The Theory of constructive Types.

(Principles of Logic and Mathematics).

Part I.

General Principles of Logic: Theory of Classes and Relations.

By

Leon Chwistek.

Introduction.

The purpose of the present paper is to show how we can build up a system of Logic and Mathematics, assuming no other primitive ideas and propositions than those of the Logical Calculus. It is to be remarked that, for foundation of Mathematics, there is hardly any other method to be found. Suppose we assume any system of mathematical axioms: we then must prove that this system contains no contradiction. To prove anything, we must have some primitive ideas and propositions. These in their turn must contain the primitive ideas and propositions of the Logical Calculus. There is no means of building up a system of Mathematics, without assuming the primitive ideas and propositions of the Logical Calculus, or their equivalents. Therefore any system of Mathematics must contain the primitive ideas and propositions of the Logical Calculus.

We shall see that numbers are classes, and classes are propositional functions. Therefore, Mathematics is a part of the theory of propositional functions. Now, the logical calculus being a part of the theory of propositional functions, it seems obvious that we can get at least a part of Mathematics without assuming any other primitive propositions than those which belong to the Logical Calculus. This part of Mathematics appears to be the most solidly founded. Other parts of Mathematics, — the theories based e. g. on the axiom of infinity or on Zermelo's axiom — are to be considered as con-

sequences of these hypotheses. In modern Mathematics two following problems seem to be of great importance:

1° Can we prove a given proposition without Zermelo's axiom? [1])

2° Given a class other than the null-class, can we determine at least one of its elements?

Problem 1. can not be fully answerd without a perfect system of symbolic Logic and Mathematics, otherwise there always remains a suspicion we may have tacitly used the axiom of Zermelo.

Problem 2. appears to be very obscure so long as we work without Symbolic Logic. Then we have no means to reject the postulate of Kronecker, stating that a number is definite when we can calculate it to as many decimal places as we choose. Now this postulate implies a serious limitation of the domain of classic analysis, not being itself clear enough [2]).

In the system of Symbolic Logic we have no other objects than propositional functions and propositions. Now, we get propositional functions from propositions by a formal processus which does not contain any ambiguity. Therefore, to have any object, it is necessary and sufficient to have a proposition from which this object is to be obtained by a wholly determined formal processus.

If we assume existence axioms, we can prove that there are objects, which perhaps cannot be determined. In a system based exlusively on the primitive propositions of the Logical Calculus, there is no means to prove the existence of the elements of a class, without having an instance of such elements [3]).

A contrary method of working is followed by Prof. Hilbert in his interesting paper: Neubegründung der Mathematik [4]). Prof. Hilbert assumes a system of axioms containing the primitive propositions of the Logical Calculus together with some purely Mathematical axioms (e. g. Zermelo's axiom); and he endeavours to prove

[1]) Cf. the important paper of Prof. Sierpiński: L'axiome de M. Zermelo et son rôle dans la Théorie des ensembles et l'analyse, Bulletin de l'Academie des Sciences, Cracovie 1919.

[2]) For this remark I am indebted to Prof. Zaremba.

[3]) In spite of a remark on p. 136 of Vol. I. of Principia, the system of Whitehead and Russell appears to be able to prove existence axioms without using any instance, as we shall see below.

[4]) Abhandlungen aus dem mathematischen Seminar der Hamburgischen Universität. Hamburg 1920.

with the help of „metamathematical" methods that they imply no contradiction. Nevertheless Hilbert must either explicitly or implicitly use the primitive propositions and ideas of the Logical Calculus. Suppose he has proved by means of these primitive ideas and propositions that a system of propositions (say p, q, r) is compatible with them. Then, he has simply proved these propositions. If he has used (explicitly or tacitly) other ideas or propositions, then he has assumed some new hypotheses which appear as more general than Zermelo's axiom etc. At any rate, the system of primitive propositions of Symbolic Logic and its consequences remains as basis of any further investigation. Note, that Hilbert does not assume the Theory of Types. Nevertheless I can hardly assume that we have a „Meta-mathematic" at our disposition, which could be really free from problems connected with the Theory of Types [1]). To see this clearly, note, that such a „Meta-mathematic cannot be essentially different from the Logical Calculus, this calculus being as a matter of fact a simple consequence of the laws of our thinking. Now, as we shall see below, we can not employ any self-consistent Logical Calculus at all, if we do not asume the Theory of Types. Therefore there seems to be no means of avoiding the said theory.

I. A Critical examination of the theory of Prof. A. N. Whitehead and Hon. B. Russell.

A. Functions.

The fundamental hypothesis of the Theory of Types of Whitehead and Russell, as developed in their classic work: Principia Mathematica [2]) consists in the statement that the idea of „all objects" is meaningless. As a matter of fact, there seems to be no means of preserving this idea, because it is easy to build up a propositional function $Q\hat{x}$ based on this idea, and being a contradictory object. Suppose all objects are possible values of a propositional function Φx, and suppose we can speak about all properties of x (i. e. all propositional functions $\Phi\hat{x}$ such that either Φx or $\sim\Phi x$, [which is

[1]) As we shall see below, there is a Meta-mathematic, dealing only with the meaning of symbols, but never with the truth or falsehood of propositions. Therefore there is no means of proving a mathematical or logical proposition with such a Metamathematic.

[2]) Vol. I. Cambridge 1910.

not-Φx]). Now let us write $(\exists\Phi)$ instead of „for some $\hat{\Phi x}$", $=$ instead of „is identical with" and $\Phi x \,.\, \psi x$ instead of „Φx and ψx". With the help of these ideas, which seem to be essential to any system of symbolic logic, we can build up the proposition:

$$(\exists\Phi) \,.\, \hat{\Phi x} = a \,.\sim \Phi a,$$

which we shall denote by Qa. Now, it is easy to see that $\hat{Qx}$ is a contradictory object, we having propositions $Q(\hat{Qx})$ and $\sim Q(\hat{Qx})$[1]. To avoid such objects, there seems to be no other means than to suppose with Whitehead and Russell, that $\hat{Qx}$ can be no possible value of the argument of Qx, the idea of „all values of the argument of $\hat{\Phi x}$" being not equivalent to the idea of „all objects". Moreover, we should assume that the idea of „all objects" is meaningless, we having a hierarchy of types of objects. Suppose we can speak about „all properties of x" i. e. about „all propositional functions $\hat{\Phi x}$ such, that either Φx or $\sim\Phi x$". We shall have to deal 1° with individuals i. e. objects being neither propositions nor propositional functions; 2°, with propositional functions whose arguments take individuals as possible values, i. e. propositional functions of the 1st type; 3°, with propositional functions whose arguments take functions of the 1st type as possible values, i. e. propositional functions of the 2d type... and so on. Such a simple hierarchy of types would be, as a matter of fact, sufficient to build up a self-consistent system of Symbolic Logic, there being no purely Logical paradoxes based on the idea of „all properties of x". Nevertheless, as this last idea does not exclude such contradictions, as Richard's paradox, or König's, it seems to be interesting to get a system of Symbolic Logic, free from such contradictions. To avoid these we must agree with Whitehead and Russell that the idea of „all properties of x" is meaningless.

Then we cannot speak about „all functions $\hat{\Phi x}$ such, that either Φx or $\sim\Phi x$", we having moreover a hierarchy of functions of different types (or, as we call them, functions of *different orders*) having x as a possible value of their argument.

We see that this seriously complicates the primitive theory of types. Now, such symbols as (x) i. e. „for all x's" or $(\exists x)$ i. e. „for some x's" have meaning *only* if x denotes individuals. To

[1]) Cf. Über die Antinomieen der Prinzipien der Mathematik, Mathematische Zeitschrift 1922.

have such symbols for functions. Whitehead and Russell build up the idea of a matrix, i. e. of a function, having no such constituent as (x) or $(\exists x)$. Such functions are to be denoted by symbols like $\Phi!\hat{x}, \psi!(\hat{x}, \hat{y})$, etc. Having these functions, we can use symbols $(\Phi), (\exists\Phi)$ for them, and build up matrices of the 2^{d} type, whose arguments are $\Phi!\hat{x}$, and of which there are no such constituents as $(x), (\exists x), (\Phi), (\exists\Phi)$. These matrices are to be denoted with symbols:

$$f!(\hat{\Phi}!\hat{z}), \quad f!(\hat{\Phi}!\hat{x}, \hat{x}) \quad \text{etc.}$$

Other functions are to be obtained from matrices, using symbols like $(x), (\exists x)$ $(\Phi), (\exists\Phi)$, e. g.

$$(x) . \psi!(x, \hat{y}), \quad (\exists x) . f!(\hat{\Phi}!\hat{x}, x),$$

$$(\Phi) . f!(\Phi!\hat{x}, \hat{x}) \quad \text{etc.}$$

This part of the Whitehead-Russellian Theory of Types, we shall call *the pure theory of types*, or the *theory of constructive types*. This theory with formal modifications is to be developed in the present paper.

The theory of Whitehead and Russell, as assumed in their „Principia Mathematica", cannot be treated as a *pure theory of types;* these authors having supplemented this theory with an „existence axiom" [1]) they call *the axiom of reducibility*, and this axiom being neither a purely logical axiom, nor a simple application of the ideas of the *pure theory of types*. This axiom states that:

$$(\exists\Phi) . \Phi!x \equiv_{x} \psi x,$$

i. e. „every function of a variable is equivalent for all values, to some predicative function" [2]), i. e. to a matrix.

Now, it is obvious that, given any function $\psi\hat{x}$, we have sometimes no means of building up a matrix equivalent to this function. So, if we affirm the existence of such a function, we must suppose that there are matrices which we cannot build up, i. e. matrices which are not constructive. Now, we can prove, by the method used by Richard, that, if *there are only constructive functions*, the

[1]) Cf. Trzy odczyty odnoszące się do pojęcia istnienia, Przegląd filozoficzny 1917.

[2]) Principia Vol I. p. 177.

axiom of reducibility is false [1]). If we assume the theory of classes and relations developed by Whitehead and Russell, and the Richard's idea of expression, we can build up a contradiction quite analogous to Richard's paradox. This theory being based on the axiom of reducibility, it seems obvious that this axiom implies contradiction [2]). Nevertheless, as we shall see at once the definition of functions of classes (or relations) given in the Principia appears to be ambiguous. If we deal simply with propositional functions, there seems to be no means to get Richard's paradox.

B. Classes.

1. The definition of a function of a class, given in Principia (*20·01) is as follows:

$$f\{\hat{z}(\psi z\} . = : (\exists \Phi) : \Phi!x . \equiv_x . \psi x : f\{\Phi!\hat{z}\}. \qquad \text{Df}$$

This definition is completed by the following convention about the scope of the symbol $\hat{z}(\psi z)$:

The scope of the symbol „$\hat{z}(\psi z)$" is the smallest proposition enclosed in dots or brackets in which „$\hat{z}(\psi z)$" occurs [3]).

Now, this convention appears to be insufficient to avoid all ambiguity. Let us take for $f\{\Phi!\hat{z}\}$ the function: $\sim(\Phi!\hat{z} = \theta!\hat{z})$, and put this function into the following proposition (*20·1), we deduce immediately from our definition of a function of a class:

$$f\{\hat{z}(\psi z)\} . \equiv : (\exists \Phi) : \Phi!x . \equiv_x . \psi x : f\{\Phi!\hat{z}\}.$$

Then we get:

$$\sim(\hat{z}(\psi z) = \theta!\hat{z}) . \equiv : (\exists \psi) : \Phi!x . \equiv_x . \psi x : \sim(\Phi!\hat{z} = \theta!\hat{z}).$$

Now, apply our convention to this last proposition. We get:

$$(1)\ \sim(\exists \Phi) : \Phi!x . \equiv_x . \psi x : (\Phi!\hat{z} = \theta!\hat{z}) : \equiv : (\exists \Phi) : \Phi!x . \equiv_x . \psi x : \sim(\Phi!\hat{z} = \theta!z).$$

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[1]) Zasada sprzeczności w świetle nowszych badań B. Russella, (Akademja Umiejętności, Kraków 1912).

[2]) Über die Antinomieen der Prinzipien der Mathematik l. c.

[3]) Cf. Principia Vol. I. p. 197 and 181.

Now let us take $\theta!\hat{z}$ for $\psi\hat{z}$. It is easy to prove the proposition:

$$(\exists\Phi) : \Phi!x \equiv_x \theta!x : \Phi!\hat{z} = \theta!\hat{z},$$

whence we get immediately by (1):

$$\sim(\exists\Phi) : \Phi!x \,.\equiv_x \theta!x : \sim(\Phi!\hat{z} = \theta!\hat{z}), \quad \text{i. e.}$$

$$\Phi!x \,.\equiv_x.\, \theta!x \,.\supset_\Phi.\, \Phi!z = \theta!z.$$

Then, assuming the convention of Whitehead and Russell we have proved that all equivalent matrices must be identical; which is a paralogism. So we cannot assume Whitehead and Russell's convention. As and I see no means of making any other useful convention, I have tried to note explicitly the scope of „$\hat{z}(\psi z)$“, i. e. I have assumed the following definition:

$$[\hat{z}(\psi z)] \,.\, f\{\hat{z}(\psi z)\} \,.=: (\exists\Phi) : \Phi!x \,.\equiv_x \psi x : f\{\Phi!\hat{z}\}. \qquad \text{Df}$$

With such a definition of a function of class we avoid all ambiguity, but it soon appears that we get no simplification of the calculus of functions. This becomes clear if we remark that, e. g. $\sim[\hat{z}(\psi z)] \,.\, f\{\hat{z}(\psi z)\}$ and $[\hat{z}(\psi z)] \,.\sim f\{\hat{z}(\psi z)\}$ are two different functions. It is to be noted that, if we do not assume that any two equivalent matrices must be identical, we have the proposition:

$$[\hat{z}(\psi z)] \,.\, [\hat{z}(\psi z)] : \hat{z}(\psi z) \neq \hat{z}(\psi z),$$

where the symbol $\neq$ is given by the definition (13·02)

$$x \neq y \,.=.\, \sim(x = y). \qquad \text{Df}$$

The most important consequence of our noting the scope of the class-symbols is as follows:

We can prove:

$$[\hat{z}(\psi z)] : [\hat{z}(\Phi z)] \,.\, \hat{z}(\psi z) = \hat{z}(\Phi z) : \equiv : \psi z \,.\equiv_z.\, \Phi z \qquad (*20.15),$$

but we cannot prove:

$$\Phi z \,.\equiv_z.\, \psi z \,.\supset : g\{[\hat{z}(\psi z)] \,.\, f![\hat{z}(\psi z)]\} \,.\equiv.\, g\{[\hat{z}(\Phi z)] \,.\, f![\hat{z}(\Phi z)]\}.$$

Therefore, given any function of the form:

$$g\{[\alpha]\, f!\alpha\},$$

we cannot take for α any class β such, that:

$$x\varepsilon\alpha \,.\equiv_x .\, x\varepsilon\beta.$$

Of course, in practice this substitution is always possible, as we have to use only extensional functions, but we must prove, for any given instance that our function is extensional. Now, for the use of extensional functions the use of the Leibnizian idea of identity is, as a matter of fact, superfluous. Hence, our definition of a function of a class seems to be useless.

2. I pass to the following difficulty of the Whitehead-Russellian theory of classes, which seems to be more essential. Let us prove *20·7 for classes of classes. Thus we must first write *12·1 in the following manner:

$$(\exists f) : f!\{\Phi!(\hat{\theta}!\hat{x})\} . \equiv_{\Phi} . g\{\Phi!(\hat{\theta}!\hat{x})\}.$$

Le us now write explicitly *20·7. We have:

$$(\exists f) : . (\exists \psi) : \psi!\alpha . \equiv_{\alpha} . \Phi!\alpha : f!(\psi!\hat{\alpha}) : \equiv_{\Phi} : (\exists \psi) : \psi!\alpha . \equiv_{\alpha} . \Phi!\alpha : g(\psi!\hat{\alpha}).$$

It is obvious that to prove such a proposition, we should have:

$$(\exists f) : f!\{\Phi!\hat{\alpha}\} . \equiv_{\Phi} . g\{\Phi!\hat{\alpha}\}.$$

Now, remark that this proposition has the following meaning:

$$(\exists f) : . f!\{(\exists \theta) : \theta!x . \equiv_{x} . \hat{\chi}!x : \Phi!(\theta!\hat{z})\} : \equiv_{\Phi} : g\{(\exists \theta) : \theta!x . \equiv_{x} . \hat{\chi}!x : \Phi!(\theta!\hat{z})\}.$$

Thus, we see that the axiom of reducibility must be assumed for variables of the type:

$$(\exists \theta) : \theta!x . \equiv_{x} . \hat{\chi}!x : \Phi!(\theta!\hat{z}).$$

Note that the same difficulty subsists, if we note explicitly the scope of the class-symbols. If we will not assume the axiom of reducibility for such functions as: $g\{(\exists \theta) : \theta!x . \equiv_{x} . \hat{\chi}!x : \Phi!(\theta!\hat{z})\}$, such a primitive proposition being of course some-what artificial, we should assume this axiom for variable functions of any type. Now, in Principia we have no other functions but those of matrices or individuals. Therefore a radical modification of the system of Whitehead and Russell seems absolutely necessary, even if we agree with the axiom of reducibility. We then meet the following difficulties:

A. Suppose we agree with such symbols as:

$$f!\{\Phi\hat{x}\} \quad g!\{f(\hat{\Phi}!\hat{x})\},$$

it is easy to see that our notation will imply contradictions. Let us take the function $f(\hat{\Phi}!\hat{z}, a)$ of the variable $\Phi!\hat{x}$, and let us write:

$$g!\{f(\hat{\Phi}!\hat{z}, a)\}.$$

Now by $*9{\cdot}15$. „if for some a, there is a proposition fa, then there is a function $f\hat{x}^{u}$, we have the following *function of* x:

$$g!\{f(\hat{\Phi}!\hat{z}, \hat{x})\}.$$

On the other hand, given any function $f(\hat{\Phi}!\hat{z}, \hat{x})$, of $\Phi!\hat{z}, x$, the expression:

$$g!\{f(\hat{\Phi}!\hat{z}, \hat{x})\}$$

denotes a *proposition*. Thus the same expression denotes a function of x and does not denote it.

The same contradiction can be constructed for matrices, if we agree that $\hat{\Phi}!a$, i. e. the function of one variable function $\Phi!\hat{x}$ of individuals, is a matrix (Cf. Principia, Vol I. p. 170). Now, we can take $f!(\hat{\Phi}!\hat{z}, a)$ for $\Phi!a$ (Cf. Principia Vol. I. p. 155). Therefore $f!(\hat{\Phi}!\hat{z}, a)$ is a matrix. To avoid this ambiguity *I* shall write $\hat{x}[\Phi\hat{x}]$ for $\Phi\hat{x}$ and $\hat{x}\hat{y}[\Phi\{\hat{x}\hat{y}\}]$ or $\hat{y}\hat{x}[\Phi\{\hat{x}, \hat{y}\}]$ for $\Phi\{\hat{x}, \hat{y}\}$.

B. If we have no other variables as matrices, we cannot use the axiom of reducibility as a general hypothesis, like Zermelo's axiom, because we cannot write with meaning:

$$(\Phi) : (\exists \psi) : \psi!x . \equiv_{x} . \Phi x.$$

If we assume functions of any type as variables, then we must have means of speaking of „all functions of the same type as a given function $\Phi\hat{x}^{u}$. As a matter of fact, it will be seen below that we can construct following expression:

$$(\Phi)_{\theta\hat{x}} : (\exists \psi) : \psi! x . \equiv_{x} . \Phi x$$

Here $(\Phi)_{\theta\hat{x}}$ means: „for *all* functions of the same type as $\Phi\hat{x}^{u}$.

Such propositions as those, given above, can be used as hypotheses, like Zermelo's axiom, therefore if we assume functions of any type as variables, there is no serious reason to have the axiom of reducibility among our primitive propositions, even if we are willing to pass over all the other objections I have stated above.

It is to be remarked that there are hardly any propositions of mathematics to be found, which require the Axiom of reduci-

bility as hypothesis. We shall have to do with other, less general hypotheses i. e. the axiom of finite numbers and the axiom of the continuum. We may also note that these hypotheses are of secondary importance.

Suppose we sacrifice the Whitehead-Russellian theory of classes to preserve the axiom of reducibility in its primitive form. Then we shall have no such propositions as $f\{\hat{z}(\Phi z)\}$ and be obliged to use the complicated method of substitution: $(\exists \psi) : \psi! x. \equiv \equiv_x . \Phi x : f\{\psi! x\}$. At any rate the following remark seems to be conclusive. To have the Theory of Types, we must speak about *functions of the same type*. Now, it is natural to have variables denoting functions of the same type. The axiom of reducibility would therefore appear to be a hypothesis, like the axiom of infinity.

C. Arguments of Whitehead and Russell. General remarks.

There are in Principia 3 arguments to prove the neccessity of the axiom of reducibility or some equivalent axiom for a system of Logic based on the Theory of Types.

The 1st argument says: There is no possibility of giving an adequate definition of identity without the axiom of reducibility [1].

The 2d argument is based on the opinion that „if we assume the existence of classes, the axiom of reducibility can be proved" [2].

The 3d argument is as follows:

„If Mathematics is to be possible, it is absolutely necessary that we should have some method of making statements, which will usually be equivalent to what we have in mind when we (inaccurately) speak of "all properties of x". Hence we must find, if possible, some method of reducing the order of propositional function without affecting the truth or falsehood of its values" [3].

The three arguments quoted above do not appear to be sufficient.

1° Of course, a general definition of identity is hardly possible without the axiom of reducibility, but such a definition is irrelevant in practice. In a system of Logic and of Mathematics we have to deal as a matter of fact with statements concerning identity either

[1]) Principia Vol I. p. 60.
[2]) Ibd. p. 60.
[3]) Ibd. p. 173.

of classes or of relations, and, as we shall see below there is a definition of identity to be given, which is quite sufficient for this purpose.

2° If classes are such objects as satisfy the postulates of Huntington's Theory of classes, then their existence does not imply the axiom of reducibility. We shall have to deal with such objects.

3° It is to be remarked that Mathematics, as developed in Principia, being in practice conformable to ordinary Mathematics, is as a matter of fact much more general. Now, it is natural to try to construct a system of Mathematics, which being more general than the system of Principia, would be in practice equivalent to ordinary Mathematics. We shall see that there is no serious difficulty in the realisation of this purpose. It is true that we shall have no such a thing as the class of inductive numbers, but we shall have nevertheless to deal with no such class as the continuum, but we shall nevertheless have to deal with the continuum, conceived as an ambiguous symbol [1]), which will allow us to develop the theory of Lebesgues' measure and other chapters of the classic theory of functions [2]). Only there is no means of constructing the theory of transfinite cardinals, without any existence-axiom. With such, we can of course prove all propositions of Principia without any difficulty.

There being no serious difficulty in our purpose, it is natural to try and to realise it. It is to be remarked, that the system of Whitehead and Russell is very useful as base of our researches. No primitive propositions are to be adopted, which are not to be found in Principia. We have to take directly from Principia all that remains true, if the axiom of reducibility is false and if functions of a given type are used as variables instead of matrices. The other propositions are of 3 kinds. The 1st class contains propositions which cannot be proved at all. The 2d class consists of propositions which can be proved by some new method, or have at least an equivalent conformable to the general ideas of our system, and which can be proved. The 3d class contains propositions which can be proved only for some classes or relations, e. g. for one-one relations.

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[1]) Cf. Hermann Weyl: Über die neue Grundlagenkrise der Mathematik, Mathematische Zeitschrift.

[2]) Miara Lebesgue'a. Archiwum Lwowskiego Towarzystwa Naukowego, Lwów 1922.

It is easy to see that my system must be much more complicated than that of Whitehead and Russell. It might be thought that any further complication must be useless to clear up the ideas on which Mathematics is to be based. But it may be erroneous to think that clear ideas are never complicated: while, we must agree that many simple ideas are, as a matter of fact, very obscure.

The system of Whitehead and Russell, being the most perfect and most ingeniously constructed system of Logic I know, I hardly conceive that any other method in working on these matters can be used. The knowledge of Principia is therefore quite sufficient to understand what is said in this paper. All the propositions used as corollaries being stated, there is as a matter of fact no essential difficulty in understanding my proofs without the knowledge of Principia.

To sum up my system is based on a most consistent application of the Russellian theory of types. Mathematical ideas are developed step by step, with the help of special hypotheses, if necessary, which affords a base for constructing the hierarchy of different stages of Mathematics. This method seems to prove that there is no one unique system, but on the contrary many exclusive systems of Mathematics.

The name „constructive types" is based on the theoretical possibility of construction of all functions belonging to a given type of my system.

II. Directions concerning the meaning and the use of symbols.

It is to be remarked that we can hardlu imagine a system of symbolic Logic without some directions concerning the meaning of symbols. Take e. g. the proposition $f\{\varphi p\}$. We having an operation consisting in taking q for p in φp, we might think that φq was one of the possible values of $f\{\varphi p\}$. Now, it is easy to see that such an interpretation of our symbols implies a contradiction. Let us write:

$$\varphi p = f q,$$

(where the Leibnizian idea of identity is assumed).

Then, by our hypothesis we must have:

$$\varphi q \equiv f q.$$

Hence we get: $\varphi p = fp . \supset . \varphi q \equiv fq.$

Now, put $r \supset r$ for φr and $r \supset p$ for fr. We get:

$$p \supset p = p \supset p . \supset : q \supset q . \equiv . q \supset p.$$

Here the hypothesis being true, we must have

$$q \supset q . \equiv . q \supset p,$$

which is evidently false.

Now, we must conclude that there is no such logical operation $f\{\varphi p\}$ as consists in getting φq out of φp. To avoid such false interpretations of symbols as we have seen above, we need some directions for their meaning.

When we assume the Theory of Types, our contradiction disappears at once. This is clear when we note that if $f\{\varphi p\}$ is a function of φp, we can take for φp any proposition of the same type as φp, e. g. q. Now, $f\{q\}$ can have no meaning, as there is no variable argument in q. Therefore $f\{\varphi p\}$ cannot be a function of φp.

The Theory of Types of Whitehead and Russell contains some philosophical theories which seem to be useless. Now, it is important to have some directions for working with symbols. Therefore, in this part I cannot follow the method of Principia.

The theory of expressions to be given below is as a matter of fact that conforms to the theory of Principia. I assume letters denoting individuals, but I do not assume any analysis of the idea of an individual, these objects never being used for any actual substitution and their only use consisting in their being of the same type. An analogous remark may be made about predicative functions [1]).

The essential difference of the actual theory from that of Principia consists in the assumption of variable functions of assigned types. Many directions, which seem to be tacitly assumed in Principia are here explicitly given.

To avoid ambiguity in the notation of the functions, I use symbols analogous to those which Whitehead and Russell use for classes and relations.

[1]) For this remark as for many other valuable pieces of advice I am most indebted to Prof. Wilkosz.

Following the theory of Prof. Sleszyński and Prof. Wilkosz [1]). I have strictly separated verbal directions from primitive propositions, and I do not assume any verbal proofs. Moreover I have not assumed any verbal primitive ideas. In what follows, such words as e. g. expressions, propositions, functions, types etc. have no intrinsic meaning, their use being purely practical and defined step by step by our directions. Note that our directions are a simple abbreviation, that we use instead of a full list of expressions belonging to our system. Without such a list, or a machine furnishing as many useful expressions as we like, there is no perfect system of Logic to be thought of. This part of the system of Logic and Mathematics may be called the real „Metamathematic". In comparison with the theory of Prof. Hilbert [2]) our idea of „Metamathematic" seems to be more precise. It is obvious that there can be no such things as verbal proofs in a system of Logic and Mathematics. Verbal proofs seem to be the common imperfection of Principia [3]) and of the theory of Hilbert.

A. Directions concerning the meaning of symbols.

0·00 Letters

0·01 Expressions p, q, r, s are elementary letters.

0·011 Elementary letters denote elementary propositions.

0·02 Expressions x, y, z are individual letters.

0·021 Individual letters denote individuals.

0·03 Expressions θ, φ, ψ, χ are primitive letters.

0·04 Expressions f, g, h, j are fundamental letters.

0·05 If λ is an elementary, (individual, primitive, or fundamental) letter, as the case may be, λ' is an elementary (or individual, or primitive, or fundamental) letter.

0·051 If λ is a letter, λ' is an essentially different letter, (like ι and i).

0·06 If λ is an individual (or primitive, or fundamental) letter, $\hat{\lambda}$ is an individual (or primitive, or fundamental) noted variable.

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[1]) For the notice of this theory and for the suggestion of the name „direction" I am indebted to Mr. Nikodym.

[2]) l. c.

[3]) Note especially the proposition * 10·221. It is to be remarked that we have no real proof of this proposition in Principia. Nevertheless, this proposition seems to be assumed as equivalent to other propositions asserted in Principia.

0·061 If λ is an individual (or primitive, or fundamental) letter, $\bar{\lambda}$ is an individual (or primitive, or fundamental) apparent variable.

0·062 An expression containing only one individual (or primitive, or fundamental) letter λ and no other letters and symbols, is an individual (or primitive, or fundamental) real variable.

0·07 Expressions α, β, γ, δ are functional class-letters.

0·071 Expressions L, M, N, T are functional relation-letters.

0·072 Functional class-letters are functional expressions with I variable.

0·073 Functional relation-letters are functional expressions with II variables.

0·08 Espressions σ, τ, $\varkappa$, ω are fundamental class-letters.

0·081 Expressions P, Q, R, S are fundamental relation-letters.

0·082 Expressions u, v, w, t are determined letters.

0·083 Fundamental and functionel class-letters, fundamental relation-letters and determined letters are fundamental letters.

0·09 Expressions a, b, c, d are pseudo-letters.

0·091 All pseudo-letters occurring in a given expression stand for functional class-letters or relation-letters.

0·092 If λ is a fundamental class-(or relation-) letter, (or a functional class- or relation-letter, or a determined, or a pseudo-letter), then λ' is a fundamental class-(or relation-) letter, (or a functional class-or relation-letter, or a determined, or a pseudo-letter).

0·10 Expressions.

0·11 If E, F are expressions denoting logical propositions, then $.E|F.$ denotes logical proposition [1]).

Remark: The dots are an essential part of the expression $.E|F.$ Note that there is no need of a further theory of dots. For this theory of dots I am indebted to Prof. Leśniewski. The idea of $p|q$ was introduced by Mr. Sheffer [2]). If we use the

[1]) Numbers 0 1·11 correspond to * 1·7·71 of Principia. Logical propositions make up the lowest type of propositions. In our system there are no other propositions. Nevertheless, if I am speaking about logical propositions, I am working with propositions which are not logical.

[2]) Transations of American Mathematical, society 1915.

definitions given by this author (to be given below) we shall have: $..p \mid q . \equiv \sim . p \vee q ..$

0·111 Any expression denoting logical proposition is a proposi-tional expression.

0·12 Propositional (or functional) expressions are significant, they have a meaning in isolation.

0·13 If λ is any primitive or fundamental variable or any functional expression and $\xi, \eta, \zeta, \vartheta$ any individual or fundamental variables or any functional expressions, then the expressions $\lambda\{\xi\}$, $\lambda\{\xi, \eta\}$, $\lambda\{\xi, \eta, \zeta\}$, $\lambda\{\xi, \eta, \zeta, \vartheta\}$ are functional patterns with I, with II, with III, or with IV arguments. Here λ is a functional sign, $\xi, \eta, \zeta, \vartheta$ are arguments belonging to λ.

0·131 In functional patterns $\lambda\{\xi\}$, $\mu\{\xi'\}$ or $\lambda\{\xi, \eta\}$, $\mu\{\xi', \eta'\}$, or $\lambda\{\xi, \eta, \vartheta\}$, $\mu\{\xi', \eta', \vartheta'\}$, or $\lambda\{\xi, \eta, \vartheta, \zeta\}$, $\mu\{\xi', \eta', \vartheta', \zeta'\}$ the arguments δ, δ' where δ is ξ, or η, or ζ, or ϑ, are corresponding arguments.

0·1311 Any functional pattern, whose functional sign is a primitive or fundamental real variable and whose arguments are individual real variables, is a propositional expression

0·132 If in a significant expression E, the fundamental variable ξ and a functional expression M having no letters in common with ξ are corresponding arguments belonging to the same functional sign: then ξ is a determined variable, or a variable determined by M.

0·14 If $E(\lambda)$, or $E(\lambda, \eta)$, or $H(\lambda, \mu, \nu)$, or $I(\lambda, \mu, \nu, \varrho)$ are any propositional expressions containing the individual, primitive, or determined real veriables: λ, or λ, μ, or λ, μ, ν, or λ, μ, ν, ϱ: thent hee xpressions: $\hat{\lambda}[E(\hat{\lambda})]$, or $\hat{\lambda}\hat{\mu}[G(\hat{\lambda}, \hat{\mu})]$, or $\hat{\lambda}\hat{\mu}\hat{\nu}[H(\hat{\lambda}, \hat{\mu}, \hat{\nu})]$, or $\hat{\lambda}\hat{\mu}\hat{\nu}\hat{\varrho}[I(\hat{\lambda}, \hat{\mu}, \hat{\nu}, \hat{\varrho})]$ are functional expressions with I, or with II, or with III, or with IV variables. Here we have turned real variables into noted variables.

0·141 If $E(\hat{\lambda})$, $G(\hat{\lambda}, \hat{\mu})$, $H(\hat{\lambda}, \hat{\mu}, \hat{\nu})$, $I(\hat{\lambda}, \hat{\mu}, \hat{\nu}, \hat{\varrho})$ are any expressions containing the noted variables $\hat{\lambda}$, or $\hat{\lambda}$, $\hat{\mu}$, or $\hat{\lambda}$, $\hat{\mu}$, $\hat{\nu}$, or $\hat{\lambda}, \hat{\mu}, \hat{\nu}, \hat{\varrho}$; and if: $\hat{\lambda}[E(\hat{\lambda})]$, $\hat{\lambda}\hat{\mu}[G(\hat{\lambda}, \hat{\mu})]$, $\hat{\lambda}\hat{\mu}\hat{\nu}[H(\hat{\lambda}, \hat{\mu}, \hat{\nu})]$, $\hat{\lambda}\hat{\mu}\hat{\nu}\hat{\varrho}[I(\hat{\lambda}, \hat{\mu}, \hat{\nu}, \hat{\varrho})]$ are functional expressions, then the expres-

sions: $E(\lambda)$, $G(\lambda, \mu)$, $H(\lambda, \mu, \nu)$, $I(\lambda, \mu \nu)$ are propositional expressions [1]).

0·15 If $E(\lambda)$ is any propositional expression containing the individual (or primitive, or determined) real variable λ, then $(\overline{\lambda})E(\overline{\lambda})$ is a propositional expression. Here we have turned a real variable into an apparent variable.

0·151 If the expressions:

$$(\overline{\lambda})(\overline{\mu})E(\overline{\lambda}, \overline{\mu}), \quad (\overline{\mu})(\overline{\lambda})E(\overline{\lambda}, \overline{\mu})$$

are propositional expressions, they denote the same logical propositions, i. e they have the same meaning [2]).

0·152 Any functional expression containing no fundamental letters and no apparent variables is a primitive functional expression.

0·16 Suppose that λ, μ, ν, ρ are any individual, (or primitive, or determined real variables) and $F(\lambda)$, or $G(\lambda, \mu)$, or $H(\lambda, \mu, \nu)$ or $I(\lambda, \mu, \nu, \rho)$ are any propositional expressions containing the variables λ, or λ, μ, or λ, μ, ν, or λ, μ, ν, ρ. Suppose that λ, λ', or μ, μ', or ν, ν', or ρ, ρ' are at the same time two different individual (or primitive, or fundamental) real variables, or that λ, (μ, ν, or ρ) is a determined real variable and λ' (μ', ν', or ρ') is any functional expression having no letters in common with λ, (μ, ν, or ρ). Then, if $F(\lambda')$, or $G(\lambda', \mu')$, or $H(\lambda', \mu', \nu')$, or $I(\lambda', \mu', \nu', \rho')$ denotes a logical proposition, $\hat{\lambda}[F(\hat{\lambda})]\{\lambda'\}$, or $\hat{\lambda}\hat{\rho}[G(\hat{\lambda}, \hat{\mu})]\{\lambda', \mu'\}$, or $\hat{\lambda}\hat{\mu}\hat{\nu}\ [H(\hat{\lambda}, \hat{\mu}, \hat{\nu})]\{\lambda', \mu', \nu'\}$, or $\hat{\lambda}\hat{\mu}\hat{\nu}\hat{\rho}\ [I(\hat{\lambda}, \hat{\mu}, \hat{\nu}, \hat{\rho})]\{\lambda', \mu', \nu', \rho'\}$ denotes the same logical proposition [3]). Here λ, λ' and μ, μ', and ν, ν' and ρ, ρ' are connexed one with another.

Note that here the alphabetical order and the order of variables in expressions $F(\lambda)$, $G(\lambda, \mu)$, $H(\lambda, \mu, \nu)$, $I(\lambda, \mu, \nu, \rho)$ are irrelevant.

0·161 If λ is a determined variable, and if ξ is an argument connexed with λ, then ξ is a determined variable.

[1]) Directions 0·13·131 correspond to ✕9·15 of Principia.

[2]) This direction correspond to ✳11·07 of Principia.

[3]) Without a direction of this kind we could not write e. g.

$$\alpha \varepsilon \hat{x}[\varphi\{x\}] = \varphi\{\alpha\}.$$

The need of a particular direction concerning this matter was first pointed out to me by Prof. Leśniewski.

0·17 Fundamental class-letters are undetermined functional signs of functional patterns with I argument.

0·171 Functional class-letters are undetermined functional signs of functional patterns with I argument being an individual variable.

0·172 Fundamental relation-letters are undetermined functional signs of functional patterns with II arguments.

0·173 Functional relation-letters are undetermined functional signs of functional patterns with II arguments being individual variables.

0·18 The functional class-letter λ stands for $\hat{\xi}[\lambda\{\hat{\xi}\}]$, where ξ is any individual letter.

0·181 The functional relation-letter λ stands for $\hat{\xi}\hat{\mu}[\lambda\{\hat{\xi},\hat{\mu}\}]$, where ξ, μ are individual letters.

0·19 Determined letters are fundamental letters determined by a functional class-letter or a functional relation-letter.

0·20 Types.

0·2 All primitive functional expressions with I (or II, or III, or IV) individual variables denote predicative functions of the same type.

0·23 If in a given functional expression we change the order of noted variables preceding th angular brackets, we get a functional expression denoting a function of the same type. E. g. expressions: $\hat{x}\hat{\varphi}[\hat{\varphi}\{\hat{x}\}]$, $\hat{\varphi}\hat{x}[\hat{\varphi}\{\hat{x}\}]$ denote functions of the same type.

0·24 If E, G are any expressions such that $.E|G.$ contains the noted variable λ and if $\hat{\lambda}[.E|G.]$ is a functional expression, then $\hat{\lambda}[.G|E.]$ denotes a function of the same type.

0·241 If E, G are any expressions such that $.E|G.$ contains the noted variable $\hat{\lambda}$, and if $\hat{\lambda}[.E|G.]$ is a functional expression, then $\hat{\lambda}[..E|G.|G.]$, denotes a function of the same type.

0.2411 If E, G are any expressions such that $\hat{\lambda}[E|p.]$, $\hat{\lambda}[.G|p.]$ denote functions of the same type, the expressions $\hat{\lambda}[E]$, $\hat{\lambda}[G]$ denote functions of the same type.

0·252 If E, G are any expressions and λ any real variable, then if $\hat{\lambda}[E]$, $\hat{\lambda}[G]$ denote functions of the same type, and if $\hat{\lambda}[.E|G.]$

is a functional expression, it denotes a function of the same type.

0·243 If $\hat{\lambda}[E]$, $\hat{\lambda}[G]$ denote functions of the same type, then if $\hat{\lambda}[.E|H.]$, $\hat{\lambda}[.G|H.]$ are functional expressions, they denote functions of the same type.

0·25 If $E(\hat{\lambda}, \hat{\mu})$, $G(\hat{\nu}, \hat{\rho})$ are expressions containing noted variables $\hat{\lambda}, \hat{\mu}$, or $\hat{\nu}, \hat{\rho}$, and if $.E(\lambda, \mu)|G(\nu, \rho).$ is a propositional expression, the expressions: $\hat{\lambda}\hat{\mu}[E(\hat{\lambda}, \hat{\mu})]$, $\hat{\nu}\hat{\rho}[G(\hat{\nu}, \hat{\rho})]$ denoting functions of the same type, then the expressions: $\hat{\lambda}[(\overline{\mu})E(\hat{\lambda}, \overline{\mu})]$, $\hat{\nu}[(\overline{\rho})G(\hat{\nu}, \overline{\rho})]$ denote functions of the same type.

0·26 If $E(\lambda)$, $G(\mu)$ are expressions denoting functions of the same type, and containing the real variables λ, or μ, the expression $\nu\{E(\mu), G(\mu)\}$, where ν is a primitive or fundamental letter absent in $E(\mu)$ and $G(\mu)$, being a propositional expression, the expressions: $\hat{\mu}E(\hat{\mu})$, $\hat{\mu}G(\hat{\mu})$ denote functions of the same type.

0·261 Given the individual, or fundamental letter, or the functional expression F and the expressions $E(\hat{\lambda})$, $G(\hat{\lambda})$ containing the noted variable $\hat{\lambda}$, then if $\hat{\lambda}E(\hat{\lambda})$, $\hat{\lambda}G(\hat{\lambda})$ denote functions of the same type, and if $E(F)$, $G(F)$ are functional expressions, these expressions denote functions of the same type.

0·27 If $E(\overline{\rho}, \hat{\lambda})$ is any expression containing the variables $\overline{\rho}$ and $\hat{\lambda}$ and H is any propositional expression, then if $\hat{\lambda}[(\overline{\rho}).H|E(\overline{\rho}, \hat{\lambda}).]$ $\hat{\lambda}[.H|(\overline{\rho})E(\overline{\rho}, \hat{\lambda}).]$ are functional expressions, they denote functions of the same type[1]).

0·271 If $E(\lambda)$ is any expression containing the fundamental real variable λ, then the expressions $\hat{\lambda}[E(\hat{\lambda})]$. $\hat{\lambda}[.\hat{\lambda}\{\xi\}|E(\hat{\lambda}).]$, respectively $\hat{\lambda}[E(\hat{\lambda})]$, $\hat{\lambda}[.\hat{\lambda}\{\xi, \eta\}|E(\hat{\lambda}).]$ denote functions of the same type, if they are functional expressions.

0·28 If E, F, G are any expressions such that the expression: $.E|.F|G..$ contains the noted variable $\hat{\lambda}$, and if the expressions: $\hat{\lambda}[.E|.F|G..]$, $\hat{\lambda}[..E|F.|G.]$ are functional expressions, they denote functions of the same type.

Remark: Statements 0·2—0·28 concern the idea of „being of the same type". In Principia, we have a definition of this idea (* 9·131). Nevertheless, it is 1° a verbal definition, 2° it seems

[1]) This direction corresponds to the definitions * 9·03·04 of Principia.

to imply vicious circle, because to define the idea of being of the same type, we must use the very same idea. Now, verbal directions are essentially different from the proper propositions of the system. Therefore, there is no advantage in putting them in the form of a definition.

0·29 If there is a significant expression, which contains functional expressions F, G as corresponding arguments of the same functional sign, then E, F denote functions of the same type [1]).

0·30 Definitions

0·3 Given any expression E, we can use instead of E any other expression Ω, if it has 1^0 no meaning in isolation, 2^0 if it contains no significant letters or expressions unless real apparent or noted variables, elementary letters or funtional expressions present in E, 3^0 if it has no such components X, Y that Ω is $X\,Y$ and X or Y is a functional (or propositional) expression, E being a functional (or propositional) expression. Then we write:

$$\Omega \underset{df}{=} E.$$

This expression is a definition. Here E is the defining expression, Ω the defined symbol.

0·31 In a defined symbol Ω we can turn real variables into noted or apparent variables and we can take a functional expression for a determined real variable, but no other modifications of the defined symbols can be allowed. [2])

0·32 If $\Omega \underset{df}{=} E$ and $F(E)$ is a propositional (or functional) expression, then $F(\Omega)$ has the same meaning as $F(E)$.

0·321 If $\Omega \underset{df}{=} E$, and if $F(E')$, $F(\Omega)$ are propositional expressions, E' having the same meaning as E, the expressions $F(E')$, $F(\Omega)$ have the same meaning.

0·33 $\sim p \underset{df}{=} .p|p.$

0·34 $.p \vee q. \underset{df}{=} ..p|q.|.p|q..$ [3])

1·01 $.p \supset q. \underset{df}{=} .\sim p \vee q.$

---

[1]) This direction corresponds partly to * 9·14 of Principia.

[2]) Without such a direction we could never be sure to avoid ambiguities as noted in Chap. I. [3]) cf. l. c.

0·40 Substitution.

0·4 In any significant expression E take for any individual (primitive or fundamental) letter used as an apparent (or noted) variable, any other individual (primitive or fundamental) letter absent in E. We get an expression E' having the same meaning as E.

0·41 In any propositional (or functional) expression E, take for any functional expression (1), or any individual (2), (or primitive (3), or undetermined fundamental (4)) real variable, in some of its occurrences, any fundamental letter absent from E, which appears after the substitution to be a determined variable (1), or any individual (2), (or primitive (3), or fundamental (4)) real variable, being absent from E. We get a propositional (or functional) expression E'.

Compatible expressions.

0·42 If two significant expressions are present in a given significant expression, they are compatible expressions.

0·4201 Any significant expressions are compatible in respect of a common elementary letter.

0·421 Two significant expressions are compatible in respect of a common individual letter, if this letter is in both expressions used as a real, (respectively noted, or apparent) variable.

0·422 Two significant expressions are compatible in respect of a common primitive or fundamental letter, if this letter is in both expressions used as real, (respectively noted, or apparent) variable, and if it occurs in both expressions as a part of a common functional expression. [1]).

0·423 Two significant expressions are compatible in respect of a common fundamental letter, if it occurs in both expressions as a real, (noted, or apparent) variable, or if it is determined in both expressions by the same expression.

0·424 Two significant expressions are compatible, if they are compatible in respect of all common letters.

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[1]) Directions 0·42—0·422 are conform to the practice of Principia.

Application to the construction of significant expressions.

0·43 If ξ, η, ζ, ϑ are any compatible functional expressions or individual letters, and λ is any primitive, or fundamental letter absent from ξ, η, ζ, ϑ, then: $\lambda\{\xi\}$, $\lambda\{\xi, \eta\}$, $\lambda\{\xi, \eta, \zeta\}$ and $\lambda\{\xi, \eta, \zeta, \vartheta\}$ are propositional expressions.

E. g. the following expressions are propositional expressions,

$$\varphi\{x\},\ \varphi\{x, y\},\ \varphi\{x, y, z\},\ \varphi\{x, y, z, x'\}$$
$$f\{x\},\ f\{x, y\},\ f\{x, y, z\},\ f\{x, y, z, x'\}$$
$$\varphi\{\hat{x}[h\{\hat{x}\}]\},\ \varphi\{x, \hat{z}[h\{\hat{z}\}]\},\ f\{\hat{\varphi}[\hat{\varphi}\{x\}]\}.$$

Subordinate expressions.

0·44 In any expression E, take for any elementary letters, any propositional expressions compatible one with another and with E. We get a subordinate expression E' [1]).

E. g. the following expressions are propositional expressions:

$$\sim.\sim\varphi\{\hat{x}[h\{\hat{x}\}]\} \vee \sim\varphi\{\hat{x}[h\{\hat{x}\}]\}.$$
$$\sim.\sim\varphi\{g\} \vee \sim\varphi\{\hat{x}[h\{\hat{x}\}]\}. \qquad [0{\cdot}41]$$
$$\sim.\sim\hat{x}[h\{\hat{x}\}]\{a\} \vee \sim(\overline{\varphi})\sim.\sim\overline{\varphi}\{\hat{x}[h\{\hat{x}\}]\} \vee \sim\overline{\varphi}\{\hat{x}[h\{\hat{x}\}]\}..$$
$$\sim.\sim g\{a\} \vee \sim(\overline{\varphi})\sim.\sim\overline{\varphi}\{g\} \vee \sim\overline{\varphi}\{\hat{x}[h\{\hat{x}\}]\}.. \qquad [0{\cdot}41]$$

0·441 Given any propositional (or functional) expression E containing an individual (or primitive) real variable ξ, or a fundamental real variable η, determined by a functional expression H, take for ξ any individual (or primitive) letter absent in E, or used in E as a real variable, and for η, any fundamental real variable absent in E, or determined by H, or any functional expression compatible with E and denoting a function of the same type as the function denoted by H, We get a subordinate propositional (or functional) expression E', compatible with E. [2])

E. g. if E is $(\overline{\varphi})\sim.\sim\overline{\varphi}\{g\} \vee \overline{\varphi}\{\hat{x}[h\{\hat{x}\}]\}.$, then E' can be: $(\overline{\varphi})\sim.\sim\overline{\varphi}\{\hat{x}[\sim h\{\hat{x}\}]\} \vee \overline{\varphi}\{\hat{x}[h\{\hat{x}\}]\}.$

Derived expressions.

0·45 In any propositional expression E, take for some elementary letters any propositional expressions compatible one with another and with E. We get a derived expression E'.

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[1]) This direction corresponds to * 9·61·62·63 * 10·13 of Principia.

[2]) Directions 0·29·341 correspond to * 9·14 of Principia.

0·451 In any propositional expression E, containing no implicit individual letters, i. e. no symbol Ω defined by an expression E' containing individual letters absent in Ω, take for all individual letters any functional expressions denoting functions of the same type and compatible one with another and with E. Then we get an expression E'. If E' is a propositional expression, it is a derived expression in respect of E.

0·452 Given any propositional expression E, containing fundamental real variables, which never occur as arguments of a functional sign (undetermined variables), take for these letters any functional expressions, compatible one with another and with E, whose variables appear after the substitution to be determined by functional expressions denoting functions of the same type as those denoted by the connexed arguments, or to be individual, or primitive variables at the same time as the connexed arguments, then we get a derived propositional expression E' [1]).

The Logical Calculus.

B. Directions concerning the use of symbols.

Any Logical Calculus must follow fundamental directions of the use of symbols, i. e. the Modus Ponens, the Law of Generalisation and the Law of Substitution. As an abbreviation, useful for avoiding the repetition of primitive propositions and proofs for functions of the same type as a given function, I also assume the Law of the Automatical construction of Assertions. To understand the use of these directions the following remarks seem to be necessary. Directions concerning the meaning of symbols enable us to have as many significant expressions as we choose. Suppose we have a list of expressions denoting logical propositions. It is interesting to have a method of discerning the expressions denoting true logical propositions from other expressions present in our list. Now, we assume some primitive propositions, which are common rules of the Logical Calculus. The expressions denoting these propositions are expressions denoting true propositions. Other

[1]) The use of the derived expressions is conform to the practice of Principia.

expressions denoting true propositions are to be got by the following directions.

0·5 Any expression asserted in the system will be preceded by the sign $\vdash$.

Modus Ponens.

0·51 If we have the assertions: $\vdash E$ and $\vdash . E \supset F .$, where E, F are any propositional expressions, we can assume the assertion $\vdash F$.

Remark: Note that in Principia, some difficulties arise by the use of the Modus Ponens. E. g. we can get the following assertions:

(1) $\vdash \omega \varepsilon \iota' \iota' \iota' x$, (2) $\vdash . \omega \varepsilon \iota' \iota' \iota' x \supset . N_0 c' \omega \cup - \omega > 2 ..$

Now, by the Modus Ponens we should have:

(3) $$\vdash . N_0 c' \omega \cup - \omega > 2.$$

Here ω is undetermined as to type; therefore we can take for ω e. g. $\iota' x$ and we get the doubtful assertion: $\vdash . N_0 c' \iota' x > 2$ [1]. In our system there is no means of proving that $N_0 c' \omega \cup - \omega > 2$ is a propositional expression [0·41·43], as we have no other undetermined variables as functional signs of certain functional patterns.

The Law of Generalisation.

0·52 If an expression E, containing an individual, primitive, or determined real variable, is asserted, then the expression E' we get from E by turning this variable into an apparent variable can be asserted [2].

The Law of Substitution.

0·53 If an expression E is asserted, then its subordinate expressions can be asserted.

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[1] Cf. Principia Vol II p. 35.

[2] Cf. Principia Vol II. p. 35.

The Law of Automatical Construction of assertions.

0·54 If an expression E is asserted, then its derived expression E' can be asserted.

Remarks: 1. Note that this direction is constantly used by Whitehead and Russell. Using it, we tacitly assume that the given proposition is got from primitive propositions concerning functions of the same type as those occurring in our proposition, by the directions 0·51·52·53.

2. These directions seem to be quite sufficient to get all true propositions denoted by expressions present in our list. Now we may hope to have complete demonstrations of our propositions [1]). It is obvious that we have no need of a definition of demonstration. I think such a definition cannot be useful in any science. At any rate I cannot agree with a theory of demonstration based on the idea of finite numbers.

C. Primitive Propositions.

[The following primitive propositions are taken from Principia with their numbers and names].

The Principle of Tautology [Taut.]

1·2 $\vdash .. p \vee p . \supset p .$

The Principle of Addition [Add.]

1·3 $\vdash . q \supset . p \vee q ..$

The Principle of Permutation [Perm.]

1·4 $\vdash .. p \vee q . \supset . q \vee p ..$

The Principle of Association [Assoc.]

1·5 $\vdash .. p \vee . q \vee r .. \supset . q \vee . p \vee r ...$

The Principle of Summation [Sum.]

1·6 $\vdash .. q \supset r . \supset .. p \vee q . \supset . p \vee r ...$

The following primitive propositions are to be assumed in Principia, if we choose the „alternative method". [2])

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[1]) On this subject cf. the important paper of Prof. Zaremba: Essai sur la théorie de la démonstration dans les sciences mathématiques. L'enseignement mathém, Nr. 1. 1916.

[2]) Cf. Principia, Vol I. p. 144.

The Principle of Deduction [Ded.]

10·1 $\vdash .(\bar{x})\, f\{\bar{x}\} \supset f\{y\}.$

The Principle of Disjunction [Disj.]

10·12 $\vdash .(\bar{x}).\, p \vee f\{\bar{x}\} .\supset .\, p \vee (\bar{x}) f\{\bar{x}\}..$

III. Logical Calculus. Functions of the same Type.

A. Logical Calculus.

With our directions and primitive propositions we get the Logical Calculus of elementary propositions by applying the method of Principia (numbers $*1 - *5$). There is no reason to repeat this here. I shall simply quote the definitions and propositions to be used below as lemmas..

2·001 $:=_{df}..$

2·02 $\vdash .\, q \supset .\, p \supset q :$

2·08 $\vdash .\, p \supset p .$

2·21 $\vdash .\sim p \supset .\, p \supset q :$

3·01 $.\, p\, .\, q .=_{df} \sim .\sim p \vee \sim q .$

3·26 $\vdash : p\, .\, q .\supset p .$

3·33 $\vdash :. p \supset q :. q \supset r : \supset .\, p \supset r :$

3·35 $\vdash : p : p \supset q : \supset q .$

4·01 $.\, p \equiv q .=_{df} : p \supset q :. q \supset p :$

4·2 $\vdash .\, p \equiv p .$

5·21 $\vdash : p \equiv q .\equiv .\; q \equiv p :$

4·22 $\vdash :. p \equiv q :. q \equiv r : \supset .\, p \equiv r :$

4·3 $\vdash : p\, .\, q .\equiv .\, q\, .\, p :$

4·32 $\vdash :. p\, .\, q : r .\equiv .\, p : q\, .\, r :.$

4·36 $\vdash : p \equiv q .\supset : p\, .\, r .\equiv .\, q\, .\, r :$

4·83 $\vdash :. p \supset q :. \sim p \supset q : \supset q .$

4·85 $\vdash : p \equiv q .\supset : r \supset p .\equiv .\, r \supset q :.$

5·32 $\vdash : p \supset .\, q \equiv r : \equiv : p\, .\, q .\equiv .\, p\, .\, r :.$

5·34 $\vdash .\sim p \supset : p\, .\, q .\equiv .\, p\, .\, r :.$ [1])

[1]) This proposition seems to be absent in Principia.

Dem. $\vdash .2\cdot21 \left[\frac{\cdot q \equiv r \cdot}{q} \right] \supset$

$\vdash . \sim p \supset . p \supset . q \equiv r :.$

$[3\cdot33.5\cdot32] \supset \vdash .$ Prop.

Analogous remarks as those given above apply to the theory of apparent variables (numbers * 10 and * 11 of Principia). Nevertheless this theory cannot be taken textually from Principia, the following modifications being necessary:

1° We omit all verbal propositions, using our directions exclusively.

2° We use the expressions given above instead of those of Principia.

3° We use other definitions.

4° Numbers * 9 and * 12 of Principia are to be omitted.

Definitions.

$10\cdot01 \quad (\exists \bar{x}) \underset{df}{=} \sim(\bar{x}) \sim$

$10\cdot011 \quad (\exists \bar{u}) \underset{df}{=} \sim(\bar{u}) \sim$

The Whitehead-Russellian definition: $(\exists \bar{x}) f\{\bar{x}\} = \sim(\bar{x}) \sim f\{\bar{x}\}$ is useless, because e. g. the expression $(\exists \bar{x}) f\{\bar{x}, y\}$ has no meaning at all. We could as a matter of fact use the expression: $(\exists \bar{x}) \hat{z}[f\{\hat{z}, y\}]\{\bar{x}\}$, but here $\hat{z}[f\{\hat{z}, y\}]\{\bar{x}\}$ would not be treated as a functional pattern, no other operations than simple substitution being allowed in defined symbols. For the same reason we cannot use the abbreviations * 10·02·03 of Principia.

$11\cdot01 \quad (\bar{x}, \bar{y}) \underset{df}{=} (\bar{x})\,(\bar{y})$

$11\cdot02 \quad (\bar{x}, \bar{y}, \bar{z}) \underset{df}{=} (\bar{x})\,(\bar{y})\,(\bar{z})$

$11\cdot021 \quad (\bar{x}, \bar{y}, \bar{z}, \bar{x}') \underset{df}{=} (\bar{x})\,(\bar{y})\,(\bar{z})\,(\bar{x}')$

$11\cdot03 \quad (\exists \bar{x}, \bar{y}) \underset{df}{=} (\exists \bar{x})\,(\exists \bar{y})$

$11\cdot04 \quad (\exists \bar{x}, \bar{y}, \bar{z}) \underset{df}{=} (\exists \bar{x}, \bar{y})\,(\exists \bar{z})$

Analogous abbreviations are to be assumed for fundamental letters. The following propositions are to be used below:

$10\cdot24 \quad \vdash . f\{y\} \supset (\exists \bar{x}) f\{\bar{x}\} .$

10·271 $\vdash .(\bar{z}) . f\{\bar{z}\} \equiv g\{\bar{z}\} . \supset . (\bar{z}) f\{\bar{z}\} \equiv (\bar{z}) g\{\bar{z}\} :$

10·281 $\vdash .(\bar{z}) . f\{\bar{z}\} \equiv g\{\bar{z}\} . \supset . (\exists \bar{z}) f\{\bar{z}\} \equiv (\exists \bar{z}) g\{\bar{z}\} :$

11·35 $\vdash .(\bar{x}, \bar{y}) . f\{\bar{x}, \bar{y}\} \supset p . \equiv . (\exists \bar{x}, \bar{y}) f\{\bar{x}, \bar{y}\} \supset p :$

B. Functions of the same type.

For expressions containing determined variables, let us use following definitions.

12·01 $\mathcal{C}\{x, y\} = \underset{df}{} (\bar{\varphi}) : \bar{\varphi}\{x\} \supset \bar{\varphi}\{x\} . \supset . \bar{\varphi}\{y\} \supset \bar{\varphi}\{y\} :$

12·02 $f_x\{y\} = \underset{df}{} . f\{y\} . \mathcal{C}\{y, x\} .$

12·021 $f_a\{u, z\} = \underset{df}{} . f\{u, z\} . \mathcal{C}\{u, a\} .$

12·022 $f_a\{z, u\} = \underset{df}{} . f\{z, u\} . \mathcal{C}\{u, a\} .$

12·03 $f_{a, x}\{u, z\} = \underset{df}{} : f\{u, z\} . \mathcal{C}\{u, a\} : \mathcal{C}\{z, x\} .$

12·031 $f_{a, x}\{z, u\} = \underset{df}{} : f\{z, u\} . \mathcal{C}\{u, a\} : \mathcal{C}\{z, x\} .$

12·0311 $f_{x, a}\{z, u\} = \underset{df}{} : f\{z, u\} . \mathcal{C}\{z, x\} : \mathcal{C}\{u, a\} .$

12·032 $f_{x, y}\{x', y'\} = \underset{df}{} : f\{x', y'\} . \mathcal{C}\{x', x\} : \mathcal{C}\{y', y\} .$

Note that here any other operations than simple substitutions are impossible. Take e. g. in 12·02 the expression $\hat{z}[.\varphi\{\hat{z}\} \vee \psi\{\hat{z}\}.]$ for f. We get the expression: $\hat{z}[.\varphi\{\hat{z}\} \vee \psi\{\hat{z}\}.]_x\{y\}$, which is no functional pattern, the letter x being present. Therefore for the further use of our symbol we must use the definition 12·02.

I pass now to the proof of the proposition: $\vdash \mathcal{C}\{x, y\}$. This proposition enable us to construct a formal method to get as many expressions as we choose denoting functions of the same type. Some applications of this method are given below [1]).

12·1 $\vdash \mathcal{C}\{x, y\}$

Dem. $\vdash$ 2·08 $\supset \vdash . \varphi\{y\} \supset \varphi\{y\}.$ (1)

[2·02] $\supset \vdash : \varphi\{y\} \supset \varphi\{y\} . \supset :$

$: \varphi\{x\} \supset \varphi\{x\} . \supset . \varphi\{y\} \supset \varphi\{y\} :.$ (2)

(1) . (2) . 0·51 $\supset \vdash : \varphi\{x\} \supset \varphi\{x\} . \supset . \varphi\{y\} \supset \varphi\{y\} :$

[0·52] $\supset \vdash .$ Prop.

[1]) The method of writing the demonstrations is textually taken from Principia.

The following propositions are immediate consequences of 12·1.

12·11 $\vdash . \mathcal{C}\{x, y\} \equiv \mathcal{C}\{y, x\}$. [5·1. 12·1. 0·51]

12·12 $\vdash : \mathcal{C}\{x, y\} . \mathcal{C}\{x, z\} . \supset \mathcal{C}\{y, z\}$. [12·1. 2·02. 0·51]

The following propositions appear to be true, if significant, by 12·1 and 2·02.

12·22 $\vdash \mathcal{C}\{\hat{x}[g\{\hat{x}\}], \hat{x}[\sim g\{\hat{x}\}]\}$

Constr [1]). $\vdash$ 12·1. 0·451·252·33·421·422·424·53 $\supset$

$\vdash \mathcal{C}\{\hat{x}[\hat{z}[g\{\hat{z}\}|\{\hat{x}\}], \hat{x}[\sim \hat{z}[g\{\hat{z}\}]\{\hat{x}\}]\}$.

[0·16] $\supset \vdash$ Prop.

In an analogous manner we get:

12·23 $\vdash \mathcal{C}\{\hat{x}\hat{y}[g\{\hat{x}, \hat{y}\}], \hat{y}\hat{x}[g\{\hat{x}, \hat{y}\}]\}$.

12·2301 $\vdash \mathcal{C}\{\hat{x}\hat{u}[g_a\{\hat{x}, \hat{u}\}], \hat{u}\hat{x}[g_a\{\hat{x}, \hat{u}\}]\}$.

12·24 $\vdash \mathcal{C}\{\hat{x}[.p|g\{\hat{x}\}.], \hat{x}[.g\{\hat{x}\}|p.]\}$.

12·241 $\vdash \mathcal{C}\{\hat{x}[.g\{\hat{x}\}|p.], \hat{x}[.g\{\hat{x}\}|\sim p.]\}$. [0.241·243·28]

12·242 $\vdash . \mathcal{C}\{f, \hat{x}[g\{\hat{x}\}]\} \supset \mathcal{C}\{\hat{x}[g\{\hat{x}\}], \hat{x}[.g\{\hat{x}\}|f\{\hat{x}\}.]\}$.

12·2421 $\vdash . \mathcal{C}\{f, \hat{x}\hat{y}[g\{\hat{x}, \hat{y}\}]\} \supset \mathcal{C}\{\hat{x}\hat{y}[g\{\hat{x}, \hat{y}\}], \hat{x}\hat{y}[.g\{\hat{x}, \hat{y}\}|f\{\hat{x}, \hat{y}\}.]\}$.

Constr. $\vdash$ 12·1·242 $\supset$

$. \mathcal{C}\{\hat{x}[g\{\hat{x}, y\}, \hat{x}[g\{\hat{x}, y\}]\} \supset \mathcal{C}\{\hat{x}[g\{\hat{x}, y\}], \hat{x}[.g\{\hat{x}, y\}|g\{\hat{x}, y\}.]\}$.

[0·26·23] $\supset \vdash . \mathcal{C}\{\hat{x}\hat{y}[g\{\hat{x}, \hat{y}\}, \hat{x}\hat{y}[g\{\hat{x}, \hat{y}\}]\} \supset$

$\mathcal{C}\{\hat{x}\hat{y}|g\{\hat{x}, \hat{y}\}], \hat{x}\hat{y}[.g\{\hat{x}, \hat{y}\}|g\{\hat{x}. \hat{y}\}.]\}$.

[0·16·4] $\supset \vdash . \mathcal{C}\{\hat{z}\hat{z}'[g\{\hat{z}, \hat{z}'\}], \hat{x}\hat{y}[g\{\hat{x}, \hat{y}\}]\} \supset$

$\mathcal{C}\{\hat{x}\hat{y}[g\{\hat{x}, \hat{y}\}, \hat{x}\hat{y}[.g\{\hat{x}, \hat{y}\}|\hat{z}\hat{z}'[g\{\hat{z}, \hat{z}'\}]\{\hat{x}, \hat{y}\}.]\}$.

[0·41] $\supset \vdash$ Prop.

12·2422 $\vdash . \mathcal{C}\{f, \hat{x}\hat{u}[g_a\{\hat{x}, \hat{u}\}]\} \supset$

$\mathcal{C}\{\hat{x}\hat{u}[g_a\{\hat{x}, \hat{u}\}], \hat{x}\hat{u}[.g_a\{\hat{x}, \hat{u}\} \vee f\{\hat{x}, \hat{u}\}.]\}$.

12·25 $\vdash . \mathcal{C}\{f, \hat{x}\hat{z}[g\{\hat{x}, \hat{z}\}]\} \supset \mathcal{C}\{\hat{x}[(\overline{y})f\{\hat{x}, \hat{y}\}], \hat{x}[(\overline{y})g\{\hat{x}, \overline{y}\}]\}$.

12·2501 $\vdash . \mathcal{C}\{f, \hat{x}\hat{v}[g_a\{\hat{x}, v\}]\} \supset$

$\mathcal{C}\{\hat{x}[(\overline{u})f_a\{\hat{x}, \overline{u}\}], \hat{x}[(\overline{u})g_a\{\hat{x}, \overline{u}\}]\}$.

12·2502 $\vdash . \mathcal{C}\{f, \hat{x}\hat{v}[g_a\{\hat{x}, \hat{v}\}]\} \supset$

$\mathcal{C}\{\hat{u}[(\overline{x})f_a\{\overline{x}, \hat{u}\}, \hat{u}[(\overline{x})g_a\{\overline{x}, \hat{u}\}]\}$.

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[1]) Here „Constr." is an abbreviation of „Construction". I use this symbol instead of „Dem" (demonstration) in proofs concerning the meaning and not the truth of propositions.

12·261 $\vdash . \mathcal{C}\{f, \hat{x}\hat{y}[g\{\hat{x}, \hat{y}\}]\} \supset$
$\mathcal{C}\{\hat{x}[f\{\hat{x}, z\}], \hat{x}[g\{\hat{x}, z\}]\}.$

12·2611 $\vdash . \mathcal{C}\{f, \hat{x}\hat{u}[g_a\{\hat{x}, \hat{u}\}]\} \supset$
$\mathcal{C}\{\hat{x}[f_a\{\hat{x}, t\}], \hat{x}[g_a\{\hat{x}, t\}]\}.$

12·2612 $\vdash . \mathcal{C}\{f, \hat{x}\hat{u}[g_a\{\hat{x}, \hat{u}\}]\} \supset$
$\mathcal{C}\{\hat{u}[f_a\{z, \hat{u}\}], \hat{u}[g_a\{z, \hat{u}\}]\}.$

12·27 $\vdash \mathcal{C}\{\hat{y}[(\overline{x}) . p | f\{\overline{x}, \hat{y}\} .], \hat{y}[. p | (\overline{x}) f\{\overline{x}, \hat{y}\} .]\}$

12·2701 $\vdash \mathcal{C}\{\hat{y}[(\overline{u}) . p | f_a\{\overline{u}, \hat{y}\} .], \hat{y}[. p | (\overline{u}) f_a\{\overline{u}, \hat{y}\} .]\}$

12·2702 $\vdash \mathcal{C}\{\hat{u}[(\overline{x}) . p | f_a\{\overline{x}, \hat{u}\} .], \hat{u}[. p | (\overline{x}) f_a\{\overline{x}, \hat{u}\} .]\}$

12·271 $\vdash \mathcal{C}\{\hat{u}[f_a\{\hat{u}\}], \hat{u}[. \hat{u}\{x\} | f_a\{\hat{u}\} .]\}$

12·2711 $\vdash \mathcal{C}\{\hat{u}[f_R\{\hat{u}\}], \hat{u}[. \hat{u}\{x\} | f_R\{\hat{u}\} .]\}$

12·2712 $\vdash \mathcal{C}\{\hat{x}[f_a\{\hat{x}, u\}], \hat{x}[. u\{\hat{x}, y\} | f_a\{\hat{x}, u\} .]\}$

12·2713 $\vdash \mathcal{C}\{\hat{x}[f_R\{\hat{x}, u\}], \hat{x}[. u\{\hat{x}, y\} | f_R\{\hat{x}, u\} .]\}$

12·28 $\vdash \mathcal{C}\{\hat{x}[. f\{\hat{x}\} | . p | q :], \hat{x}[: f\{\hat{x}\} | p . | q .]\}$

12·2801 $\vdash \mathcal{C}\{\hat{x}[. f\{\hat{x}\} | . g\{\hat{x}\} | q :], \hat{x}[: f\{\hat{x}\} | g\{\hat{x}\} . | q .]\}$

12·2802 $\vdash \mathcal{C}\{\hat{x}[. f\{\hat{x}\} | . g\{\hat{x}\} | h\{\hat{x}\} :], \hat{x}[: f\{\hat{x}\} | g\{\hat{x}\} . | h\{\hat{x}\} .]\}$

12·2803 $\vdash \mathcal{C}\{\hat{x}[. p | . g\{\hat{x}\} | h\{\hat{x}\} :], \hat{x}[: p | g\{\hat{x}\} . | h\{\hat{x}\} .]\}$

12·2804 $\vdash \mathcal{C}\{\hat{x}[. p | . q | h\{\hat{x}\} :], \hat{x}[: p | q . | h\{\hat{x}\} .]\}$

I pass to the following applications of propositions: 12·22—12·2804.

12·3 $\vdash \mathcal{C}\{\hat{x}[. f\{\hat{x}\} \vee g\{\hat{x}\} .], \hat{x}[. f\{\hat{x}\} | g\{\hat{x}\} .]\}$

Constr. $\vdash$ 12·242. 0·34 $\supset \vdash$ Prop.

12·311 $\vdash \mathcal{C}\{\hat{x}[: f\{\hat{x}\} \vee g\{\hat{x}\} . \vee h\{\hat{x}\} .], \hat{x}[. f\{\hat{x}\} \vee . g\{\hat{x}\} \vee h\{\hat{x}\} :]\}$

Constr. $\vdash . 12·3.0.243 . \supset .$

$\vdash \mathcal{C}\{\hat{x}[: f\{\hat{x}\} \vee g\{\hat{x}\} . | h\{\hat{x}\} .], \hat{x}[: f\{\hat{x}\} | g\{\hat{x}\} . | h\{\hat{x}\} .]\}.$

$[12·3·2802] \supset \mathcal{C}\{[: f\{\hat{x}\} \vee g\{\hat{x}\} . \vee h\{\hat{x}\} .], \hat{x}[. f\{\hat{x}\} | . g\{\hat{x}\} | h\{\hat{x}\} :]\}$ (1)

$\vdash [12·3.0·243·24] \supset .$

$\mathcal{C}\{x | . f\{\hat{x}\} | . g\{\hat{x}\} | h\{\hat{x}\} :], \hat{x}[. f\{\hat{x}\} \vee . g\{\hat{x}\} \vee g\{\hat{x}\} :]\}.$ (2)

$\vdash . (1). (2). 12.12 . \supset \vdash$ Prop.

12·312 $\vdash \mathcal{C}\{\hat{x}[. g\{\hat{x}\} \vee f\{\hat{x}\} .], \hat{x}[. \sim g\{\hat{x}\} \vee f\{\hat{x}\} .]\}$

Constr. $\vdash . \mathcal{C}\{\hat{x}[. \hat{z}[g\{\hat{z}\}]\{\hat{x}\} \vee f\{\hat{x}\} .], \hat{x}[. \hat{z}[g\{\hat{z}\}]\{\hat{x}\} \vee f\{\hat{x}\} .]\}$
$. \mathcal{C}\{\hat{z}[g\{\hat{z}\}], \hat{z}[g\{\hat{z}\}]\} .$ [0·43·44·14·15]
$[12·22.0·54·441] \supset$

$\vdash . \mathcal{C}\{\hat{x}[.\hat{z}[g\{\hat{z}\}]\{\hat{x}\} \vee f\{\hat{x}\} .], \hat{x}[.\hat{z}[\sim g\{\hat{z}\}]\{\hat{x}\} \vee f\{\hat{x}\} .]\}.$
$\mathcal{C}\{\hat{z}[\sim g\{\hat{z}\}], \hat{z}[g\{\hat{z}\}]\}.$
[3·27.0·51] $\supset$
$\vdash \mathcal{C}\{\hat{x}[.\hat{z}[g\{\hat{z}\}]\{\hat{x}\} \vee f\{\hat{x}\} .], \hat{x}[.\hat{z}[\sim g\{\hat{z}\}]\{\hat{x}\} \vee f\{\hat{x}\}.]\}$
[0·16] $\supset \vdash .$ Prop.

12·313 $\vdash \mathcal{C}\{\hat{x}[.g\{\hat{x}\} \vee f\{\hat{x}\} .], \hat{x}[:g\{\hat{x}\} \vee g\{\hat{x}\} . \vee f\{\hat{x}\} .]\}$
Constr. as 12·3.

12·32 $\vdash \mathcal{C}\{\hat{x}[.g\{\hat{x}\} . f\{\hat{x}\} .], \hat{x}[.g\{\hat{x}\} \vee . g\{\hat{x}\} . f\{\hat{x}\} :]\}$
Constr. $\vdash$ 12·22.3·01 $\supset$
$\vdash \mathcal{C}\{\hat{x}[.g\{\hat{x}\} . f\{\hat{x}\} .], \hat{x}[.\sim g\{\hat{x}\} \vee \sim f\{\hat{x}\} .]\}$
[12·313.0·54.12·12] $\supset$
$\vdash \mathcal{C}\{\hat{x}[.g\{\hat{x}\} . f\{\hat{x}\} .], \hat{x}[:\sim g\{\hat{x}\} \vee \sim g\{\hat{x}\} . \vee \sim f\{\hat{x}\}.]\}$
[12·311.0·54.12·12] $\supset$
$\vdash \mathcal{C}\{\hat{x}[.g\{\hat{x}\} . f\{\hat{x}\} .], \hat{x}[.\sim g\{x\} \vee . \sim g\{\hat{x}\} \vee \sim f\{\hat{x}\} :]\}$ (1)
[12·312.0·54.12·12.3·01] $\supset$
$\vdash \mathcal{C}\{\hat{x}[.g\{\hat{x}\} . f\{\hat{x}\} .], \hat{x}[.\sim g\{\hat{x}\} \vee . g\{\hat{x}\} . f\{\hat{x}\} :]\}$
[12·312.0·54.12·12] $\supset \vdash$ Prop.

12·33 $\vdash \mathcal{C}\{\hat{x}[.u\{\hat{x}\} : f_\alpha\{u, \hat{x}\} . f_\alpha\{v, \hat{x}\} :], \hat{x}[f_\alpha\{v, \hat{x}\}]\}$
Constr. $\vdash .$ 12·241.0·411.12·22 $\supset$
$\vdash \mathcal{C}\{\hat{x}[.f_\alpha\{u, \hat{x}\} . f_\alpha\{u, \hat{x}\} .], \hat{x}[f_\alpha\{u, \hat{x}\}]\}$
[0·41.12·1] $\supset \vdash \mathcal{C}\{\hat{x}[.f_\alpha\{u, \hat{x}\} . f_\alpha\{v, \hat{x}\} .], \hat{x}[f_\alpha\{v, \hat{x}\}]\}$ (1)
$\vdash .$ 12·2712 $\supset$
$\vdash \mathcal{C}\{\hat{x}\hat{u}[.\hat{u}\{\hat{x}\} : f_\alpha\{\hat{u}, \hat{x}\} . f_\alpha\{v, \hat{x}\} :], \hat{x}\hat{u}[.f_\alpha\{\hat{u}, \hat{x}\} . f_\alpha\{v, \hat{x}\} .]\}$
[12·2611] $\supset$
$\vdash \mathcal{C}\{\hat{x}[.u\{\hat{x}\} : f_\alpha\{u, \hat{x}\} . f_\alpha\{v, \hat{x}\} :], \hat{x}[.f_\alpha\{u, \hat{x}\} . f_\alpha\{v, \hat{x}\} .]\}$
[(1). 12·12] $\supset \vdash .$ Prop.
This proposition is used in 120·152 (Part II) and in other propositions.

12·34 $\vdash \mathcal{C}\{\hat{x}\hat{z}[.f\{\hat{x}\} . g\{\hat{x}, \hat{z}\} .], \hat{x}\hat{z}[:f\{\hat{x}\} . g\{\hat{x}, \hat{z}\} . \supset f\{\hat{z}\} .]\}$
Constr. $\vdash$ (1) ad 12·32 $\supset$
$\vdash \mathcal{C}\{\hat{x}[.f\{\hat{x}\} . g\{\hat{x}, y\} .], \hat{x}[.\sim f\{\hat{x}\} \vee . \sim f\{\hat{x}\} \vee \sim g\{\hat{x}, y\} :]\}$
[12·2401] $\supset$
$\vdash \mathcal{C}\{\hat{x}[.f\{\hat{x}\} . g\{\hat{x}, y\} .], \hat{x}[:\sim f\{\hat{x}\} \vee \sim g\{\hat{x}, y\} . \vee \sim f\{\hat{x}\}.]\}$
[12·3] $\supset$
$\vdash \mathcal{C}\{\hat{x}[.f\{\hat{x}\} . g\{\hat{x}, y\} .], \hat{x}[.\sim\sim . \sim f\{\hat{x}\} \vee \sim g\{\hat{x}, y\} . \vee \sim f\{\hat{x}\} .]\}$

$[12{\cdot}3.3{\cdot}01] \supset$

$\vdash \mathcal{C}\{\hat{x}\,[\,.f\{\hat{x}\}\,.g\{\hat{x},y\}\,.\,],\hat{x}\,[\,.\sim.f\{\hat{x}\}\,.g\{\hat{x},y\}\,.\vee f\{\hat{x}\}\,.\,]\}$

$[1{\cdot}01] \supset \vdash \mathcal{C}\{\hat{x}\,[\,.f\{\hat{x}\}\,.\,g\{\hat{x},y\}\,.\,],\hat{x}\,[\,:f\{\hat{x}\}\,.g\{\hat{x},y\}\,.\supset f\{\hat{x}\}\,.\,]\}$

$[0{\cdot}26] \supset$

$\vdash \mathcal{C}\{\hat{x}\,\hat{y}\,[\,.f\{\hat{x}\}\,.\hat{g}\{\hat{x},y\}\,.\,],\hat{x}\,\hat{y}\,[:f\{\hat{x}\}\,.g\{\hat{x},\hat{y}\}\,.\supset f\{\hat{x}\}\,.\,]\}$

$[12{\cdot}261] \supset \vdash \mathcal{C}\{\hat{y}\,[\,.f\{z\}\,.g\{z,\hat{y}\}\,.\,],\hat{y}\,[:f\{z\}\,.g\{z,\hat{y}\}\,.\supset f\{z\}\,.\,]\}$

$[0.41] \supset \vdash \mathcal{C}\{\hat{y}\,[\,.f\{x\}\,.g\{x,\hat{y}\}\,.\,],\hat{y}\,[:f\{x\}\,.g\{x,\hat{y}\}\,.\supset f\{x'\}\,.\,]\}$

$[0{\cdot}26] \supset$

$\vdash \mathcal{C}\{\hat{x}\,\hat{y}\,[\,.f\{\hat{x}\}\,.g\{\hat{x},\hat{y}\}\,.\,],\hat{x}\,\hat{y}\,[:f\{\hat{x}\}\,.g\{\hat{x},\hat{y}\}\,.\supset .f\{x'\}\,.\,]\}$

$[12{\cdot}261] \supset$

$\vdash \mathcal{C}\{\hat{x}\,[\,.f\{\hat{x}\}\,.g\{\hat{x},y\}\,.\,],\hat{x}\,[:f\{\hat{x}\}\,.g\{\hat{x},y'\}\,.\supset .f\{\hat{x}\}\,.\,]\}$

$[0{\cdot}26] \supset \vdash .$ Prop.

IV. Classes. Relations. Identity. Descriptions.

A. Classes. Relations.

There is no difference at all between a function with I variable and a class, or between a function with II variables and a relation [1]). The theory of classes and relations is based on the following definitions.

20·01 $\omega_{(\alpha)} \underset{df}{=} \hat{u}\,[\omega_a\{\hat{u}\}]$

21·01 $R_{(a,\,b)} \underset{df}{=} \hat{u}\,\hat{v}\,[R_{a,\,b}\{\hat{u},\hat{v}\}]$

21·011 $R_{(\alpha)} \underset{df}{=} R_{(a,\,a)}$

21·0111 $R_{(x,\,a)} \underset{df}{=} \hat{y}\,\hat{v}\,[R_{x,\,a}\{\hat{y},\hat{v}\}]$

21·012 $R_{(a,\,x)} \underset{df}{=} \hat{u}\,\hat{y}\,[R_{a,\,x}\{\hat{u},\hat{y}\}]$

The idea of a class (of a relation) can be defined as follows:

20·03 $\mathrm{Cls}_{(\alpha)} \underset{df}{=} \hat{u}[\,\mathcal{C}\{\hat{u},\alpha\}]$

21·03 $\mathrm{Rel}_{(M)} \underset{df}{=} \hat{u}[\,\mathcal{C}\{\hat{u},M\}\,]$

21·031 $\mathrm{Rel}_{R_{(a,\,x)}} \underset{df}{=} \hat{u}[\,\mathcal{C}\{\hat{u},R_{(a,\,x)}\}]$

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[1]) Cf. Zasady Czystej Teorji Typów, Przegląd filozoficzny 1922.

The calculus of classes and relations is based on the following definitions:

22·01 $. \alpha \subset \beta . = _{df} (\overline{x}) . \alpha \{\overline{x}\} \supset \beta \{\overline{x}\}.$

22·011 $. \sigma_{(\alpha)} \subset \tau_{(\alpha)} . = _{df} (\overline{u}) . \sigma_{(\alpha)} \{\overline{u}\} \supset \tau_{(\alpha)} \{\overline{u}\}.$

22·02 $. \alpha \cap \beta . = _{df} \hat{x} [. \alpha \{\hat{x}\} . \beta \{\hat{x}\} .]$

22·021 $. \sigma_{(\alpha)} \cap \tau_{(\alpha)} = _{df} \hat{u} [. \sigma_{(\alpha)} \{\hat{u}\} . \tau_{(\alpha)} \{\hat{u}\} .]$

22·03 $. \alpha \cup \beta . = _{df} \hat{x} [. \alpha \{\hat{x}\} \vee \beta \{\hat{x}\} .]$

22·031 $. \sigma_{(\alpha)} \cup \tau_{(\alpha)} . = _{df} \hat{u} [. \sigma_{(\alpha)} \{\hat{u}\} \vee \tau_{(\alpha)} \{\hat{u}\} .]$

22·04 $- \alpha = _{df} \hat{x} [\sim \alpha \{\hat{x}\}]$

22·041 $- \sigma_{(\alpha)} = _{df} \hat{u} [\sim \sigma_{(\alpha)} \{\hat{u}\}]$

22·05 $. \alpha - \beta . = _{df} . \alpha \cap - \beta$

22·051 $. \sigma_{(\alpha)} - \tau_{(\alpha)} . = _{df} . \sigma_{(\alpha)} \cap - \tau_{(\alpha)} .$

23·01 $. M \subset N . = _{df} (\overline{x}) (\overline{y}) . M \{\overline{x}, \overline{y}\} \supset N \{\overline{x}, \overline{y}\}.$

23·011 $. P_{(\alpha, \beta)} \subset Q_{(\alpha, \beta)} . = _{df} (\overline{u}) (\overline{v}) . P_{(\alpha, \beta)} \{\overline{u}, \overline{v}\} \supset Q_{(\alpha, \beta)} \{\overline{u}, \overline{v}\}.$

23·012 $. P_{(\alpha, \beta)} \subset Q_{(\alpha, \beta)} . = _{df} (\overline{y}) (\overline{v}) . P_{(\alpha, \beta)} \{\overline{y}, \overline{v}\} \supset Q_{(\alpha, \beta)} \{\overline{y}, \overline{v}\}.$

23·02 $. M \cap N . = _{df} \hat{x} \hat{y} [. M \{\hat{x}, \hat{y}\} . N \{\hat{x}, \hat{y}\} .]$

23·021 $. P_{(\alpha, \beta)} \cap Q_{(\alpha, \beta)} . = _{df} \hat{u} \hat{v} [. P_{(\alpha, \beta)} \{\hat{u}, \hat{v}\} . Q_{(\alpha, \beta)} \{\hat{u}, \hat{v}\} .]$

23·022 $. P_{(\alpha, \alpha)} \cap Q_{(\alpha, \alpha)} . = _{df} \hat{y} \hat{v} [. P_{(\alpha, \alpha)} \{\hat{y}, \hat{v}\} . Q_{(\alpha, \alpha)} \{\hat{y}, \hat{v}\} .]$

23·03 $. M \cup N . = _{df} \hat{x} \hat{y} [. M \{\hat{x}, \hat{y}\} \vee M \{\hat{x}, \hat{y}\} .]$

23·031 $. P_{(\alpha, \beta)} \cup Q_{(\alpha, \beta)} . = _{df} \hat{u} \hat{v} [. P_{(\alpha, \beta)} \{\hat{u}, \hat{v}\} \vee Q_{(\alpha, \beta)} \{\hat{u}, \hat{v}\} .]$

23·032 $. P_{(\alpha, \alpha)} \cup Q_{(\alpha, \alpha)} . = _{df} \hat{y} \hat{v} [. P_{(\alpha, \alpha)} \{\hat{y}, \hat{v}\} \vee Q_{(\alpha, \alpha)} \{\hat{y}, \hat{v}\} .]$

23·04 $- M = _{df} \hat{x} \hat{y} [\sim M \{\hat{x}, \hat{y}\}]$

23·041 $- P_{(\alpha, \beta)} = _{df} \hat{u} \hat{v} [\sim P_{(\alpha, \beta)} \{\hat{u}, \hat{v}\}]$

23·042 $- P_{(\alpha, \alpha)} = _{df} \hat{y} \hat{v} [\sim P_{(\alpha, \alpha)} \{\hat{y}, \hat{v}\}]$

23·05 $. M - N . = . M \cap - N.$

23·051 $. P_{(a,\,b)} - Q_{(a,\,b)} \underset{df}{=} . P_{(a,\,b)} \cap - Q_{(a,\,b)}$.

23·052 $. P_{(x,\,a)} - Q_{(x,\,a)} \underset{df}{=} . P_{(x,\,a)} \cap - Q_{(x,\,a)}$

Remark that Whitehead and Russell use for relations instead of $\subset, \cap, \cup, -$ the symbols: $\dot{\subset}, \dot{\cap}, \dot{\cup}, \dot{-}$; nevertheless, as these symbols have no meaning in isolation, we having only such expressions as $\alpha \subset \beta$, or $M \subset N$, etc. the use of different symbols in both cases is, as a matter of fact, superfluous. —

B. Identity.

The definition of the identity of two classes (relations) is as follows:

13·01 $. \alpha = \beta . \underset{df}{=} (\overline{x}) . \alpha \{\overline{x}\} \equiv \beta \{\overline{x}\}$.

13·011 $. \sigma_{(a)} = \tau_{(a)} . \underset{df}{=} (\overline{u}) . \sigma_{(a)} \{\overline{u}\} \equiv \tau_{(a)} \{\overline{u}\}$.

13·012 $. M = N . \underset{df}{=} (\overline{x}) (\overline{y}) . M \{\overline{x}, \overline{y}\} \equiv N \{\overline{x}, \overline{y}\}$.

13·013 $. P_{(a\ b)} = Q_{(a.\ b)} . \underset{df}{=} (\overline{u}) (\overline{v} . P_{(a,\ b)} \{\overline{u}, \overline{v}\} \equiv Q_{(a,\ b)} \{\overline{u}, \overline{v}\}$.

13·014 $. P_{(x,\ a)} = Q_{(x,\ a)} . \underset{df}{=} (\overline{y}) (\overline{v}) . P_{(x,\ a)} \{\overline{y}, \overline{v}\} \equiv Q_{(x.\ a)} \{\overline{y}, \overline{v}\}$.

13·02 $. \alpha \neq \beta . \underset{df}{=} \sim . \alpha = \beta$. etc.

13·022 $. M \neq N . \underset{df}{=} \sim . M = N$. etc.

We see that identity of classes (and of relations) is essentially different from the Leibnizian identity used by Whitehead and Russell. With our definitions we have no such proposition as $: \alpha = \beta . \supset . f \{\alpha\} \equiv f \{\beta\}$: but, as a matter of fact, this proposition is completely useless, as we need only another and less general proposition, to be given below. I begin with the following abbreviation:

13·015 $. u \underset{a,\,b}{=} v . \underset{df}{=} .: u = v : \mathcal{C}\{u, a\} : \mathcal{C}\{v, b\}$.

13·016 $. u \underset{\sigma(a),\ \tau(a)}{=} v . \underset{df}{=} .: u = v : \mathcal{C}\{u, \sigma_{(a)}\} : \mathcal{C}\{v, \tau_{(a)}\}$.

etc.

Now, *I* pass to the definition of an extensional function:

13·04 extens $[\varkappa_{(a)}] \underset{df}{=} \hat{f} [(\overline{u}, \overline{v}) : \overline{u} \underset{a,\,a}{=} \overline{v} . \supset : \hat{f}_a \{\overline{u}\} \equiv \hat{f}_a \{\overline{v}\} : \mathcal{C}\{\hat{f}, \varkappa_{(a)}\} :]$

13·041 extens $[R_{(a,\ b)}] \underset{df}{=} \hat{f} [(\overline{u}, \overline{v}, \overline{w}, \overline{t}) :. \overline{u} \underset{a,\,a}{=} \overline{v} .: \overline{w} \underset{b,\,b}{=} \overline{t} :$

$\supset : \hat{f}_{a,\ b} \{\overline{u}, \overline{w}\} \equiv \hat{f}_{a.\ b} \{\overline{v}, \overline{t}\} : \mathcal{C}\{\hat{f}, R_{(a,\ b)}\} :]$

13·042 $\text{extens}\,[R_{(\varkappa(a),b)}] =_{df} \hat{f}\,[(\overline{u,v,w,t}) :. \overline{u}_{\varkappa(a),\varkappa(a)} = \overline{v} .\, \overline{w}_{b,b} = \overline{t} :$

$\supset : \hat{f}_{\varkappa(a),b}\,\{\overline{u,w}\} \equiv \hat{f}_{\varkappa(a),b}\,\{\overline{v,t}\} : \mathcal{C}\{\hat{f}, R_{(\varkappa(a),b)}\} :]$

13·043 $\text{extens}\,[R_{(P(a,c),b)}] =_{df} \hat{f}\,[\overline{u,v,w,t}) : \overline{u}_{P(a,c),P(a,c)} = \overline{v} :. \overline{w}_{b,b} = \overline{t} :$

$\supset : \hat{f}_{P(a,c),b}\,\{\overline{u,w}\} \equiv \hat{f}_{P(a,c),b}\,\{\overline{v,t}\} : \mathcal{C}\{\hat{f}, R_{(P(a,c),b)}\} :]$

Now we can prove the following proposition:

13·12 $\vdash : u = v . \supset : f_a\{u\} . \text{extens}\,[\varkappa_{(a)}]\{f\} . \equiv . f_a\{v\} . \text{extens}\,[\varkappa_{(a)}]\{f\} :$

Dem $\vdash . 5\cdot34 \supset$

$\vdash : \sim \text{extens}\,[\varkappa_{(a)}]\{f\} \supset$

$. : f_a\{u\} . \text{extens}\,[\varkappa_{(a)}]\{f\} . \equiv . f_a\{v\} . \text{extens}\,[\varkappa_{(a)}]\{f\} : \supset$

$: u = v . \supset : f_a\{u\} . \text{extens}\,[\varkappa_{(a)}]\{f\} . \equiv . f_a\{v\} . \text{extens}\,[\varkappa_{(a)}]\{f\} :: \quad (1)$

$\vdash . 4\cdot2 \supset$

$\vdash . \text{extens}\,[\varkappa_{(a)}]\{f\} \supset (\overline{u,v}) : \overline{u}_{a,a} = \overline{v} . \supset : f_a\{\overline{u}\} \equiv f_a\{\overline{v}\} : \mathcal{C}\{f, \varkappa_{(a)}\} :.$

$[10\cdot1] \supset : u =_{a,a} v \supset : f_a\{\overline{u}\} \equiv f_a\{\overline{v}\} : \mathcal{C}\{f, \varkappa_a\} :.$

$[5\cdot32.12\cdot1] \supset$

$: u = v . \supset : f_a\{\overline{u}\} . \text{extens}\,[\varkappa_{(a)}]\{f\} . \equiv . f_a\{v\} . \text{extens}\,[\varkappa_a]\{f\} :.$

$[(1) . 4\cdot83] \supset \vdash . \text{Prop.}$

I add the following important propositions:

13·16 $\vdash : u = v . \equiv . v = u :$

13·17 $\vdash :. u = v :. v = w : \supset . u = w :$

As we shall have to do only with extensional functions, the following definitions are very important:

20·02 $u \varepsilon \varkappa_{(a)} =_{df} . \varkappa_{(a)}\{u\} . \text{extens}\,[\varkappa_{(a)}]\{\varkappa_{(a)}\} .$

21·02 $u\, R_{(a,b)}\, v =_{df} . R_{(a,b)}\{u,v\} . \text{extens}\,[R_{(a,b)}]\{R_{(a,b)}\} .$

21·021 $u\, R_{(\varkappa(a),b)}\, v =_{df} . R_{(\varkappa(a),b)}\{u,v\} . \text{extens}\,[R_{(\varkappa(a),b)}]\{R_{(\varkappa(a),b)}\} .$

21·022 $u\, R_{(P(a,b),c)}\, v =_{df} . R_{(P(a,b),c)}\{u,v\} . \text{extens}\,[R_{(P(a,b),c)}]\{R_{(P(a,b),c)}\} .$

Note that with our definition of $R_{(a,b)}$ no conventions based on the alphabetical order of letters, or on the order of letters occurring in a given expression are needed. [1])

The definition 20·02 (21·02) enable us to prove the following propositions by a simple substitution in 13·12 of the symbol $u \varepsilon \varkappa_{(a)}$, $(u\, R_{(a,b)}\, w)$.

[1]) Cf. Principia I. p. 211.

13·121 $\vdash : u = v . \supset . u \in \kappa_{(a)} \equiv v \in \kappa_{(a)} :$

13·122 $\vdash : u = v . \supset . u R_{(a, b)} w \equiv v R_{(a, b)} w :$

With these propositions we can establish the whole theory of identity and the Calculus of Classes and Relations as contained in numbers * 13, * 22, * 23 of Principia. Numbers * 20 and * 21 of Principia are to be omitted.

C. Huntington's Postulates.

Now, let us assume the following definitions:

24·01 $\mathrm{V} = \underset{df}{\hat{x}} [\mathcal{C}\{\hat{x}, x'\}]$

24·011 $\mathrm{V}_{(a)} = \underset{df}{\hat{u}} [\mathcal{C}\{\hat{u}, a\}]$

24·02 $\Lambda = \underset{df}{.} \alpha - \mathrm{V} .$

24·021 $\Lambda_{(a)} = \underset{df}{.} \kappa_{(a)} - \mathrm{V}_{(a)} .$

25·01 $\dot{\mathrm{V}} = \underset{df}{\hat{x}\, \hat{y}} [\mathcal{C}\{\hat{x}, \hat{y}\}]$

25·011 $\dot{\mathrm{V}}_{(a, b)} = \underset{df}{\hat{u}\, \hat{v}} [. \mathcal{C}\{\hat{u}, a\} . \mathcal{C}\{v, b\} .]$

25·0111 $\dot{\mathrm{V}}_{(x, a)} = \hat{x}\, \hat{u} [. \mathcal{C}\{\hat{x}, x'\} . \mathcal{C}\{\hat{u}, a\} .]$

25·02 $\dot{\Lambda} = \underset{df}{.} M - \dot{\mathrm{V}} .$

25·021 $\dot{\Lambda}_{(a, b)} = \underset{df}{.} R_{(a, b)} - \dot{\mathrm{V}}_{(a, b)} .$

25·022 $\dot{\Lambda}_{(x, a)} = \underset{df}{.} R_{(x, a)} - \dot{\mathrm{V}}_{(x, a)} .$

24·03 $\exists ! \alpha = \underset{df}{(\exists \overline{x})} \alpha \{\overline{x}\}$

24·031 $\exists ! \kappa_{(a)} = \underset{df}{(\exists \overline{u})} \overline{u} \in \kappa_{(a)} .$

25·03 $\exists ! M = \underset{df}{(\exists \overline{x, y})} M \{\overline{x, y}\}$

25·031 $\exists ! R_{(a, b)} = \underset{df}{(\exists \overline{u, v})} \overline{u}\, R_{(a, b)}\, \overline{v}$

15·032 $\exists ! R_{(\kappa_{(a)}, b)} = \underset{df}{(\exists \overline{u, v})} \overline{u}\, R_{(\kappa_{(a)}, b)}\, \overline{v}$

With these definitions we can construct a complete proof of Huntington's postulates by simple application of our definitions and of the Logical Calculus.

We have:

22·37 $\vdash . \alpha \cup \beta . \in \mathrm{Cls}_{(\alpha \cup \beta)}$

22·36 $\vdash . \alpha \cap \beta . \in \mathrm{Cls}_{(\alpha \cup \beta)}$

24·24 $\vdash : \alpha \cup \Lambda . = \alpha .$

24·26 $\vdash : \alpha \cap \Lambda . = \alpha .$

22·57 $\vdash : \alpha \cup \beta . = \beta \cup \alpha :$

22·51 $\vdash : \alpha \cap \beta . = . \beta \cap \alpha :$

22·69 $\vdash :. \alpha \cup \beta . \cap . \alpha \cup \gamma : = . \alpha \cup . \beta \cap \gamma :.$

22·68 $\vdash :. \alpha \cap \beta . \cap . \alpha \cap \gamma : = . \alpha \cap . \beta \cup \gamma :.$

24·21 $\vdash : \alpha \cap - \alpha . = \Lambda .$

24·22 $\vdash : \alpha \cup - \alpha . = V .$

24·1 $\vdash . \Lambda \neq V .$

Analogous propositions are to be asserted for $\sigma_{(a)}$, $\tau_{(a)}$; an analogous set of propositions is to be stated for relations: M, N, or $R_{(a,b)}$, $P_{(a,b)}$ etc.

D. Descriptions.

I cannot agree with the theory of descriptions of Principia this theory being based on modifications of defined symbols, which are not simple substitutions. Now, a general theory of descriptions seems to be superfluous, as we need only the following abbreviations:

14·01 $. u = R^{\epsilon}_{(a,b)} v . =_{df} . u R_{(a,b)} v . (\overline{w}) . \overline{w} R_{(a,b)} v \supset . \overline{w} = u :$

14·011 $. R^{\epsilon}_{(a,b)} v = u . =_{df} . u R_{(a,b)} v . (\overline{w}) . \overline{w} R_{(a,b)} v \supset . u = \overline{w} :$

14·012 $. R^{\epsilon}_{(a,b)} v = u . =_{df} (\exists \overline{w}) : R^{\epsilon}_{(a,b)} v = \overline{w} :. \overline{w} = R^{\epsilon}_{(a,b)} u :$

14·02 $R^{\epsilon}_{(a,b)} v \varepsilon \varkappa_{(a)} =_{df} (\exists \overline{w}) : R^{\epsilon}_{(a,b)} v = \overline{w} : \overline{w} \varepsilon \varkappa_{(a)} .$

14·021 $R^{\epsilon}_{(a,b)} v \; P_{(a,c)} u =_{df} (\exists \overline{w}) : R^{\epsilon}_{(a,b)} v = \overline{w} : \overline{w} P_{(a,c)} u .$

14·022 $u P_{(c,a)} R^{\epsilon}_{(a,b)} v =_{df} (\exists \overline{w}) : R^{\epsilon}_{(a,b)} v = \overline{w} : u P_{(c,a)} \overline{w} .$

With these definitions we can prove the following propositions:

14·1 $\vdash : u = R^{\epsilon}_{(a,b)} v . \equiv . R^{\epsilon}_{(a,b)} v = u :$

Dem. $\vdash$ 13·17 $\supset \vdash : u = w . \equiv . w = u :$

[4·85] $\supset \vdash : w R_{(a,b)} v \supset . u = w : \equiv . w R_{(a,b)} v \supset . w = u : .$

[10·271] $\supset \vdash . (\overline{w}) . \overline{w} R_{(a,b)} v \supset . u = \overline{w} : \equiv (\overline{w}) . \overline{w} R_{(a,b)} v \supset . \overline{w} = u : .$

[4·36] $\supset \vdash .$ Prop.

14·11 $\vdash : R^{\epsilon}_{(a,b)} v = R^{\epsilon}_{(a,b)} u . \equiv . R^{\epsilon}_{(a,b)} u = R^{\epsilon}_{(a,d)} v :$

Dem $\vdash$ 14·1. 4·3 $\supset$

$\vdash :. R^{\epsilon}_{(a,b)} v = w : w = R^{\epsilon}_{(a,b)} u : \equiv : R^{\epsilon}_{(a,b)} u = w :. w = R^{\epsilon}_{(a,b)} v :.$

[10·281] $\vdash (\exists \overline{w}) : R'_{(a,b)} v = \overline{w} : \overline{w} = R_{(a,b)} u : \equiv : (\exists \overline{w}) : R_{(a,b)} u = \overline{w} : \overline{w} = R'_{(a,b)} v :$
[14·012] $\supset \vdash$. Prop.

14·12 $\vdash : . R'_{(a,b)} v = w : . R'_{(a,b)} u = w : \supset . R'_{(a,b)} v = R'_{(a,b)} u :$
Dem $\vdash$.14·1.10·24 $\supset \vdash$ Hp $\supset (\exists \overline{t}) : R'_{(a,b)} v = \overline{t} : . \overline{t} = R'_{(a,b)} u :$
[14·012] $\supset \vdash$. Prop.

14·13 $\vdash : . R'_{(a,b)} v = t : . R_{(a,b)} v = u : \supset . u = t :$
$\underset{a,a}{}$
Dem. $\vdash$ 14·011.3·27.10·1 $\supset$
$\vdash : R'_{(a,b)} v = u . \supset : u \; R_{(a,b)} v . t \; R_{(a,b)} v . \supset . u = t :$ (1)
$\underset{a,a}{}$
$\vdash$ 3·27 $\supset \vdash : R'_{(a,b)} v = t . \supset t \; R_{(a,b)} v .$ (2)
$\vdash$. (1). (2). 4·32. $\supset \vdash$. Hp $\supset$.
$: u \; R_{(a,b)} v . t \; R_{(a,b)} v . \supset . u = t : t \; R v :$
[3·35·27] $\supset \vdash$. Prop.

14·2 $\vdash : . t = u : R'_{(a,b)} v = u : \supset . R'_{(a,b)} v = t :$
$\underset{a,a}{}$
Dem $\vdash$ 13·122 $\supset \vdash . t = u . \supset . u \; R_{(a,b)} v \equiv t \; R_{(a,b)} v :$ (1)
$\underset{a,a}{}$
$\supset \vdash : t = u . \supset : w \; R_{(a,b)} v \supset . w = u : \equiv . w \; R_{(a,b)} v \supset . w = t : :$
$\vdash$. 4·22. $\supset$. $(\overline{w})$. $\overline{w} \; R_{(a,b)} v \supset . \overline{w} = u : \equiv (\overline{w}) \; \overline{w} \; R_{(a,b)} v \supset . \overline{w} = t : :$
[(1)] $\supset \vdash$. Prop.

14·21 $\vdash : . R'_{(a,b)} u = R'_{(a,b)} v : . R'_{(a,b)} v = R'_{(a,b)} t : \supset . R'_{(a,b)} u = R'_{(a,b)} t :$
Dem $\vdash$ 4·32 $\supset \vdash : : R'_{(a,b)} u = w : . w = R'_{(a,b)} v : : . R'_{(a,b)} v =$
$= w' : . w' = R'_{(a,b)} t : . \supset : . R'_{(a,b)} u = w : . w =$
$= R'_{(a,b)} v : : . R'_{(a,b)} v = w' : . w' = R'_{(a,b)} t : :$
[4·32] $\supset : R'_{(a,b)} u = w : . w = R'_{(a,b)} v : : . R'_{(a,b)} v = w' : . w' = R'_{(a,b)} t :$
[14·13] $\supset : R'_{(a,b)} u = w : . w' = w : : w' = R'_{(a,b)} t :$
$\underset{a,a}{}$
[14·2] $\supset : R'_{(a,b)} u = w : . w = R'_{(a,b)} t :$
[14·13] $\supset . R'_{(a,b)} u = R'_{(a,b)} t .$
[11·35] $\supset . \vdash$. Prop.

Our method of dealing with descriptions implies the following modification of definitions of the theory of relations. Take e. g. the definition of the Converses of relations. In Principia we have first the definition: $\mathrm{Cnv} = \hat{Q} \, \hat{P} [. x \, Q \, y \equiv_{x,y} y \, P \, x .]$ Df., and by this definition, the converse P is $\mathrm{Cnv}' P$. — Nevertheless it is to be remarbed that the relation Cnv is never used unless in $\mathrm{Cnv}' P$. Therefore it seems better to have no such relation as Cnv, but simply the relation $\mathrm{Cnv}' P$. — This method will be applied to all analogous problems. Then we have:

31·01 $\mathrm{Cnv}^{\iota} M = \underset{df}{\hat{y}\,\hat{x}\,[M\{\hat{x}, \hat{y}\}]}$

31·011 $\mathrm{Cnv}^{\iota} P_{(a,b)} = \underset{df}{\hat{v}\,\hat{u}\,[P_{(a,b)}\{\hat{u}, \hat{v}\}]}$

31·012 $\mathrm{Cnv}^{\iota} P_{(x,a)} = \underset{df}{\hat{u}\,\hat{y}\,[P_{(x,a)}\{\hat{y}, \hat{u}\}]}$

31·02 $\overline{M} = \underset{df}{\mathrm{Cnv}^{\iota} M}$ et *c.*

32·01 $\vec{M}^{\iota} y = \underset{df}{\hat{x}\,[M\{\hat{x}, y\}]}$

32·011 $\vec{P}_{(a,b)}{}^{\iota} v = \underset{df}{\hat{u}\,[P_{(a,b)}\{\hat{u}, v\}]}$

32·012 $\vec{P}_{(x,a)}{}^{\iota} v = \underset{df}{\hat{y}\,[P_{(x,a)}\{\hat{y}, v\}]}$

32·02 $\overline{M}^{\iota} y = \underset{df}{\hat{x}\,[M\{y, \hat{x}\}]}$ etc.

33·01 $D^{\iota} M = \underset{df}{\hat{x}\,[(\exists\, \overline{y})\, M\{\hat{x}, \overline{y}\}]}$

33·011 $D^{\iota} P_{(a,b)} = \underset{df}{\hat{u}\,[(\exists\, \overline{v})\, P_{(a,b)}\{\hat{u}, \overline{v}\}]}$

33·02 $D^{\iota} M = \underset{df}{\hat{x}\,[(\exists\, \overline{y})\, M\{\overline{y}, \hat{x}\}]}$

33·021 $D^{\iota} P_{(a,b)} = \underset{df}{\hat{u}\,[(\exists\, \overline{v})\, P_{(a,b)}\{\overline{v}, \hat{u}\}]}$

33·03 $C^{\iota} M = \underset{df}{\hat{x}\,[(\exists\, \overline{y}).\, M\{\hat{x}, \overline{y}\} \vee M\{\overline{y}, \hat{x}\}.]}$

33·031 $C^{\iota} P_{(a,b)} = \underset{df}{\hat{u}\,[(\exists\, \overline{v}).\, P_{(a,b)}\{u, \overline{v}\} \vee P_{(a,b)}\{\overline{v}, \hat{u}\}.]}$

This method enable us to get the Logic of Relations without any difficulty. —

Errata.

p. 18. l. 15., read „necessity“ instead of „neccessity“.
p. 20. l. 27., read „hardly“ instead of „hardlu“.
p. 22. l. 26., read „If“ instead of „Il“.
p. 23. l. 16., read „functional“ instead of „functionel“.
p. 23. footnote l. 5 read „Transactions of the American Mathematical Society, 1913., instead of „Trausactions of American Mathematical, society 1913“.
p. 24. l. 29. read „then the expressions“ instead of thent hee expressions“.

p. 26. l. 22., read „the“ instead of „th“.

p. 28. l. 4., read „advantage“ instead of „adventage“.

p. 29. l. 9., read „or primitive (3)) real variable, or any fundamental real variable, which never occurs as argument or constituent of an argument of a functional sign“, instead of „, or primitive (3), or undetermined fundamental (4)) real variable“.