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SURFACE, CONGRUENCE, COMPLEX.

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In the article Curve [785], the subject was treated from an historical point of view for the purpose of showing how the leading ideas of the theory were successively arrived at. These leading ideas apply to surfaces, but the ideas peculiar to surfaces are scarcely of the like fundamental nature, being rather developments of the former set in their application to a more advanced portion of geometry; there is consequently less occasion for the historical mode of treatment. Curves in space were briefly considered in the same article, and they will not be discussed here; but it is proper to refer to them in connexion with the other notions of solid geometry. In plane geometry the elementary figures are the point and the line; and we then have the curve, which may be regarded as a singly infinite system of points, and also as a singly infinite system of lines. In solid geometry the elementary figures are the point, the line, and the plane; we have, moreover, first, that which under one aspect is the curve and under another aspect the developable (or torse), and which may be regarded as a singly infinite system of points, of lines, or of planes; and secondly, the surface, which may be regarded as a doubly infinite system of points or of planes, and also as a special triply infinite system of lines. (The tangent lines of a surface are a special complex.) As distinct particular cases of the first figure, we have the plane curve and the cone: and as a particular case of the second figure, the ruled surface, regulus, or singly infinite system of lines; we have, besides, the congruence or doubly infinite system of lines, and the complex or triply infinite system of lines. And thus crowds of theories arise which have hardly any analogues in plane geometry; the relation of a curve to the various surfaces which can be drawn through it, and that of a surface to the various curves which can be drawn upon it, are different in kind from those which in plane geometry most nearly correspond to them,—the relation of a system of points to the different curves through them and that of a curve to the systems of points upon it. In particular, there is nothing in plane geometry to correspond to the theory of the curves of curvature of a surface. Again, to the single theorem of plane geometry, that a line is the shortest distance between two points, there correspond in solid geometry two extensive and difficult theories,—that of the geodesic lines on a surface and that of the minimal surface, or surface of minimum area, for a given boundary. And it would be easy to say more in illustration of the great extent and complexity of the subject.

Surfaces in General; Torses, &c.

1. A surface may be regarded as the locus of a doubly infinite system of points,—that is, the locus of the system of points determined by a single equation $U=(*\between x, y, z, 1)^n, =0$, between the Cartesian coordinates (to fix the ideas, say rectangular coordinates) x, y, z; or, if we please, by a single homogeneous relation $U=(*\between x, y, z, w)^n, =0$, between the quadriplanar coordinates x, y, z, w. The degree n of the equation is the order of the surface; and this definition of the order agrees with the geometrical one, that the order of the surface is equal to the number of the intersections of the surface by an arbitrary line. Starting from the foregoing point definition of the surface, we might develop the notions of the tangent line and the tangent plane; but it will be more convenient to consider the surface *ab initio* from the more general point of view in its relation to the point, the line, and the plane.

2. Mention has been made of the plane curve and the cone; it is proper to recall that the *order* of a plane curve is equal to the number of its intersections by an arbitrary line (in the plane of the curve), and that its *class* is equal to the number of tangents to the curve which pass through an arbitrary point (in the plane of the curve). The cone is a figure correlative to the plane curve: corresponding to the plane of the curve we have the vertex of the cone, to its tangents the generating lines of the cone, and to its points the tangent planes of the cone. But from a different point of view, we may consider the generating lines of the cone as corresponding to the points of the curve and its tangent planes as corresponding to the tangents of the curve. From this point of view, we define the order of the cone as equal to the number of its intersections (generating lines) by an arbitrary plane through the vertex, and its class as equal to the number of the tangent planes which pass through an arbitrary line through the vertex. And in the same way that a plane curve has singularities (singular points and singular tangents), so a cone has singularities (singular generating lines and singular tangent planes).

3. Consider now a surface in connexion with an arbitrary line. The line meets the surface in a certain number of points, and, as already mentioned, the *order* of the surface is equal to the number of these intersections. We have through the line a certain number of tangent planes of the surface, and the *class* of the surface is equal to the number of these tangent planes.

But, further, through the line imagine a plane; this meets the surface in a curve the order of which is equal (as is at once seen) to the order of the surface. Again, on the line imagine a point; this is the vertex of a cone circumscribing the surface, and the class of this cone is equal (as is at once seen) to the class of the surface.

The tangent lines of the surface, which lie in the plane, are nothing else than the tangents of the plane section, and thus form a singly infinite series of lines; similarly, the tangent lines of the surface, which pass through the point, are nothing else than the generating lines of the circumscribed cone, and thus form a singly infinite series of lines. But, if we consider those tangent lines of the surface which are at once in the plane and through the point, we see that they are finite in number; and we define the *rank* of a surface as equal to the number of tangent lines which lie in a given plane and pass through a given point in that plane. It at once follows that the class of the plane section and the order of the circumscribed cone are each equal to the rank of the surface, and are thus equal to each other. It may be noticed that for a general surface $(* \between x, y, z, w)^n, = 0$, of order n without point singularities the rank is $a, = n(n-1)$, and the class is $n', = n(n-1)^2$; this implies (what is, in fact, the case) that the circumscribed cone has line singularities, for otherwise its class, that is, the class of the surface, would be $a(a-1)$, which is not $= n(n-1)^2$.

4. In the last preceding number, the notions of the tangent line and the tangent plane have been assumed as known, but they require to be further explained in reference to the original point definition of the surface. Speaking generally, we may say that the points of the surface consecutive to a given point on it lie in a plane which is the tangent plane at the given point, and conversely the given point is the point of contact of this tangent plane, and that any line through the point of contact and in the tangent plane is a tangent line touching the surface at the point of contact. Hence we see at once that the tangent line is any line meeting the surface in two consecutive points, or—what is the same thing—a line meeting the surface in the point of contact, counting as two intersections, and in $\overline{n-2}$ other points. But, from the foregoing notion of the tangent plane as a plane containing the point of contact and the consecutive points of the surface, the passage to the true definition of the tangent plane is not equally obvious. A plane in general meets the surface of the order n in a curve of that order without double points; but the plane may be such that the curve has a double point, and when this is so the plane is a tangent plane having the double point for its point of contact. The double point is either an acnode (isolated point), then the surface at the point in question is convex towards (that is, concave away from) the tangent plane; or else it is a crunode, and the surface at the point in question is then concavo-convex, that is, it has its two curvatures in opposite senses (see *infra*, No. 16). Observe that, in either case, any line whatever in the plane and through the point meets the surface in the points in which it meets the plane curve, namely, in the point of contact, which *qua* double point counts as two intersections, and in $\overline{n-2}$ other points; that is, we have the preceding definition of the tangent line.

5. The complete enumeration and discussion of the singularities of a surface is a question of extreme difficulty which has not yet been solved*. A plane curve has

* In a plane curve, the only singularities which need to be considered are those that present themselves in Plücker's equations: for every higher singularity whatever is equivalent to a certain number of nodes, cusps, inflexions, and double tangents. As regards a surface, no such reduction of the higher singularities has as yet been made.

point singularities and line singularities; corresponding to these, we have for the surface isolated point singularities and isolated plane singularities, but there are besides continuous singularities applying to curves on or torses circumscribed to the surface, and it is among these that we have the non-special singularities which play the most important part in the theory. Thus the plane curve represented by the general equation $(*\between x, y, z)^n = 0$, of any given order n, has the non-special line singularities of inflexions and double tangents; corresponding to this, the surface represented by the general equation $(*\between x, y, z, w)^n = 0$, of any given order n, has, not the isolated plane singularities, but the continuous singularities of the spinode curve or torse and the node-couple curve or torse. A plane may meet the surface in a curve having (1) a cusp (spinode) or (2) a pair of double points; in each case, there is a singly infinite system of such singular tangent planes, and the locus of the points of contact is the curve, the envelope of the tangent planes the torse. The reciprocal singularities to these are the nodal curve and the cuspidal curve: the surface may intersect or touch itself along a curve in such wise that, cutting the surface by an arbitrary plane, the curve of intersection has, at each intersection of the plane with the curve on the surface, (1) a double point (node) or (2) a cusp. Observe that these are singularities not occurring in the surface represented by the general equation $(*\between x, y, z, w)^n = 0$ of any order; observe further that, in the case of both or either of these singularities, the definition of the tangent plane must be modified. A tangent plane is a plane such that there is in the plane section a double point in addition to the nodes or cusps at the intersections with the singular lines on the surface.

6. As regards isolated singularities, it will be sufficient to mention the point singularity of the conical point (or cnicnode) and the corresponding plane singularity of the conic of contact (or cnictrope). In the former case, we have a point such that the consecutive points, instead of lying in a tangent plane, lie on a quadric cone, having the point for its vertex; in the latter case, we have a plane touching the surface along a conic, that is, the complete intersection of the surface by the plane is made up of the conic taken twice and of a residual curve of the order $n-4$.

7. We may, in the general theory of surfaces, consider either a surface and its reciprocal surface, the reciprocal surface being taken to be the surface enveloped by the polar planes (in regard to a given quadric surface) of the points of the original surface; or—what is better—we may consider a given surface in reference to the reciprocal relations of its order, rank, class, and singularities. In either case, we have a series of unaccented letters and a corresponding series of accented letters, and the relations between them are such that we may in any equation interchange the accented and the unaccented letters; in some cases, an unaccented letter may be equal to the corresponding accented letter. Thus, let n, n' be as before the order and the class of the surface, but, instead of immediately defining the rank, let a be used to denote the class of the plane section and a' the order of the circumscribed cone; also let S, S' be numbers referring to the singularities. The form of the relations is $a = a'$ (= rank of surface); $a' = n(n-1) - S$; $n' = n(n-1)^2 - S$; $a = n'(n'-1) - S'$; $n = n'(n'-1)^2 - S'$. In these last equations S, S' are merely written down to denote proper corresponding combinations of the several numbers referring to the singularities collectively denoted

by S, S' respectively. The theory, as already mentioned, is a complex and difficult one, and it is not the intention to further develop it here.

8. A developable or torse corresponds to a curve in space, in the same manner as a cone corresponds to a plane curve: although capable of representation by an equation $U = (*\between x, y, z, w)^n, = 0$, and so of coming under the foregoing point definition of a surface, it is an entirely distinct geometrical conception. We may indeed, *qua* surface, regard it as a surface characterized by the property that each of its tangent planes touches it, not at a single point, but along a line; this is equivalent to saying that it is the envelope, not of a doubly infinite series of planes, as is a proper surface, but of a singly infinite system of planes. But it is perhaps easier to regard it as the locus of a singly infinite system of lines, each line meeting the consecutive line, or, what is the same thing, the lines being tangent lines of a curve in space. The tangent plane is then the plane through two consecutive lines, or, what is the same thing, an osculating plane of the curve, whence also the tangent plane intersects the surface in the generating line counting twice, and in a residual curve of the order $n-2$. The curve is said to be the edge of regression of the developable, and it is a cuspidal curve thereof; that is to say, any plane section of the developable has at each point of intersection with the edge of regression a cusp. A sheet of paper bent in any manner without crumpling gives a developable; but we cannot with a single sheet of paper properly exhibit the form in the neighbourhood of the edge of regression: we need two sheets connected along a plane curve, which, when the paper is bent, becomes the edge of regression and appears as a cuspidal curve on the surface.

It may be mentioned that the condition which must be satisfied in order that the equation $U = 0$ shall represent a developable is $H(U) = 0$; that is, the Hessian or functional determinant formed with the second differential coefficients of U must vanish in virtue of the equation $U = 0$, or—what is the same thing—$H(U)$ must contain U as a factor. If in Cartesian coordinates the equation is taken in the form $z - f(x, y) = 0$, then the condition is $rt - s^2 = 0$ identically, where r, s, t denote as usual the second differential coefficients of z in regard to x, y respectively.

9. A ruled surface or regulus is the locus of a singly infinite system of lines, where the consecutive lines do not intersect; this is a true surface, for there is a doubly infinite series of tangent planes,—in fact, any plane through any one of the lines is a tangent plane of the surface, touching it at a point on the line, and in such wise that, as the tangent plane turns about the line, the point of contact moves along the line. The complete intersection of the surface by the tangent plane is made up of the line counting once and of a residual curve of the order $n-1$. A quadric surface is a regulus in a twofold manner, for there are on the surface two systems of lines each of which is a regulus. A cubic surface may be a regulus (see No. 11 *infra*).

Surfaces of the Orders 2, 3, *and* 4.

10. A surface of the second order or a quadric surface is a surface such that every line meets it in two points, or—what comes to the same thing—such that every plane section thereof is a conic or quadric curve. Such surfaces have been studied

from every point of view. The only singular forms are when there is (i) a conical point (cnicnode), when the surface is a cone of the second order or quadricone; (ii) a conic of contact (cnictrope), when the surface is this conic; from a different point of view it is a *surface aplatie* or flattened surface. Excluding these degenerate forms, the surface is of the order, rank, and class each $=2$, and it has no singularities. Distinguishing the forms according to *reality*, we have the ellipsoid, the hyperboloid of two sheets, the hyperboloid of one sheet, the elliptic paraboloid, and the hyperbolic paraboloid (see Geometry, Analytical, [790]). A particular case of the ellipsoid is the sphere; in abstract geometry, this is a quadric surface passing through a given quadric curve, the circle at infinity. The tangent plane of a quadric surface meets it in a quadric curve having a node, that is, in a pair of lines; hence there are on the surface two singly infinite sets of lines. Two lines of the same set do not meet, but each line of the one set meets each line of the other set; the surface is thus a regulus in a twofold manner. The lines are real for the hyperboloid of one sheet and for the hyperbolic paraboloid; for the other forms of surface they are imaginary.

11. We have next the surface of the third order or cubic surface, which has also been very completely studied. Such a surface may have isolated point singularities (cnicnodes or points of higher singularity), or it may have a nodal line; we have thus $21+2, =23$ cases. In the general case of a surface without any singularities, the order, rank, and class are $=3$, 6, 12 respectively. The surface has upon it 27 lines, lying by threes in 45 planes, which are triple tangent planes. Observe that the tangent plane is a plane meeting the surface in a curve having a node. For a surface of any given order n there will be a certain number of planes each meeting the surface in a curve with 3 nodes, that is, triple tangent planes; and, in the particular case where $n=3$, the cubic curve with 3 nodes is of course a set of 3 lines; it is found that the number of triple tangent planes is, as just mentioned, $=45$. This would give 135 lines, but through each line we have 5 such planes, and the number of lines is thus $=27$. The theory of the 27 lines is an extensive and interesting one; in particular, it may be noticed that we can, in thirty-six ways, select a system of 6×6 lines, or "double sixer," such that no 2 lines of the same set intersect each other, but that each line of the one set intersects each line of the other set.

A cubic surface having a nodal line is a ruled surface or regulus; in fact, any plane through the nodal line meets the surface in this line counting twice and in a residual line, and there is thus on the surface a singly infinite set of lines. There are two forms; but the distinction between them need not be referred to here.

12. As regards quartic surfaces, only particular forms have been much studied. A quartic surface can have at most 16 conical points (cnicnodes); an instance of such a surface is Fresnel's wave surface, which has 4 real cnicnodes in one of the principal planes, 4×2 imaginary ones in the other two principal planes, and 4 imaginary ones at infinity,—in all 16 cnicnodes; the same surface has also 4 real $+$ 12 imaginary planes each touching the surface along a circle (cnictropes),—in all 16 cnictropes. It was easy by a mere homographic transformation to pass to the more general surface called the tetrahedroid; but this was itself only a particular form of the general surface

with 16 cnicnodes and 16 cnictropes first studied by Kummer. Quartic surfaces with a smaller number of cnicnodes have also been considered.

Another very important form is the quartic surface having a nodal conic; the nodal conic may be the circle at infinity, and we have then the so-called anallagmatic surface, otherwise the cyclide (which includes the particular form called Dupin's cyclide). These correspond to the bicircular quartic curve of plane geometry. Other forms of quartic surface might be referred to.

Congruences and Complexes.

13. A congruence is a doubly infinite system of lines. A line depends on four parameters and can therefore be determined so as to satisfy four conditions; if only two conditions are imposed on the line, we have a doubly infinite system of lines or a congruence. For instance, the lines meeting each of two given lines form a congruence. It is hardly necessary to remark that, imposing on the line one more condition, we have a ruled surface or regulus; thus we can in an infinity of ways separate the congruence into a singly infinite system of reguli or of torses (see *infra*, No. 16).

Considering in connexion with the congruence two arbitrary lines, there will be in the congruence a determinate number of lines which meet each of these two lines; and the number of lines thus meeting the two lines is said to be the *order-class* of the congruence. If the two arbitrary lines are taken to intersect each other, the congruence lines which meet each of the two lines separate themselves into two sets,—those which lie in the plane of the two lines and those which pass through their intersection. There will be in the former set a determinate number of congruence lines which is the *order* of the congruence, and in the latter set a determinate number of congruence lines which is the *class* of the congruence. In other words, the order of the congruence is equal to the number of congruence lines lying in an arbitrary plane, and its class to the number of congruence lines passing through an arbitrary point.

The following systems of lines form each of them a congruence:—(A) lines meeting each of two given curves; (B) lines meeting a given curve twice; (C) lines meeting a given curve and touching a given surface; (D) lines touching each of two given surfaces; (E) lines touching a given surface twice, or, say, the bitangents of a given surface.

The last case is the most general one; and conversely, for a given congruence, there will be in general a surface having the congruence lines for bitangents. This surface is said to be the *focal surface* of the congruence; the general surface with 16 cnicnodes first presented itself in this manner as the focal surface of a congruence. But the focal surface may degenerate into the forms belonging to the other cases A, B, C, D.

14. A complex is a triply infinite system of lines,—for instance, the tangent lines of a surface. Considering an arbitrary point in connexion with the complex, the com-

plex lines which pass through the point form a cone; considering a plane in connexion with it, the complex lines which lie in the plane envelope a curve. It is easy to see that the class of the curve is equal to the order of the cone; in fact, each of these numbers is equal to the number of complex lines which lie in an arbitrary plane and pass through an arbitrary point of that plane; and we then say *order* of complex = order of curve; *rank* of complex = class of curve = order of cone; *class* of complex = class of cone. It is to be observed that, while for a congruence there is in general a surface having the congruence lines for bitangents, for a complex there is not in general any surface having the complex lines for tangents; the tangent lines of a surface are thus only a special form of complex. The theory of complexes first presented itself in the researches of Malus on systems of rays of light in connexion with double refraction.

15. The analytical theory as well of congruences as of complexes is most easily carried out by means of the six coordinates of a line; viz. there are coordinates (a, b, c, f, g, h) connected by the equation $af + bg + ch = 0$, and therefore such that the ratios $a : b : c : f : g : h$ constitute a system of four arbitrary parameters. We have thus a congruence of the order n represented by a single homogeneous equation of that order $(*\between a, b, c, f, g, h)^n = 0$ between the six coordinates; two such relations determine a congruence. But we have in regard to congruences the same difficulty as that which presents itself in regard to curves in space: it is not every congruence which can be represented completely and precisely by *two* such equations.

The linear equation $(*\between a, b, c, f, g, h) = 0$ represents a congruence of the first order or linear congruence; such congruences are interesting both in geometry and in connexion with the theory of forces acting on a rigid body.

Curves of Curvature; Asymptotic Lines.

16. The normals of a surface form a congruence. In any congruence, the lines consecutive to a given congruence line do not in general meet this line; but there is a determinate number of consecutive lines which do meet it; or, attending for the moment to only one of these, say the congruence line is met by a consecutive congruence line. In particular, each normal is met by a consecutive normal; this again is met by a consecutive normal, and so on. That is, we have a singly infinite system of normals each meeting the consecutive normal, and so forming a torse; starting from different normals successively, we obtain a singly infinite system of such torses. But each normal is in fact met by two consecutive normals, and, using in the construction first the one and then the other of these, we obtain two singly infinite systems of torses each intersecting the given surface at right angles. In other words, if in place of the normal we consider the point on the surface, we obtain on the surface two singly infinite systems of curves such that for any curve of either system the normals at consecutive points intersect each other; moreover, for each normal the torses of the two systems intersect each other at right angles; and therefore for each point of the surface the curves of the two systems intersect each other at right angles. The two systems of curves are said to be the curves of curvature of the surface.

The normal is met by the two consecutive normals in two points which are the centres of curvature for the point on the surface; these lie either on the same side of the point or on opposite sides, and the surface has at the point in question like curvatures or opposite curvatures in the two cases respectively (see *supra*, No. 4).

17. In immediate connexion with the curves of curvature, we have the so-called asymptotic curves (Haupt-tangenten-linien). The tangent plane at a point of the surface cuts the surface in a curve having at that point a node. Thus we have at the point of the surface two directions of passage to a consecutive point, or, say, two elements of arc; and, passing along one of these to the consecutive point, and thence to a consecutive point, and so on, we obtain on the surface a curve. Starting successively from different points of the surface we thus obtain a singly infinite system of curves; or, using first one and then the other of the two directions, we obtain two singly infinite systems of curves, which are the curves above referred to. The two curves at any point are equally inclined to the two curves of curvature at that point, or—what is the same thing—the supplementary angles formed by the two asymptotic lines are bisected by the two curves of curvature. In the case of a quadric surface, the asymptotic curves are the two systems of lines on the surface.

Geodesic Lines.

18. A geodesic line (or curve) is a shortest curve on a surface; more accurately, the element of arc between two consecutive points of a geodesic line is a shortest arc on the surface. We are thus led to the fundamental property that, at each point of the curve, the osculating plane of the curve passes through the normal of the surface; in other words, any two consecutive arcs PP', $P'P''$ are *in plano* with the normal at P'. Starting from a given point P on the surface, we have a singly infinite system of geodesics proceeding along the surface in the direction of the several tangent lines at the point P; and, if the direction PP' is given, the property gives a construction by successive elements of arc for the required geodesic line.

Considering the geodesic lines which proceed from a given point P of the surface, any particular geodesic line is or is not again intersected by the consecutive generating line; if it is thus intersected, the generating line is a shortest line on the surface up to, but not beyond, the point at which it is first intersected by the consecutive generating line; if it is not intersected, it continues a shortest line for the whole course.

In the analytical theory both of geodesic lines and of the curves of curvature, and in other parts of the theory of surfaces, it is very convenient to consider the rectangular coordinates x, y, z of a point of the surface as given functions of two independent parameters p, q; the form of these functions of course determines the surface, since by the elimination of p, q from the three equations we obtain the equation in the coordinates x, y, z. We have for the geodesic lines a differential equation of the second order between p and q; the general solution contains two arbitrary constants, and is thus capable of representing the geodesic line which can be drawn from a given point in a given direction on the surface. In the case of a quadric surface, the solution involves hyperelliptic integrals of the first kind, depending on the square root of a sextic function.

Curvilinear Coordinates.

19. The expressions of the coordinates x, y, z in terms of p, q may contain a parameter r, and, if this is regarded as a given constant, these expressions will as before refer to a point on a given surface. But, if p, q, r are regarded as three independent parameters, x, y, z will be the coordinates of a point in space, determined by means of the three parameters p, q, r; these parameters are said to be the curvilinear coordinates, or (in a generalized sense of the term) simply the coordinates of the point. We arrive otherwise at the notion by taking p, q, r each as a given function of x, y, z; say we have $p=f_1(x, y, z)$, $q=f_2(x, y, z)$, $r=f_3(x, y, z)$, which equations of course lead to expressions for p, q, r each as a function of x, y, z. The first equation determines a singly infinite set of surfaces: for any given value of p we have a surface; and similarly the second and third equations determine each a singly infinite set of surfaces. If, to fix the ideas, f_1, f_2, f_3 are taken to denote each a rational and integral function of x, y, z, then two surfaces of the same set will not intersect each other, and through a given point of space there will pass one surface of each set; that is, the point will be determined as a point of intersection of three surfaces belonging to the three sets respectively; moreover, the whole of space will be divided by the three sets of surfaces into a triply infinite system of elements, each of them being a parallelepiped.

Orthotomic Surfaces; Parallel Surfaces.

20. The three sets of surfaces may be such that the three surfaces through any point of space whatever intersect each other at right angles; and they are in this case said to be orthotomic. The term curvilinear coordinates was almost appropriated by Lamé, to whom this theory is chiefly due, to the case in question: assuming that the equations $p=f_1(x, y, z)$, $q=f_2(x, y, z)$, $r=f_3(x, y, z)$ refer to a system of orthotomic surfaces, we have in the restricted sense p, q, r as the curvilinear coordinates of the point.

An interesting special case is that of confocal quadric surfaces. The general equation of a surface confocal with the ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1 \text{ is } \frac{x^2}{a^2+\theta}+\frac{y^2}{b^2+\theta}+\frac{z^2}{c^2+\theta}=1;$$

and, if in this equation we consider x, y, z as given, we have for θ a cubic equation with three real roots p, q, r, and thus we have through the point three real surfaces, one an ellipsoid, one a hyperboloid of one sheet, and one a hyperboloid of two sheets.

21. The theory is connected with that of curves of curvature by Dupin's theorem. Thus in any system of orthotomic surfaces, each surface of any one of the three sets is intersected by the surfaces of the other two sets in its curves of curvature.

22. No one of the three sets of surfaces is altogether arbitrary: in the equation $p = f_1(x, y, z)$, p is not an arbitrary function of x, y, z, but it must satisfy a certain partial differential equation of the third order. Assuming that p has this value, we have $q = f_2(x, y, z)$ and $r = f_3(x, y, z)$ determinate functions of x, y, z, such that the three sets of surfaces form an orthotomic system.

23. Starting from a given surface, it has been seen (No. 16) that the normals along the curves of curvature form two systems of torses intersecting each other, and also the given surface, at right angles. But there are, intersecting the two systems of torses at right angles, not only the given surface, but a singly infinite system of surfaces. If at each point of the given surface we measure off along the normal one and the same distance at pleasure, then the locus of the points thus obtained is a surface cutting all the normals of the given surface at right angles, or, in other words, having the same normals as the given surface; and it is therefore a parallel surface to the given surface. Hence the singly infinite system of parallel surfaces and the two singly infinite systems of torses form together a set of orthotomic surfaces.

The Minimal Surface.

24. This is the surface of minimum area—more accurately, a surface such that, for any indefinitely small closed curve which can be drawn on it round any point, the area of the surface is less than it is for any other surface whatever through the closed curve. It at once follows that the surface at every point is concavo-convex; for, if at any point this was not the case, we could, by cutting the surface by a plane, describe round the point an indefinitely small closed plane curve, and the plane area within the closed curve would then be less than the area of the element of surface within the same curve. The condition leads to a partial differential equation of the second order for the determination of the minimal surface: considering z as a function of x, y, and writing as usual p, q, r, s, t for the first and second differential coefficients of z in regard to x, y respectively, the equation (as first shown by Lagrange) is $(1+q^2)r - 2pqs + (1+p^2)t = 0$, or, as this may also be written,

$$\frac{d}{dy}\frac{q}{\sqrt{1+p^2+q^2}} + \frac{d}{dx}\frac{p}{\sqrt{1+p^2+q^2}} = 0.$$

The general integral contains of course arbitrary functions, and, if we imagine these so determined that the surface may pass through a given closed curve, and if, moreover, there is but one minimal surface passing through that curve, we have the solution of the problem of finding the surface of minimum area within the same curve. The surface continued beyond the closed curve is a minimal surface, but it is not of necessity or in general a surface of minimum area for an arbitrary bounding curve not wholly included within the given closed curve. It is hardly necessary to remark that the plane is a minimal surface, and that, if the given closed curve is a plane curve, the plane is the proper solution; that is, the plane area within the given closed curve is less than the area for any other surface through the same curve. The given closed curve is not of necessity a single curve: it may be, for instance, a skew polygon of four or more sides.

The partial differential equation was dealt with in a very remarkable manner by Riemann. From the second form given above it appears that we have $\frac{q\,dx - p\,dy}{\sqrt{1+p^2+q^2}}$ = a complete differential, or, putting this $= d\zeta$, we introduce into the solution a variable ζ, which combines with z in the forms $z \pm i\zeta$ ($i = \sqrt{-1}$ as usual). The boundary conditions have to be satisfied by the determination of the conjugate variables η, η' as functions of $z + i\zeta$, $z - i\zeta$, or, say, of Z, Z' respectively. By writing S, S' to denote $x + iy$, $x - iy$ respectively, Riemann obtains finally two ordinary differential equations of the first order in S, S', η, η', Z, Z', and the results are completely worked out in some very interesting special cases.

The memoirs on various parts of the general subject are very numerous; references to many of them will be found in Salmon's *Treatise on the Analytic Geometry of Three Dimensions*, 4th ed., Dublin, 1882 (the most comprehensive work on solid geometry); for the minimal surface (which is not considered there) see Memoirs XVII. and XXVI. in Riemann's *Gesammelte mathematische Werke*, Leipsic, 1876; the former—"Ueber die Fläche vom kleinsten Inhalt bei gegebener Begrenzung," as published in *Gött. Abhandl.*, vol. XIII. (1866—67)—contains an introduction by Hattendorff giving the history of the question.