phi\{\sigma'\} | \Phi\{\hat{\tau}\}]\}$

$[0·41·441 . 0·243] \supset \mathfrak{C}\{\hat{\tau}[. \Phi\{\sigma'\} | \Phi\{\hat{\tau}\} .], \hat{\tau}[: \Phi\{(\tau' + \sigma')\} \, \mathfrak{C}\{\tau', \sigma', 1\} . | \Phi\{\hat{\tau}\} .]\}$

$[0·23·26·261] \supset \vdash \mathfrak{C}\{\hat{\sigma}[. \Phi\{\hat{\sigma}\} | \Phi\{\tau\} .], \hat{\sigma}[: \Phi\{\tau' + \hat{\sigma})\} \, \mathfrak{C}\{\tau', \hat{\sigma}, 1\} . | \Phi\{\tau\}]\}$

$[0·2411] \supset \vdash$ Prop.

We prove in a similar manner:

120·444 $\vdash \mathfrak{C}\{ \Phi, \hat{\sigma}[. \Phi\{(\tau' \times \hat{\sigma})\} . \mathfrak{C}\{\tau', \hat{\sigma}, 1\}]\}$

120·445 $\vdash \mathfrak{C}\{ \Phi, \hat{\sigma}[\, \Phi\{\tau^{\hat{\sigma}})\} \, \mathfrak{C}\{\tau, \hat{\sigma}, 1\} .]\}$

120·446 $\vdash \mathfrak{C}\{ \Phi, \hat{\sigma}[. \Phi\{\tau^{(\tau' + \hat{\sigma})}\} . \mathfrak{C}\{\tau, \tau', \hat{\sigma}, 1\} .]\}$

120·447 $\vdash \mathfrak{C}\{\hat{\sigma}[(\bar{\tau}) . N_0 C \text{induct}\{(\bar{\tau} + \hat{\sigma})\} . \mathfrak{C}\{\bar{\tau}, \hat{\sigma}, 1\} .], N_0 C \text{induct}\}$

Constr.

$\vdash 12·3121·3122 . \supset \vdash \mathfrak{C}\{\hat{\sigma}[(\bar{h}) : (\tau' + \hat{\sigma}) \varepsilon_1 \bar{h} | (\bar{\tau}) . (\bar{\tau} + \hat{\sigma}) \varepsilon_1 \bar{h} | \mathfrak{C}\{\tau', \bar{\tau}, \hat{\sigma}, 1\} :$

$| \mathfrak{C}\{\bar{h}, \Phi\} .], N_0 C \text{induct}\}$

$[0·27·12·2421] \supset \vdash \mathfrak{C}\{\hat{\sigma}[(\bar{h}) (\bar{\tau}) : (\tau' + \hat{\sigma}) \varepsilon_1 \bar{h} | . (\bar{\tau} + \hat{\sigma}) \varepsilon_1 \bar{h} | \mathfrak{C}\{\tau', \bar{\tau}, \hat{\sigma}, 1\} :$

$| \mathfrak{C}\{\bar{h}, \Phi\} .], N_0 C \text{induct}\}$

$[12·5] \supset \vdash \mathfrak{C}\{\hat{\sigma}[(\bar{h}) (\bar{\tau}) (\bar{\tau}') : (\bar{\tau}' + \hat{\sigma}) \varepsilon_1 \bar{h} | . (\bar{\tau} + \hat{\sigma}) \varepsilon_1 \bar{h}' | \mathfrak{C}\{\bar{\tau}', \bar{\tau}, \hat{\sigma}, 1\} :$

$| \mathfrak{C}\{\bar{h}, \Phi\} .], N_0 C \text{induct}\}$

$[0·24·27 . 12·51 .] \supset \vdash \mathfrak{C}\{\hat{\sigma}[(\bar{\tau})(\bar{h}) : (\bar{\tau} + \hat{\sigma}) \varepsilon_1 \bar{h} | (\bar{\tau}') . (\bar{\tau} + \hat{\sigma}) \varepsilon_1 \bar{h} | \mathfrak{C}\{\bar{\tau}', \bar{\tau}, \hat{\sigma}, 1\} :$

$| \mathfrak{C}\{\bar{h}, \Phi\} .], N_0 C \text{induct}\}$

$[(1) . 120·443] \supset \vdash$ Prop.

120·45 $\vdash . H(g, 0) . \sigma \varepsilon_1 g . \tau \varepsilon_1 N_0 C \text{induct} . \mathfrak{C}\{g, \Phi\} . \supset (\sigma + \tau) \varepsilon_1 g$

Dem $\vdash . 110·6·02 . 103·23 . \supset \vdash \text{Hp} \supset (\sigma + 0) \varepsilon_1 g$ (1)

$\vdash . 120·121 . 110·56 . \supset \vdash \text{Hp} \supset : (\sigma + \sigma') \varepsilon_1 g \supset (\sigma + (\sigma' + 1)) \varepsilon_1 g : \mathfrak{C}\{\sigma', 1\}$. (2)

$\vdash . (1) . (2) . \supset \vdash \text{Hp} \supset H(\hat{\sigma}'[. g\{(\sigma + \hat{\sigma}')\} . \mathfrak{C}\{\hat{\sigma}', \sigma, 1\} .], 0)$

$[120·1·443] \qquad \supset g\{(\sigma + \tau)\}$

$[\text{Hp} . 20·042 .] \supset \vdash$ Prop.

Remark that this proposition is much more general than its correlate of Principia. We shall see its importance for the further development of our theory.

$120{\cdot}4502 \quad \vdash \sigma, \tau \varepsilon_1 N_0 C\,\text{induct} \supset (\sigma + \tau)\, \varepsilon_1 N_0 C\,\text{induct}$

Dem $\vdash 120{\cdot}45 \supset \vdash . \text{Hp} . H(g, 0) . \mathfrak{C}\{g, \Phi\} . \supset (\sigma + \tau) \varepsilon_1 g$

$[120{\cdot}1] \supset \vdash$ Prop.

$120{\cdot}451 \quad \vdash . (\overline{\sigma}, \overline{\tau}) :. \sigma'' \underset{\Omega}{=} (\overline{\sigma} + \overline{\tau}) : H(g, 0) . \supset \overline{\sigma}, \overline{\tau} \varepsilon_1 (g \underset{1}{-} \iota'_1 \Lambda'') : \exists'(\sigma'' + 1).$

$\mathfrak{C}\{\sigma', \tau', 1\} : (\sigma'' + 1) \underset{\Omega}{=} (\sigma' + \tau') : \supset \sigma', \tau' \varepsilon_1 (g \underset{1}{-} \iota'_1 \Lambda'')$

$[120{\cdot}414{\cdot}124{\cdot}425 . 110{\cdot}42 . 119{\cdot}32{\cdot}11 .]$

$120{\cdot}452 \quad \vdash . (\sigma + \tau) \varepsilon_1 (N_0 C\,\text{induct} \underset{1}{-} \iota'_1 \Lambda'') . H(g, 0) . \supset \sigma, \tau \varepsilon_1 (g \underset{1}{-} \iota' \Lambda'')$

Dem $\vdash . 120{\cdot}451{\cdot}12 . 110{\cdot}62 . \supset$

$\vdash \text{Hp} \supset H(\hat{\sigma}'[(\overline{\sigma}'', \overline{\tau}'') :. \hat{\sigma}' \underset{\Omega}{=} \Lambda'' . \vee . \hat{\sigma}' \underset{\Omega}{=} (\overline{\sigma}'' + \overline{\tau}'') :. \mathfrak{C}\{\hat{\sigma}', 1\} . \supset .$

$(g \underset{1}{-} \iota'_1 \Lambda'') \{\overline{\tau}''\} . (g \underset{1}{-} \iota'_1 \Lambda'') \{\overline{\sigma}''\} .], 0)$

$[120{\cdot}12] \supset \vdash \text{Hp} \supset :. \sigma' \underset{\Omega}{=} \Lambda'' . \vee . \sigma' \underset{\Omega}{=} (\sigma + \tau) : \supset \sigma, \tau \varepsilon_1 (g \underset{1}{-} \iota'_1 \Lambda'').$

$\supset :. (\sigma + \tau) \underset{\Omega}{=} \Lambda'' . \vee . (\sigma + \tau) \underset{\Omega}{=} (\sigma + \tau) : \supset \sigma, \tau \varepsilon_1 (g \underset{1}{-} \iota'_1 \Lambda'').$

$\supset \vdash$ Prop.

$120{\cdot}46 \quad \vdash . \sigma \varepsilon_1 (NC \cup_1 \iota'_1 \Lambda'') . \tau \varepsilon_1 NC\,\text{induct} . H(g, \sigma) . \supset (\sigma + \tau) \varepsilon_1 g$

Dem $\vdash 110{\cdot}6 \supset \vdash \text{Hp} \supset g\{(\sigma + 0)\} \quad (1)$

$\vdash 110{\cdot}56 \qquad \supset \vdash \text{Hp} \supset H(\hat{\tau}'[. g\{(\sigma + \hat{\tau}')\} . \mathfrak{C}\{\hat{\tau}', 1\}], 0)$

$[\text{Hp}] \qquad \supset g\{(\sigma + \tau)\}$

$[\text{Hp} . 20{\cdot}042 .] \supset \vdash$ Prop.

$120{\cdot}4601 \quad \vdash . \sigma' \varepsilon_1 (NC \cup_1 \iota'_1 \Lambda'') . (\overline{g}) : H(\overline{g}, \sigma') \supset \tau \varepsilon_1 \overline{g} : \mathfrak{C}\{\overline{g}, \Phi\} :$

$\supset (\exists \overline{\sigma}) : \tau \underset{\Omega}{=} (\sigma' + \overline{\sigma}) : \mathfrak{C}\{\overline{\sigma}, 1\}.$

Dem $\vdash 110{\cdot}61 \supset \vdash \text{Hp} \supset . \sigma' \underset{\Omega}{=} (\sigma' + 0).$

$\supset (\exists \overline{\sigma}) : \sigma' \underset{\Omega}{=} (\sigma' + \overline{\sigma}) : \mathfrak{C}\{\overline{\sigma}, 1\}.$

$\vdash 110{\cdot}56 \supset \vdash \text{Hp} \supset : \tau \underset{\Omega}{=} (\sigma' + \sigma) . \supset . (\tau + 1) \underset{\Omega}{=} (\sigma' + (\sigma + 1)) :$

$\supset . (\exists \overline{\sigma}) : \tau \underset{\Omega}{=} (\sigma' + \overline{\sigma}) : \mathfrak{C}\{\overline{\sigma}, 1\} . \supset (\exists \overline{\sigma}) : (\tau + 1) \underset{\Omega}{=} (\sigma' + \overline{\sigma} : \mathfrak{C}\{\overline{\sigma}, 1\} . \quad (2)$

$\vdash . (1) . (2) . \supset \vdash \text{Hp} \supset$

$H(\hat{\tau}[. (\exists \overline{\sigma}) : \hat{\tau} \underset{\Omega}{=} (\sigma' + \overline{\sigma}) : \mathfrak{C}\{\overline{\sigma}, \hat{\tau}, 1\} . \vee (\Phi \cap_1 \Lambda_{(1)}) \{\hat{\tau}\} .], 0)$

$[\text{Hp}] \supset \vdash$ Prop

$120{\cdot}461 \quad \vdash . \sigma' \varepsilon_1 N_0 C\,\text{induct} . (\overline{g}) : H(\overline{g}, \sigma') \supset \tau \varepsilon_1 \overline{g} : \mathfrak{C}\{\overline{g}, \Phi\} :$

$\supset (\exists \overline{\sigma}) . \overline{\sigma} \varepsilon_1 N_0 C\,\text{induct} : \tau \underset{\Omega}{=} (\sigma' + \overline{\sigma}) :$

Dem $\vdash 120{\cdot}11 \supset \vdash . \text{Hp} . H(g, 0) . \mathfrak{C}\{g, \Phi\} . \supset H(g, \sigma')$

$\supset \vdash \text{Hp} \supset :. H(g, \sigma') \supset \tau \varepsilon_1 g . \supset . H(g, 0) \supset \tau \varepsilon_1 g :. \mathfrak{C}\{g, \Phi\}.$

$\supset \tau \varepsilon_1 N_0 C\,\text{induct}$

[120·4601] $\supset . \tau\varepsilon_1 N_0 C$induct $(\exists\bar{\sigma}) : \tau \underset{\Omega}{=} (\sigma' + \bar{\sigma}) : \mathfrak{C}\{\bar{\sigma}', 1\} :$ (1)

$\vdash .(1).120{\cdot}452. \supset \vdash . \mathrm{Hp} . \exists'\tau . \supset (\exists\bar{\sigma}) . \bar{\sigma}\varepsilon_1 N_0 C$induct$: \tau \underset{\Omega}{=} (\sigma' + \bar{\sigma}) :$ (2)

$\vdash \mathrm{Hp} \supset \vdash . \mathrm{Hp} : \tau \underset{\Omega}{=} \wedge'' : \supset \wedge'' \varepsilon_1 N_0 C$induct$: \tau \underset{\Omega}{=} (\sigma' + \wedge'') :$

$\supset (\exists\bar{\sigma}) . \bar{\sigma}\varepsilon_1 N_0 C$induct$: \tau \underset{\Omega}{=} (\sigma' + \bar{\sigma}) :$ (2)

$\vdash . (1) . (2) . \supset \vdash$ Prop.

120·4611 $\vdash . \sigma'\varepsilon_1 N_0 C$induct $. (\exists\bar{\sigma}) . \bar{\sigma}\varepsilon_1 N_0 C$induct$: \tau \underset{\Omega}{=} (\sigma' + \bar{\sigma}) :$

$\supset (\bar{g}) : H(\bar{g}, \sigma') \supset \tau\varepsilon_1 \bar{g} : \mathfrak{C}\{\bar{g}, \Phi\}.$

Dem $\vdash . \mathfrak{C}\{g, \Phi\} . H(g, \sigma') . \sigma', \sigma\varepsilon_1 N_0 C$induct$: \tau = (\sigma' + \sigma) : \supset g\{(\sigma' + 0)\}$ (1)

[110·56] $\supset . g\{(\sigma' + \tau')\} \supset g\{(\sigma' + (\tau' + 1))\}.$ (2)

$\supset H(\hat{\tau}'[. g\{(\sigma' + \hat{\tau}')\} . \mathfrak{C}\{\hat{\tau}', 1\} .], 0)$

[120·11] $\supset g\{(\sigma' + \sigma)\}$

[Hp] $\supset g\{\tau\}$ (3)

$\vdash (3) \supset \vdash . \sigma', \sigma\varepsilon_1 N_0 C$induct$: \tau \underset{\Omega}{=} (\sigma' + \sigma) : \supset : H(g, \sigma') \supset \tau\varepsilon_1 g : \mathfrak{C}\{g, \Phi\}.$

$\supset \vdash$ Prop.

120·463 $\vdash \sigma'\varepsilon_1 N_0 C$induct $\supset . (\exists\bar{\sigma}) . \bar{\sigma}\varepsilon_1 N_0 C$induct$: \tau \underset{\Omega}{=} (\sigma' + \bar{\sigma}) : \equiv$

$(\bar{g}) : H(\bar{g}, \sigma) \supset \tau\varepsilon_1 \bar{g} : \mathfrak{C}\{\bar{g}, \Phi\} :$

Dem $\vdash . 120{\cdot}461{\cdot}4611 . \supset \vdash$ Prop.

120·47 $\vdash \tau\varepsilon (N_0 C$ induct $\underset{1}{-} \iota^{\mathfrak{c}}{}_1 0) \equiv (\bar{g}) : H(\bar{g}, 1) \supset \tau\varepsilon_1 \bar{g} : \mathfrak{C}\{\bar{g}, \Phi\}.$

[120·423·463]

120·471 $\vdash (\exists\bar{\sigma}) . \bar{\sigma}\varepsilon_1 (N_0 C$induct $\underset{1}{-} \iota^{\mathfrak{c}}{}_1 0) . \bar{\sigma}\varepsilon_1 f . \equiv$

$(\exists\bar{\sigma}) . \bar{\sigma}\varepsilon N_0 C$ induct $. (\bar{\sigma} + 1)\varepsilon_1 f :$

[120·423] .

120·48 $\vdash . \sigma\varepsilon_1 N_0 C$induct$: \sigma \geqslant \tau : \supset \tau\varepsilon_1 (N_0 C$induct $\underset{1}{-} \iota^{\mathfrak{c}}{}_1 \wedge'')$

[120·452 . 117·31 .]

120·481 $\vdash . \varkappa\varepsilon' Cls_0$ induct$: \omega \underset{a}{\subset} \varkappa : \exists' Nc^{\mathfrak{c}}\omega . \supset \omega\varepsilon' Cls_0$ induct

[120·21·48]

120·49 $\vdash . \sigma\varepsilon_1 (NC \underset{1}{-} N_0 C$induct$) . \tau\varepsilon (N_0 C$induct $\underset{1}{-} \iota^{\mathfrak{c}}{}_1 \wedge'') .$

$\supset . \sigma > \tau .$

[120·48·44]

To prove the theorem 120·13, I assume the following temporary definition:

120·013 $H_0(g, 0) \underset{df}{=} . 0\varepsilon_1 g . (\bar{\tau}) : \bar{\tau}\varepsilon_1 g . \bar{\tau}\varepsilon_1 N_0 C$induct $. \supset$

$(\bar{\tau} + 1)\varepsilon_1 g : \mathfrak{C}\{g, \Phi\}.$

We now have:

120·13 $\vdash . \sigma\varepsilon_1 N_0 C\,\text{induct} . H_0(g,0) . \supset \sigma\varepsilon_1 g$

Dem $\vdash . 120{\cdot}124{\cdot}48 . \supset$

$\vdash . \text{Hp} . \sim\sigma\varepsilon_1 g . \supset : \tau\varepsilon_1 g : \tau \leqslant \sigma : \supset \tau\varepsilon_1 N_0 C\,\text{induct} .$

$[22{\cdot}621] \supset : \tau\varepsilon_1 g : \tau \leqslant \sigma : \equiv . \tau\varepsilon_1 g : \tau < \sigma : \tau\varepsilon_1 N_0 C\,\text{induct} :$ (1)

$\vdash . 117{\cdot}501 . 120{\cdot}14 . (1) . \supset \vdash . \text{Hp} . \sim\sigma\varepsilon_1 g . \supset 0\varepsilon_1 g : 0 \leqslant \sigma :$ (2)

$\vdash . (1) . 120{\cdot}429 . \supset \vdash . \text{Hp} . \sim\sigma\varepsilon_1 g . \supset$

$: \tau\varepsilon_1 g : \tau \leqslant \sigma : \supset . (\tau+1)\varepsilon_1 g : (\tau+1) \leqslant \sigma :.$ (3)

$\vdash . (2) . (3) . 120{\cdot}11 . \supset \vdash . \text{Hp} . \sim\sigma\varepsilon_1 g . \supset \sigma\varepsilon_1 g$

$\supset \vdash$ Prop.

120·23 $\vdash . \Lambda'\varepsilon' h . J(h) . \mathfrak{C}\{h, \hat{\omega}[\Phi\{Nc^{\text{c}}\hat{\omega}\}]\} . \supset . Cls_0\,\text{induct} \subset_{\Omega} h .$ [120·222·13]

120·24 $\vdash \omega\varepsilon' Cls_0\,\text{induct} \equiv (\bar{h}) :: \Lambda'\varepsilon'\bar{h} . J(\bar{h}) . \supset \omega\varepsilon'\bar{h} :$
$\mathfrak{C}\{\bar{h}, \hat{\omega}[\Phi\{Nc^{\text{c}}\hat{\omega}\}]\} . \supset \omega\varepsilon'\bar{h} .$ [120·23·22]

120·2601 $J_0(h) \underset{df}{=} (\bar{\varkappa})(\bar{u}) : \bar{\varkappa}\varepsilon' h . \bar{\varkappa}\varepsilon' Cls_0\,\text{induct} . \text{extens}(\bar{\varkappa}) .$
$\supset (\bar{\varkappa} \cup_a \iota^{\text{c}}\bar{u})\varepsilon' h : \mathfrak{C}\{\bar{u}, a\} .$

120·261 $\vdash . \omega\varepsilon_1 Cls_0\,\text{induct} . \Lambda'\varepsilon' h . J_0(h) . \mathfrak{C}\{h\ \hat{\omega}[\Phi\{Nc^{\text{c}}\hat{\omega}\}]\} \supset \omega\varepsilon' h$

Dem $\vdash . \text{Hp} . \sim\omega\varepsilon' h . \supset : \varkappa\varepsilon' h : Nc^{\text{c}}\varkappa \leqslant Nc^{\text{c}}\omega : \equiv .$
$\varkappa\varepsilon' h : Nc^{\text{c}}\varkappa < Nc^{\text{c}}\omega : \varkappa\varepsilon' Cls_0\,\text{induct} :$ (1)

$\vdash . 117{\cdot}501 . 120{\cdot}14 . (1) . \supset \vdash . \text{Hp} . \sim\omega\varepsilon' h . \supset . \Lambda'\varepsilon' h : Nc^{\text{c}}\Lambda' \leqslant Nc^{\text{c}}\omega :$ (2)

$\vdash . 101{\cdot}1005 . (1) . 120{\cdot}429 . \supset \vdash . \text{Hp} . \sim\omega\varepsilon' h . \supset$

$: \varkappa\varepsilon' h : Nc^{\text{c}}\varkappa \leqslant Nc^{\text{c}}\omega : \supset . (\varkappa \cup_a \iota^{\text{c}}{}_a u)\varepsilon' h : Nc^{\text{c}}(\varkappa \cup_a \iota^{\text{c}} u) \leqslant Nc^{\text{c}}\omega :.$ (3)

$\vdash . (2) . (3) . 120{\cdot}23 . \supset \vdash . \text{Hp} . \sim\omega\varepsilon' h . \supset \omega\varepsilon' h$

$\supset \vdash$ Prop.

120·473 $\vdash . \sigma\varepsilon_1(N_0 C\,\text{induct} \underset{1}{-} \iota^{\text{c}}{}_0 0) . 1\varepsilon_1 g . (\bar{\tau}) : \bar{\tau}\varepsilon_1 g .$
$\bar{\tau}\varepsilon_1 (N_0 C\,\text{induct} \underset{1}{-} \iota^{\text{c}}{}_1 0) . \supset (\bar{\tau}+1)\varepsilon_1 g : \mathfrak{C}\{g, \Phi\} . \supset \sigma\varepsilon_1 g$

This proposition is to be proved by the same method as 120·13.

120·491 $\vdash \sim\varkappa\varepsilon' Cls_0\,\text{induct} \equiv (\bar{\sigma}) . \bar{\sigma}\varepsilon_1 NC\,\text{induct} \supset$
$(\exists\bar{\omega}) : \bar{\omega} \subset_a \varkappa : \bar{\omega}\varepsilon'\bar{\sigma} :$

[120·49·429·13·121·21 . 101·11 . 117·42.]

120·493 $\vdash \varkappa\varepsilon' Cls_0\,\text{induct} \supset : Nc^{\text{c}}\omega < Nc^{\text{c}}\varkappa \equiv (\exists\bar{\omega}') . \bar{\omega}'\varepsilon' Nc^{\text{c}}\omega :$
$\bar{\omega}' \subset_a \varkappa : \sim\bar{\omega}'\varepsilon'\iota^{\text{c}}{}_{\Omega}\varkappa :$ [120·481·426 . 73·37·3 . 103·14.]

120·501 $\vdash \sigma, \tau\varepsilon_1 N_0 C\,\text{induct} \supset (\sigma\times\tau)\varepsilon_1 N_0 C\,\text{induct}$

Dem $\vdash . 120{\cdot}14 . 103{\cdot}23 . 113{\cdot}601 . \supset$

$\vdash \text{Hp} \supset : (\sigma\times 0) \underset{\Omega}{=} 0 . \vee : (\sigma\times 0) \underset{\Omega}{=} \sigma :. \sigma \underset{\Omega}{=} \Lambda'' :.$

[120·12] $\vdash . \sigma\varepsilon_1 N_0 C$ induct . $H(g, 0) . \supset g\{(\sigma \times 0)\}$ (1)

$\vdash$ 113 $\supset \vdash . \sigma \underset{\Omega}{\neq} 0 . \supset . (\sigma\times(\tau'+1)) \underset{\Omega}{=} ((\sigma\times\tau')+\sigma)$. (2)

$\vdash . (2) . 120·45 . \supset \vdash : \sigma \underset{\Omega}{\neq} 0 :$ Hp . $H(g,0) . \mathfrak{C}\{g, \Phi\} . \supset$

$. g\{(\sigma\times\tau')\} \supset g\{(\sigma\times(\tau'+1))\}$. (3)

$\vdash (1).(3). \supset \vdash : \sigma \underset{\Omega}{\neq} 0 :$ Hp . $H(g,0) . \mathfrak{C}\{g, \Phi\}$.

$\supset H(\hat{\tau}'[.g\{(\sigma\times\hat{\tau}')\} . \mathfrak{C}\{\hat{\tau}'.1)\}.], 0)$

[120·11] $\supset g\{(\sigma\times\tau)\}$ (4)

$\vdash$ 113·601 $\supset \vdash : \sigma \underset{\Omega}{=} 0 :$ Hp $\supset (\sigma\times\tau)\varepsilon_1 N_0 C$ induct (5)

$\vdash . (4) . (5) . \supset \vdash$ Prop.

This is the second fundamental theorem concerning the group-properties of inductive numbers.

120·51 $\vdash . \sigma, \sigma', \tau\varepsilon_1 N_0 C$ induct: $\tau \underset{\Omega}{\neq} 0 : \exists'(\tau\times\sigma) : (\tau\times\sigma) \underset{\Omega}{=} (\tau\times\sigma') :$

$\supset . \sigma \underset{\Omega}{=} \sigma'$.

[120·436·44 . 113·43·602·203 . 110·6.]

120·511 $\vdash . \sigma, \tau\varepsilon_1 N_0 C$ induct : $\tau \underset{\Omega}{\neq} 0 : \exists'\tau : (\tau\times\sigma) \underset{\Omega}{=} \tau : \supset . \sigma \underset{\Omega}{=} 1$.

[120·51 . 113·621.]

120·512 $\vdash (\sigma\times\tau)\varepsilon_1((N_0 C \text{induct} \underset{1}{-} \iota^{\epsilon}{}_1 0) \underset{1}{-} \iota^{\epsilon}{}_1 \wedge'')$

$\supset \sigma, \tau\varepsilon_1((N_0 C \text{induct} \underset{1}{-} \iota^{\epsilon}{}_1 0) \underset{1}{-} \iota^{\epsilon}{}_1 \wedge'')$

[120·48 . 113·602·203 . 117·62.]

120·513 $\vdash . \tau\varepsilon_1((N_0 C \text{induct} \underset{1}{-} \iota^{\epsilon}{}_1 0) \underset{1}{-} \iota^{\epsilon}{}_1 \wedge'') : (\tau\times\sigma) = \tau :$

$\supset . \sigma \underset{\Omega}{=} 1$. [120·511·512]

The axiom of infinity is to be defined as follows:

120·03 $\text{Infin}_0 ax \underset{df}{=} (\bar{\sigma}) . \bar{\sigma}\varepsilon_1 N_0 C \text{induct} \supset \exists'\bar{\sigma}$.

120·031 $\text{Infin}\, ax \underset{df}{=} (\bar{\sigma}) . \bar{\sigma}\varepsilon_1 NC \text{induct} \supset \exists'\bar{\sigma}$.

We have the following propositions:

120·321 $\vdash . \sigma \underset{\Omega}{\neq} (\sigma+1) . \supset \exists'\sigma$ [110·4]

120·322 $\vdash \sigma\varepsilon_1 N_0 C \text{induct} \supset . \exists'\sigma \equiv . \sigma \underset{\Omega}{\neq} (\sigma+1)$: [120·32·321]

120·33 $\vdash \text{Infin}_0 ax \equiv (\bar{\sigma}) . \bar{\sigma}\varepsilon_1 N_0 C \text{induct} \supset . \bar{\sigma} \underset{\Omega}{\neq} (\bar{\sigma}+1)$: [120·322]

125·12 $\vdash \text{Infin}_0 ax \equiv (\bar{\sigma}) . \bar{\sigma}\varepsilon_1 N_0 C \text{induct} \supset \exists'(\bar{\sigma}+1)$.

[120·423)

125·15 $\vdash \text{Infin}_0 ax \equiv (\bar{\varkappa}) . \bar{\varkappa}\varepsilon_1 Cls_0 \text{induct} \supset \exists_a(\underset{a}{-}\bar{\varkappa})$.

[120·13·21·426 . 125·12.]

125·21 $\vdash$ $\text{Infin}_0\, ax \equiv \sim (\underset{a}{-} \wedge') \, \varepsilon' Cls_0 \, \text{induct}$ [125·15]

125·211 $\vdash$ $\text{Infin}\, ax \equiv \sim (\underset{a}{-} \wedge') \, \varepsilon' Cls \, \text{induct}$

I shall use the following abbreviations:

120·51401 $(\sigma + \sigma' + \tau) \underset{df}{=} ((\sigma + \sigma') + \tau)$

120·5141 $(\sigma + \sigma' - \tau) \underset{df}{=} ((\sigma + \sigma') - \tau)$

120·5142 $(\sigma - \sigma' - \tau) \underset{df}{=} ((\sigma - \sigma'(- \tau)$

120·5143 $(\sigma - \sigma' - \tau' + \tau'') \underset{df}{=} ((\sigma - \sigma') - \tau' + \tau'')$

120·5144 $\vdash$ $\sigma \varepsilon_1 N_0 C \text{induct} \supset . (\sigma - \sigma) \underset{\Omega}{=} 0.$

[110·6 . 120·412·48 .]

120·5145 $\vdash$ $. \sigma' \varepsilon_1 N_0 C \text{induct} . \exists'(\sigma' - (\sigma + 1)) . \supset .$

$(\sigma' - (\sigma + 1)) \underset{\Omega}{=} (\sigma' - \sigma - 1).$

Dem $\vdash$. 120·411·48 . 119·34 . $\supset \vdash$ Hp $\supset . (\sigma' - (\sigma + 1) + (\sigma + 1)) \underset{\Omega}{=} \sigma'.$

[110·51·56] $\supset . ((\sigma' - (\sigma + 1) + 1) + \sigma) \underset{\Omega}{=} \sigma'.$ (1)

[110·6 . 120·5144 .] $\supset . (\sigma' - (\sigma + 1) + 1) \underset{\Omega}{=} (\sigma' - \sigma).$ (2)

$\vdash$. (1) . (2) . 119·45 . $\supset \vdash$ Prop.

120·5146 $\vdash$ $. \sigma' \varepsilon_1 N_0 C \text{induct} . \exists'(\sigma' - (\sigma + 1)) . \supset$

$. (\sigma' + 1 - (\sigma + 1)) \underset{\Omega}{=} (\sigma' - \sigma).$

Dem $\vdash$ 110·51 $\supset \vdash$ Hp $\supset . (\sigma' + 1 - (\sigma + 1)) \underset{\Omega}{=} (1 + \sigma' - (\sigma + 1)).$

[119·32] $\underset{\Omega}{=} (1 + (\sigma' - (\sigma + 1))).$

[120·5145] $\underset{\Omega}{=} (1 + (\sigma' - \sigma - 1)).$

[110·51 . (120·5141) . (120·5143) .] $\underset{\Omega}{=} (\sigma' - \sigma - 1 + 1).$

[119·34] $\supset \vdash$ Prop.

120·515 $\vdash$. $\text{Infin}\, ax .$

$(\overline{g}) :. H(\overline{g}, 0) : g \underset{1}{\subset} (N_0 C \text{induct} \underset{1}{-} \iota'_1 \wedge'') : \supset \overline{g}\{\sigma\} : \mathfrak{C}\{\overline{g}, N_0 C \text{induct}\} :$

$\supset \sigma \varepsilon_1 NC \text{induct}$

Dem $\vdash$. 3·47 . (120·001) . $\supset \vdash$. $H(g, 0) . H(g', 0) . \supset H((g \cap_1 g'), 0)$ (1)

$\vdash$ (1) $\supset \vdash$ Hp $\supset : H(g', 0) : g' \underset{1}{\subset} (N_0 C \text{induct} \underset{1}{-} \iota'_1 \wedge'') : \mathfrak{C}\{g', N_0 C \text{induct}\}.$

$\supset . H(g, 0) \supset (g \cap_1 g')\{\sigma\} :$

$\supset g\{\sigma\} :$

$\supset \vdash$ Hp $\supset : H(g', 0) : g' \underset{1}{\subset} (N_0 C \text{induct} \underset{1}{-} \iota'_1 \wedge'') : \mathfrak{C}\{g', N_0 C \text{induct}\}.$

$\supset \sigma \varepsilon_1 NC \text{induct}.$ (2)

$\vdash$ 120·1011 $\supset \vdash$ Hp $\supset . \sigma \varepsilon_1 NC$induct $\supset \sigma\varepsilon_1 (N_0 C$induct $\underset{1}{-} \iota^{\prime}{}_1 \Lambda'')$.
[Transp] $\supset \vdash . \sim \sigma\varepsilon_1(N_0C$induct $\underset{1}{-} \iota^{\prime}{}_1 \Lambda'')$. Hp . $\supset \sim \sigma\varepsilon_1 NC$induct
$\supset \vdash$ Hp $\supset (\exists \bar{g}) . H(\bar{g}, 0) : \bar{g} \underset{1}{\subset} (N_0 C$induct $\underset{1}{-} \iota^{\prime} \Lambda'') : \mathfrak{C}\{\bar{g}, N_0 C$induct$\}$ (3)
$\vdash$. (2) (3) . $\supset \vdash$ Prop.
120·51501 $\vdash$ Infin$ax \supset . (\bar{g}) :. H(\bar{g}, 0) : \bar{g} \underset{1}{\subset} (N_0C$induct $\underset{1}{-} \iota^{\prime}{}_1 \Lambda'')$:
$\supset \bar{g}\{\sigma\}$ $\mathfrak{C}\{\bar{g}. N_0 C$induct$\}$. $\equiv \sigma\varepsilon_1 NC$induct. [120·515 . 2 05.]
I shall use the following abbreviations:
120·5151 $\sigma^{\sigma'+\tau} \underset{df}{=} \sigma^{(\sigma'+\tau)}$
120·5152 $\sigma^{\sigma'-\tau} \underset{df}{=} \sigma^{(\sigma'-\tau)}$
120·51521 $g\{\sigma + \tau\} \underset{df}{=} g\{(\sigma+\tau)\}$
120·5153 $g_*(\tau', \sigma) \underset{df}{=} . g\{\tau' + 2^{\sigma'-\sigma}\} . \sim g\{\tau' + 2^{\sigma'+1-\sigma}\} . \mathfrak{C}\{\tau', \sigma, 1\}$.
120·516 $\vdash$. Infin$ax . \sigma'\varepsilon_1 NC$induct. $H(g, 0)$:
$g \underset{1}{\subset} (N_0C$induct $\underset{1}{-} \iota^{\prime}{}_1 \Lambda'')$: $\mathfrak{C}\{g, N_0C$induct$\}$.
$\supset . g\{2^{\sigma'}\} \supset g\{2^{\sigma'+1}\}$.
$\vdash$ 110·61 $\supset \vdash$. Hp $g_*(0,0)$. $\supset g_*(0,0)$
[10 24] $\supset (\exists \bar{\tau}') g_*(\bar{\tau}', 0)$ (1)
$\vdash$. 120·5146 . 110·51 . $\supset \vdash$. Hp . $g_*(0,0)$. $\supset$
$g_*(\tau'', \sigma) . \sim g\{\tau'' + 2^{\sigma'-\sigma} + 2^{\sigma'-(\sigma+1)}\} . \supset \sim g\{\tau'' + 2^{\sigma'-(\sigma+1)} + 2^{(\sigma'+1)-(\sigma+1)}\}$. (2)
[Hp . 113·66 . 116·52 32 . 120·5145] $\supset g\{\tau'' + 2^{\sigma'-(\sigma+1)} + 2^{\sigma'-(\sigma+1)}\}$. (3)
[(2) . (3) . 10·24 .] $\supset (\exists \bar{\tau}') g_*(\bar{\tau}', (\sigma+1))$ (4)
$\vdash$. 120·5146 . 113·66 . 116·52·32 . $\supset \vdash$. Hp . $g_*(0,0)$. $\supset$
: $g_*(\tau'', \sigma) . g\{\tau'' + 2^{\sigma'-\sigma} + 2^{\sigma'-(\sigma+1)}\} . \supset \sim g\{\tau'' + 2^{\sigma'-\sigma} + 2^{\sigma'+1-(\sigma+1)}\}$.
[10·24] $\supset (\exists \bar{\tau}') g_*(\bar{\tau}', (\sigma+1))$ (5)
$\vdash$. (4) . (5) . $\supset \vdash$. Hp . $g_*(0,0)$. $\supset . g_*(\tau'', \sigma) \supset (\exists \bar{\tau}') g_*(\bar{\tau}', (\sigma+1))$.
[10·24] $\supset . (\exists \bar{\tau}') g_*(\bar{\tau}', \sigma) \supset (\exists \bar{\tau}') g_*(\bar{\tau}', (\sigma+1))$.
[(1) . 120·515·447 .] $\supset (\exists \bar{\tau}') g_*(\bar{\tau}', \sigma')$
[(120·5153) . 120 5144 .] $\supset (\exists \bar{\tau}') . g\{\bar{\tau}' + 2^0\} . \sim g\{\bar{\tau}' + 2^1\} . \mathfrak{C}\{\bar{\tau}', 1\}$.
[116·32·301] $\supset (\exists \bar{\tau}') . g\{\bar{\tau}' + 1\} . \sim g\{\bar{\tau}' + 2\} . \mathfrak{C}\{\bar{\tau}', 1\})$.
[Transp] $\supset \vdash$ Hp $\supset \sim g_*(0,0)$
$\supset \vdash$ Prop.
120 517 $\vdash$. Infin$ax . \sigma'\varepsilon_1 NC$induct . $H(g, 0) . \mathfrak{C}\{g, N_0C$induct$\}$.
$\supset g\{2^{\sigma'}\}$
Dem $\vdash$ 116 301 $\supset \vdash$ Hp $\supset g\{2^0\}$ (1)

$\vdash$ 120·516 $\supset \vdash$. Hp : $g \underset{1}{\subset} (N_0C\text{induct} \underset{1}{-} \iota^{\varsigma}{}_1\Lambda'') : \supset$

: $g\{2^{\sigma}\} . \sigma\varepsilon_1 NC\text{induct} . \supset g\{2^{\sigma+1}\}$. (2)

$\vdash$. (1) . (2) . 120·13 . $\supset \vdash$. Hp: $g \underset{1}{\subset} (N_0C\text{induct} \underset{1}{-} \iota^{\varsigma}{}_1\Lambda'') : \supset g\{2^{\sigma}\}$

[120·51501] $\supset \vdash$ Prop

120·518 $\vdash$ $\sim$ Infinax . $\mathfrak{C}\{\tau, \sigma. 1\} . \supset \tau^{\sigma}\varepsilon_1 NC$induct

Dem $\vdash$ 1.5·211 $\supset \vdash$ Hp $\supset Nc^{\varsigma}(\underset{a}{-}\Lambda')\varepsilon_1 NC$induct (1)

[120·121] $\supset \Lambda''\varepsilon_1 NC$induct (2)

$\vdash$. (1) . (2) . 120·48 . $\supset \vdash$ Prop.

120·52 $\vdash$ $\sigma\varepsilon_1 NC$ induct $\supset 2^{\sigma}\varepsilon_1 NC$ induct

$\vdash$ 120·517·518 $\supset \vdash$ Prop.

120·523 $\vdash$. Infinax . $\sigma, \tau\varepsilon_1 NC$induct . $\supset \sigma^{\tau}\varepsilon_1 NC$induct

$\vdash$ 117·661 $\supset \vdash$ Hp $\supset . \sigma \leqslant 2^{\sigma}$.

[117·591] $\supset . \sigma^{\tau} \leqslant (2^{\sigma})^{\tau}$.

[120·501 . Hp . 116·63 .] $\supset . \sigma^{\tau} \leqslant 2^{(\sigma\times\tau)}$.

[120·501·52·48] $\supset \vdash$ Prop.

120·524 $\vdash$ $\sigma, \tau\varepsilon_1 NC$induct $\supset \sigma^{\tau}\varepsilon_1 NC$induct

[120·523·518]

This is the third and last fundamental theorem concerning the group properties of inductive numbers.

IX. Some remarks concerning Finite and Infinite.

I assume the following definition of finite classes:

122·001 $\text{fin}(\varkappa) \underset{df}{=} . \text{extens}(\varkappa) . (\bar{u}) . \sim \bar{u}\varepsilon_a\varkappa . \supset . Nc^{\varsigma}\varkappa \underset{\Omega}{\neq} Nc^{\varsigma}(\varkappa \cup_a \iota^{\varsigma}{}_a\bar{u})$:.

We see that all inductive classes are finite classes.

To prove that all finite classes are inductive classes, we need the multiplicative axiom, unless we assume that $(N_0C\text{induct} \underset{1}{-} NC\text{induct})$ is not a null-class. With this last hypothesis we get finite numbers, which are not inductive numbers; but we can prove that any Φ-order inductive class, being no N_0Cinduct-order inductive class must be an infinite class in a sufficiently high order. Now, as the multiplicative axiom can be proved in the system of Nominalism, as remarked above, we see that in this system we have no other finite numbers, as inductive numbers.

Note that Infinax by no means implies the existence of $\aleph_0$. The existence of any aleph must be assumed separately. — It is easy to see that if we deal with alephs, we assume the reducibility of the corresponding classes, and by this method we get a system which is practically equivalent to the simplified theory of types. We then see that Cantor's theory is closely connected with the simplified Theory of Types.

The fundamental idea of this work being that there are no other primitive propositions than those belonging to the Logical calculus, we are obliged never to deal with an hypothesis without having a parallel system based on a contradictory hypothesis. Now it is interesting to see what is to be done, if we assume an hypothesis inconsistent with the multiplicative axiom. Such an hypothesis being somewhat connected with the ideology of Realism, we can deal with it by means of the simplified Theory of Types. This matter will form the subject of a separate paper[1]). Here I wish to expound only the fundamental ideas of this work. On the other hand, I shall prove the fundamental proposition of Nominalism which I have mentioned above. I begin with this proof.

A. Nominalism.

I shall use the following definitions:

13·41 $\quad i \underset{df}{=} \hat{\varkappa}\hat{v}[(\bar{u}) : \varkappa\{\bar{u}\} \equiv (\hat{v} \underset{L}{=} \bar{u}) : \mathfrak{C}\{\bar{u}, \hat{v}, V\} . \mathfrak{C}\{\hat{\varkappa}, \omega''''_{(V)}\}.]$

13·42 $\quad (i_V\alpha) \underset{df}{=} \hat{u}[(\hat{u} \underset{L}{=} \alpha) . \mathfrak{C}\{\hat{u}, \alpha, V\}.]$

The direction 0·4, enabling us to take functions in any type for all individuals occurring explicitly or implicitly in a given expression, we ean use any function instead of a. Then our Theory of Cardinals applies to classes of any type. Now we shall have to deal with classes of the type $[i^{cc}]^{V}_{\omega''''_{(V)}}\ \omega''''_{(V)}$ instead of K and with corresponding cardinals and inductive numbers. This Theory enables us to prove the fundamental theorem of Nominalism, which I call the theorem of M. Greniewski. I use the following abbreviations:

13·43 $\quad \beta_{*} \underset{df}{=} \hat{\varkappa}[.\beta\{\hat{\varkappa}\} \mid V\{\hat{\varkappa}\}.]$

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[1]) „Über die Hypothesen der Mengenlehre", to be printed in „Mathematische Zeitschrift".

13·44 $\beta_{**} \underset{df}{=} \hat{\varkappa}[.\beta_*\{\hat{\varkappa}\} \mid V\{\hat{\varkappa}\}.]$

13·45 $\mathrm{int}(\omega) \underset{df}{=} (\overline{u}): \omega\{\overline{u}\}. \sim (\overline{u} \underset{L}{=} V): \supset$

$(\exists\overline{v}).(\overline{u} \underset{L}{=} \overline{v}_*).\omega\{\overline{v}\}.\mathfrak{C}\{\overline{v}, V\}: \mathfrak{C}\{\overline{u}, V\}.$

13·46 $\mathrm{id}(\omega) \underset{df}{=} (\overline{u}, \overline{v}): (\overline{u} \underset{L}{=} \overline{v}).\omega_V\{\overline{v}\}.\supset \omega_V\{\overline{u}\}.$

13·47 $\max(\omega) \underset{df}{=} \hat{u}[.\omega_V\{\hat{u}\}. \sim \omega_V\{\hat{u}_*\}.$

$(\overline{v}): \omega_V\{\overline{v}\}. \sim (\overline{v} \underset{L}{=} \hat{u}).\supset \omega_V\{\overline{v}_*\}:]$

13·48 $\overline{i}^{cc}\varkappa \underset{df}{=} [i^{cc}|^{V}_{\omega''''_{(V)}} \varkappa$

13·481 $\overline{\bigwedge} \underset{df}{=} (\bigwedge_{(V)} \bigcap_V \omega''''_{(V)})$

I denote by $\overline{\Omega}$ the function we get from Ω, if we use functions of the type $\overline{i}^{cc}\varkappa$ instead of functions of the type K. I shall also use the abbreviation:

13·482 $\bigwedge''' \underset{df}{=} (\bigwedge_{(\omega''''_{(V)})} \bigcup_{\omega''''_{(V)}} \overline{\Omega}).$

The interval determined by σ'' is to be defined as follows:

13·49 $\mathrm{Int}(\sigma) \underset{df}{=} \hat{\omega}[:\sigma \underset{\Lambda'''}{=} Nc^c(i^{cc}\hat{\omega} \bigcup_{\omega''''_{(V)}} \bigwedge'''): \hat{\omega}\{V\}.$

$(\exists\overline{v}).\max(\hat{\omega})\{\overline{v}\}.\mathrm{int}(\hat{\omega}).\mathrm{id}(\hat{\omega}).\mathfrak{C}\{\hat{\omega}, (i_V V)\}.]$

We have the following lemmas:

13·5 $\vdash \mathrm{Int}(1)\{(i_V V)\}$ [13·15]

13·51 $\vdash \mathrm{id}(\omega) \supset \mathrm{id}(.\omega_{(V)} \bigcup (i_V \alpha)_{(V)}.)$ [13·17]

13·512 [(13·49)] $\vdash \mathrm{Int}(\sigma)\{\omega\} \equiv : \sigma \underset{\Lambda'''}{=} Nc^c(i^{cc}\hat{\omega} \bigcup_{\omega''''_{(V)}} \bigwedge'''): \omega\{V\}.$

$(\exists\overline{v}).\max(\omega)\{\overline{v}\}.\mathrm{int}(\omega).\mathrm{id}(\omega).\mathfrak{C}\{\omega, (i_V V)\}.$

13·513 $\vdash .\mathrm{int}(\omega).\max(\omega)\{\beta\}.\supset \mathrm{int}(.\omega_{(V)} \bigcup (i_V\beta_*)_{(V)}.)$

Dem $\vdash .\mathrm{Hp}.\omega\{\alpha\}. \sim (\alpha \underset{L}{=} V).\supset (\exists\overline{v}).(\alpha \underset{L}{=} \overline{v}_*).\omega\{\overline{v}\}.$ (1)

$\vdash .\mathrm{Hp}.(i_V\beta_*)\{\alpha\}. \sim (\alpha \underset{L}{=} V).\supset (\alpha \underset{L}{=} \beta_*).\omega\{\beta\}$

$\supset (\exists\overline{v}).(\alpha \underset{L}{=} \overline{v}_*).\omega\{\overline{v}\}.$ (2)

$\vdash$ (1).(2).3·44 $\supset \vdash$ Prop.

13·514 $\vdash .\mathrm{Intax}.\omega\{V\}.\mathrm{int}(\omega).\mathrm{id}(\omega).\max(\omega)\{\beta\}.\mathfrak{C}\{\omega, (i_V V)\}.$

$\supset \max(.\omega_{(V)} \bigcup (i_V\beta_*)_{(V)}.)\{\beta_*\}$

Dem $\vdash \mathrm{Hp} \supset \sim (i_V\beta_*)\{\beta_{**}\}$ (1)

$\supset .\beta_{**} = \bigwedge.$ (2)

$\vdash$ 20·14 $\supset \vdash (\beta_{**} \underset{L}{=} V) \supset .\beta_{**} = V.$ (3)

$\vdash (2).(3).\supset \vdash \text{Hp}.\supset \sim(\beta_{**} \underset{L}{=} V)$ (4)

$\vdash .13{\cdot}513.\text{Intax}.\text{Hp}.\supset.(\beta_{**} \underset{L}{=} \alpha_*).\omega\{\alpha_*\}.\supset.(\beta_* \underset{L}{=} \alpha).\omega\{\alpha\}:$

$[(13{\cdot}46)] \qquad \supset \omega\{\beta_*\}.$

$[\text{Hp}] \qquad \supset \omega\{\beta_*\}.\sim\omega\{\beta_*\}$

$[\text{Transp}] \supset.\omega\{\alpha_*\} \supset \sim(\beta_{**} \underset{L}{=} \alpha_*)$

$[\text{Hp}] \qquad \supset: \sim(\alpha \underset{L}{=} V).\omega\{\alpha\}.\supset \sim(\beta_{**} \underset{L}{=} \alpha)$ (5)

$\vdash (1)(3)(5).\supset \vdash \text{Hp} \supset \sim.\omega_{(V)} \cup (i_V\beta_*)_{(V)}.\{\beta_{**}\}$ (6)

$[\text{Hp}] \qquad \supset.\omega_{(V)} \cup (i_V\beta_*)_{(V)}.\{\gamma\}.\sim(\beta_* \underset{L}{=} \gamma).\supset$

$.\omega_{(V)} \cup (i_V\beta_*)_{(V)}.\{\gamma_*\}$ (7)

$\vdash (6)(7).\supset \vdash \text{Prop}.$

13·52 $\vdash \text{Hp } 13{\cdot}514 : \sigma \underset{\Lambda'''}{=} Nc'(\bar{i}''\omega \cup_{\omega''''_{(V)}} \Lambda''') : \supset$

$.(\sigma+1) = Nc(\bar{i}''.\omega_{(V)} \cup (i_V\beta_*)_{(V)}. \cup_{\omega''''_{(V)}} \Lambda''').$

[110·631]

13·521 $\vdash .\text{Intax}.\exists'\text{Int}(\sigma).\supset \exists'\text{Int}(\sigma+1)$

[13·514·513·52]

13·53 $\vdash .\text{Intax}.\sigma\varepsilon_1(NC\text{induct} \underset{1}{-} \iota'_1 0).\supset \exists'\text{Int}(\sigma)$

[120·47 . 13·521·5·512]

13·54 $\vdash \text{Intax} \supset \text{Infinax}$ [13·53]

This is the theorem of Mr. Greniewski.

B. Realism and hyperrealism.

Let us assume the following definition:

$\text{Transcax} \underset{df}{=} (\bar{\varkappa}).\sim \bar{\varkappa}\varepsilon'\, Cls\,\text{induct} \underset{a}{\equiv} (-\bar{\varkappa})\varepsilon'\, Cls\,\text{induct}.$

If we assume Transcax, we can prove without any difficulty that $V_{(a)}$ is a finite class, i. e. a class which is not similar to any proper part of itself, not being an inductive class. We also prove the proposition:

$\text{Transcax} \supset \text{Infinax}.$

It is easy to see that Transcax is not consistent with Multax. Nevertheless there is an hypothesis, which is practically as much fruitful as the multiplicative axiom, being consistent with the Transcax. To get this hypothesis, I shall use the idea of self-comparable classes (intspec). We have the following definition of this idea:

$$\text{intspec}(\varkappa) \underset{df}{=} (\overline{\varkappa'}, \overline{\varkappa''}) :, \overline{\varkappa'} \underset{a}{\subset} \varkappa :. \overline{\varkappa''} \underset{a}{\subset} \varkappa : \supset . Nc^{\epsilon}\overline{\varkappa'} \text{ spec } Nc^{\epsilon}\overline{\varkappa''} :$$

With this definition we can build up the following definition of the Axiom of Affinity (Affinax)

$$\text{Affinax} \underset{df}{=} (\overline{\varkappa}, \overline{\omega}) : \text{int spec}(\overline{\omega}) . \text{int spec}(\overline{\varkappa}) . \supset$$

$$: Nc^{\epsilon}\overline{\varkappa} \text{ spec } Nc^{\epsilon}\overline{\omega} . \supset . Nc^{\epsilon}|Cls^{\epsilon}\overline{\varkappa} \text{ spec } Nc^{\epsilon}(\iota^{\epsilon\epsilon}\overline{\omega}) :.$$

We see that Affinax cannot be applied to classes, which are not self-comparable. Therefore we never can prove with this axiom the multiplicative axiom or some equivalent axiom. It is easy to see, although it can by no means be proved that Affinax is consistent with Transcax. If we take Transcax for Infinax and Affinax for Multax, we get a system, which is as well founded as Cantor's system, and which enables us to have a generalised Arithmetic and Mathematical Analysis.

Additional errata to Part I.

p. 20, l. 34, read ψp for fq
p. 21, l. 2, 3 and 4 read ψ for f
p. 23, l. 16 read „Fundamental class-letters“ for „Fundamental and functional class-letters“.
p. 24, l. 26 read $G(\lambda, \mu)$ for $E(\lambda, \eta)$
p. 25, footnote read 0·14·141 for 0·13·131
*9·15 for $\times$ 9·15
p. 25, footnote 3 read $\hat{x}$ for x, y for α
p. 26, l. 22 read „noted individual variables“ for „noted variables“
p. 26, l. 24 read $\hat{x}\hat{y}[\varphi\{\hat{x},\hat{y}\}], \hat{y}\hat{x}[\varphi\{\hat{x},\hat{y}\}]$ for $\hat{x}\hat{\varphi}[\varphi\{\hat{x}\}], \hat{\varphi}\hat{x}[\hat{\varphi}\{\hat{x}\}]$
p. 28, l. 8 read „are to be used in E as denoting“ for „denote“
p. 28, l. 17 read „expression or a real variable, E“ for „expression, E“
p. 28, l. 23 read „we can make any substitution for a letter in all its occurrences, allowed in the defining symbol“ for „we can take a functional expression for a determined real variable“.

p. 28, l. 28 read „$F(E')$ is a propositional expression" for „$F(E')$, $F(\Omega)$ are propositional expressions"
p. 29, l. 15 read „E', if all determined variables of E are determined variables of E'." for „E'."
p. 29, l. 17 read „in respect of this expression" for „expressions"
p. 29, l. 26 read „fundamental indetermined" for „fundamental"
p. 29, footnote read „conformable" for „conform"
p. 29, l. 32 read „and if" for „or if"
p. 30, l. 12 read „in respect of E" for „one with another and with E"
p. 31, l. 4 and l. 11 read „any fundamental letters or any functional expressions" for „any functional expressions"
p. 32, l. 7 read „any compatible propositional" for „any propositional"
p. 37, l. 12 is to be cut out
p. 42, l. 2 read $\underset{df}{=}$ for $=$
p. 42, l. 10 read $=$ for $\underset{df}{=}$ and $\underset{df}{=}$ for $=$
p. 42, l. 25 read $u_{(a)} = v_{(a)}$ for $u = v$
p. 43, read $\underset{a,\,a}{=}$ for $=$
p. 43, l. 13, 15 read u' for u and $\bar{u}$ and v' for v and $\bar{v}$
p. 44, l. 11 read $- V$ for $.\, \alpha - V.$
p. 44, l. 12 read $- V_{(a)}$ for $.\, \varkappa_{(a)} - V_{(a)}\,.$
p. 45, l. 2 read „$= \wedge$" for „$= \alpha$"
p. 45, l. 21 read $.\, R^{c}{}_{(a,\,b)} = R^{c}{}_{(a,\,b)}\, u\,.$ for $.\, R^{c}{}_{(a,\,b)}\, v = u\,.$
p. 47, l. 11 and 12, read $\mathit{\Pi}$ for D

Errata to Part II.

p. 96, l. 3 read „is" for „in"
p. 103, l. 10 read extens_{a}^{A} for extens_{A}^{a}
p. 117, l. 6 and 26 read $\underset{\Omega}{=}$ for $=$
p. 119, last line read $\hat{u}''\, \hat{v}'[(\exists \bar{Q}').\, \hat{u}''[Z_1]_{a}^{A} \bar{Q}' : \bar{Q}' \underset{a}{=} ((P \overset{a}{\underset{a,A}{|}} Z_2) \overset{A}{\underset{a}{|}} ((\underset{A}{\downarrow} \hat{v}') \overset{a}{\underset{\blacktriangleright}{\upharpoonleft}} \omega)) :] :]$

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