796.

SERIES.

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A SERIES is a set of terms considered as arranged in order. Usually the terms are or represent numerical magnitudes, and we are concerned with the sum of the series. The number of terms may be limited or without limit; and we have thus the two theories, finite series and infinite series. The notions of convergency and divergency present themselves only in the latter theory.

Finite Series.

1. Taking the terms to be numerical magnitudes, or say numbers, if there be a definite number of terms, then the sum of the series is nothing else than the number obtained by the addition of the terms; e.g. $4+9+10=23$, $1+2+4+8=15$. In the first example there is no apparent law for the successive terms; in the second example there is an apparent law. But it is important to notice that in neither case is there a determinate law: we can in an infinity of ways form series beginning with the apparently irregular succession of terms 4, 9, 10, or with the apparently regular succession of terms 1, 2, 4, 8. For instance, in the latter case we may have a series with the general term 2^n, when for $n=0, 1, 2, 3, 4, 5, \ldots$ the series will be $1, 2, 4, 8, 16, 32, \ldots$; or a series with the general term $\frac{1}{6}(n^3+5n+6)$, where for the same values of n the series will be $1, 2, 4, 8, 15, 26, \ldots$ The series may contain negative terms, and in forming the sum each term is of course to be taken with the proper sign.

2. But we may have a given law, such as either of those just mentioned, and the question then arises, to find the sum of an indefinite number of terms, or say of n terms (n standing for any positive integer number at pleasure) of the series. The expression for the sum cannot in this case be obtained by actual addition; the formation by addition of the sum of two terms, of three terms, &c., will, it may be, suggest (but it cannot do more than suggest) the expression for the sum of n terms

of the series. For instance, for the series of odd numbers $1+3+5+7+\ldots$, we have $1=1$, $1+3=4$, $1+3+5=9$, &c. These results at once suggest the law, $1+3+5+\ldots+(2n-1)=n^2$, which is in fact the true expression for the sum of n terms of the series; and this general expression, once obtained, can afterwards be verified.

3. We have here the theory of finite series: the general problem is, u_n being a given function of the positive integer n, to determine as a function of n the sum $u_0+u_1+u_2+\ldots+u_n$, or, in order to have n instead of $n+1$ terms, say the sum $u_0+u_1+u_2+\ldots+u_{n-1}$.

Simple cases are the three which follow.

(i) The arithmetic series,

$$a+(a+b)+(a+2b)+\ldots+(a+\overline{n-1})\,b\,;$$

writing here the terms in the reverse order, it at once appears that twice the sum is $=2a+\overline{n-1}b$ taken n times: that is, the sum $=na+\frac{1}{2}n(n-1)\,b$. In particular, we have an expression for the sum of the natural numbers

$$1+2+3+\ldots+n=\tfrac{1}{2}n\,(n+1),$$

and an expression for the sum of the odd numbers

$$1+3+5+\ldots+(2n-1)=n^2.$$

(ii) The geometric series,

$$a+ar+ar^2+\ldots+ar^{n-1};$$

here the difference between the sum and r times the sum is at once seen to be $=a-ar^n$, and the sum is thus $=a\dfrac{1-r^n}{1-r}$; in particular, the sum of the series

$$1+r+r^2+\ldots+r^{n-1}=\frac{1-r^n}{1-r}.$$

(iii) But the harmonic series,

$$\frac{1}{a}+\frac{1}{a+b}+\frac{1}{a+2b}+\ldots+\frac{1}{a+(n-1)\,b},$$

or say $\frac{1}{1}+\frac{1}{2}+\frac{1}{3}\ldots+\dfrac{1}{n}$, does not admit of summation; there is no algebraical function of n which is equal to the sum of the series.

4. If the general term be a given function u_n, and we can find v_n a function of n such that $v_{n+1}-v_n=u_n$, then we have $u_0=v_1-v_0$, $u_1=v_2-v_1$, $u_2=v_3-v_2,\ldots$, $u_n=v_{n+1}-v_n$; and hence $u_0+u_1+u_2+\ldots+u_n=v_{n+1}-v_0$,—an expression for the required sum. This is in fact an application of the Calculus of Finite Differences. In the notation of this calculus $v_{n+1}-v_n$ is written Δv_n; and the general inverse problem, or problem of integration, is from the equation of differences $\Delta v_n=u_n$ (where u_n is a given function of n) to find v_n. The general solution contains an arbitrary constant, $v_n=V_n+C$; but this disappears in the difference $v_{n+1}-v_0$. As an example consider the series

$$u_0+u_1+\ldots+u_n=0+1+3+\ldots+\tfrac{1}{2}n\,(n+1);$$

here, observing that

$$n(n+1)(n+2)-(n-1)n(n+1)=n(n+1)(\overline{n+2}-\overline{n-1}),\ =3n(n+1),$$

we have

$$v_{n+1}=\tfrac{1}{6}n(n+1)(n+2);$$

and hence

$$1+3+6+\ldots+\tfrac{1}{2}n(n+1)=\tfrac{1}{6}n(n+1)(n+2),$$

as may be at once verified for any particular value of n.

Similarly, when the general term is a factorial of the order r, we have

$$1+\frac{r+1}{1}+\ldots+\frac{n(n+1)\ldots(n+r-1)}{1.2\ \ \ldots\ \ r}=\frac{n(n+1)\ldots(n+r)}{1.2\ \ \ldots(r+1)}.$$

5. If the general term u_n be any rational and integral function of n, we have

$$u_n=u_0+\frac{n}{1}\Delta u_0+\frac{n(n-1)}{1.2}\Delta^2 u_0+\ldots+\frac{n(n-1)\ldots(n-p+1)}{1.2\ldots p}\Delta^p u_0,$$

where the series is continued only up to the term depending on p, the degree of the function u_n, for all the subsequent terms vanish. The series is thus decomposed into a set of series which have each a factorial for the general term, and which can be summed by the last formula; thus we obtain

$$u_0+u_1\ldots+u_n=(n+1)u_0+\frac{(n+1)n}{1.2}\Delta u_0+\ldots+\frac{(n+1)n(n-1)\ldots(n-p+1)}{1.2.3\ldots(p+1)}\Delta^p u_0,$$

which is a function of the degree $p+1$.

Thus for the before-mentioned series $1+2+4+8+\ldots$, if it be assumed that the general term u_n is a cubic function of n, and writing down the given terms and forming the differences, 1, 2, 4, 8; 1, 2, 4; 1, 2; 1, we have

$$u_n=1+\frac{n}{1}+\frac{n(n-1)}{1.2}+\frac{n(n-1)(n-2)}{1.2.3}\ \{=\tfrac{1}{6}(n^3+5n+6),\ \text{as above}\};$$

and the sum

$$u_0+u_1+\ldots+u_n=n+1+\frac{(n+1)n}{1.2}+\frac{(n+1)n(n-1)}{1.2.3}+\frac{(n+1)n(n-1)(n-2)}{1.2.3.4},$$
$$=\tfrac{1}{24}(n^4+2n^3+11n^2+34n+24).$$

As particular cases we have expressions for the sums of the powers of the natural numbers—

$$1^2+2^2+\ldots+n^2=\tfrac{1}{6}n(n+1)(2n+1);\ \ 1^3+2^3+\ldots+n^3=\tfrac{1}{4}n^2(n+1)^2:$$

observe that the latter $=(1+2\ldots+n)^2$; and so on.

6. We may, from the expression for the sum of the geometric series, obtain by differentiation other results: thus

$$1+r+r^2+\ldots+r^{n-1}=\frac{1-r^n}{1-r}$$

gives

$$1+2r+3r^2+\ldots+(n-1)\,r^{n-2}=\frac{d}{dr}\,\frac{1-r^n}{1-r},\;=\frac{1-nr^{n-1}+(n-1)\,r^n}{(1-r)^2};$$

and we might in this way find the sum $u_0+u_1r+\ldots+u_nr^n$, where u_n is any rational and integral function of n.

7. The expression for the sum $u_0+u_1+\ldots+u_n$ of an indefinite number of terms will in many cases lead to the sum of the infinite series $u_0+u_1+\ldots$; but the theory of infinite series requires to be considered separately. Often in dealing apparently with an infinite series $u_0+u_1+\ldots$ we consider rather an indefinite than an infinite series, and are not in any wise really concerned with the sum of the series or the question of its convergency: thus the equation

$$\left(1+mx+\frac{m\,(m-1)}{1\,.\,2}\,x^2+\ldots\right)\left(1+nx+\frac{n\,(n-1)}{1\,.\,2}\,x^2+\ldots\right)$$
$$=1+(m+n)\,x+\frac{(m+n)\,(m+n-1)}{1\,.\,2}\,x^2+\ldots$$

really means the series of identities

$$(m+n)=m+n,$$
$$\frac{(m+n)\,(m+n-1)}{1\,.\,2}=\frac{m\,(m-1)}{1\,.\,2}+2\,\frac{m}{1}\,\frac{n}{1}+\frac{n\,(n-1)}{1\,.\,2},\ \text{\&c.},$$

obtained by multiplying together the two series of the left-hand side. Again, in the method of generating functions we are concerned with an equation $\phi(t)=A_0+A_1t+\ldots+A_nt^n+\ldots$, where the function $\phi(t)$ is used only to express the law of formation of the successive coefficients.

It is an obvious remark that, although according to the original definition of a series the terms are considered as arranged in a determinate order, yet in a finite series (whether the number of terms be definite or indefinite) the sum is independent of the order of arrangement.

Infinite Series.

8. We consider an infinite series $u_0+u_1+u_2+\ldots$ of terms proceeding according to a given law, that is, the general term u_n is given as a function of n. To fix the ideas the terms may be taken to be positive numerical magnitudes, or say numbers continually diminishing to zero; that is, $u_n>u_{n+1}$, and u_n is, moreover, such a function of n that, by taking n sufficiently large, u_n can be made as small as we please.

Forming the successive sums $S_0=u_0$, $S_1=u_0+u_1$, $S_2=u_0+u_1+u_2,\ldots$, these sums S_0, S_1, $S_2,\ldots$ will be a series of continually increasing terms, and if they increase up to a determinate finite limit S (that is, if there exists a determinate numerical magnitude S such that, by taking n sufficiently large, we can make $S-S_n$ as small as we please), S is said to be the sum of the infinite series. To show that we can actually have an infinite series with a given sum S, take u_0 any number less than S, then $S-u_0$ is positive, and taking u_1 any numerical magnitude less than $S-u_0$, then $S-u_0-u_1$ is positive. And going on continually in this manner we obtain a series $u_0+u_1+u_2+\ldots$, such that for any value of n however large $S-u_0-u_1\ldots-u_n$ is positive; and if as n increases this difference diminishes to zero, we have $u_0+u_1+u_2+\ldots$, an infinite series having S for its sum. Thus, if $S=2$, and we take $u_0<2$, say $u_0=1$; $u_1<2-1$, say $u_1=\frac{1}{2}$; $u_2<2-1-\frac{1}{2}$, say $u_2=\frac{1}{4}$; and so on, we have $1+\frac{1}{2}+\frac{1}{4}+\ldots=2$; or, more generally, if r be any positive number less than 1, then $1+r+r^2+\ldots=\frac{1}{1-r}$, that is, the infinite geometric series with the first term $=1$, and with a ratio $r<1$, has the finite sum $\frac{1}{1-r}$. This, in fact, follows from the expression $1+r+r^2\ldots+r^{n-1}=\frac{1-r^n}{1-r}$ for the sum of the finite series; taking $r<1$, then as n increases r^n decreases to zero, and the sum becomes more and more nearly $=\frac{1}{1-r}$.

9. An infinite series of positive numbers can, it is clear, have a sum only if the terms continually diminish to zero; but it is not conversely true that, if this condition be satisfied, there will be a sum. For instance, in the case of the harmonic series $1+\frac{1}{2}+\frac{1}{3}+\ldots$, it can be shown that by taking a sufficient number of terms the sum of the finite series may be made as large as we please. For, writing the series in the form $1+\frac{1}{2}+(\frac{1}{3}+\frac{1}{4})+(\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8})+\ldots$, the number of terms in the brackets being doubled at each successive step, it is clear that the sum of the terms in any bracket is always $>\frac{1}{2}$; hence by sufficiently increasing the number of brackets the sum may be made as large as we please. In the foregoing series, by grouping the terms in a different manner $1+(\frac{1}{2}+\frac{1}{3})+(\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7})+\ldots$, the sum of the terms in any bracket is always <1; we thus arrive at the result that ($n=3$ at least) the sum of 2^n terms of the series is $>1+\frac{1}{2}n$ and $<n$.

10. An infinite series may contain negative terms; suppose in the first instance that the terms are alternately positive and negative. Here the absolute magnitudes of the terms must decrease down to zero, but this is a sufficient condition in order that the series may have a sum. The case in question is that of a series $v_0-v_1+v_2-\ldots$, where $v_0, v_1, v_2, \ldots$ are all positive and decrease down to zero. Here, forming the successive sums $S_0=v_0$, $S_1=v_0-v_1$, $S_2=v_0-v_1+v_2, \ldots$, $S_0, S_1, S_2, \ldots$ are all positive, and we have $S_0>S_1$, $S_1<S_2$, $S_2>S_3, \ldots$, and $S_{n+1}-S_n$ tends continually to zero. Hence the sums $S_0, S_1, S_2, \ldots$ tend continually to a positive limit S in such wise that $S_0, S_2, S_4, \ldots$ are each of them greater and $S_1, S_3, S_5, \ldots$ are each of them less than S; and we thus have S as the sum of the series. The series $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\ldots$ will serve as an example. The case just considered includes the apparently more general one where the series consists of alternate groups of positive and negative terms respectively; the terms of the same group may be united into a single term $\pm v_n$, and the original series will have a sum only if the resulting series $v_0-v_1+v_2\ldots$ has a sum, that is, if the positive partial sums $v_0, v_1, v_2, \ldots$ decrease down to zero.

The terms at the beginning of a series may be irregular as regards their signs; but, when this is so, all the terms in question (assumed to be finite in number) may be united into a single term, which is of course finite, and instead of the original series only the remaining terms of the series need be considered. Every infinite series whatever is thus substantially included under the two forms,—terms all positive and terms alternately positive and negative.

11. In brief, the sum (if any) of the infinite series $u_0 + u_1 + u_2 + \ldots$ is the finite limit (if any) of the successive sums u_0, $u_0 + u_1$, $u_0 + u_1 + u_2, \ldots$; if there is no such limit, then there is no sum. Observe that the assumed order $u_0, u_1, u_2, \ldots$ of the terms is part of and essential to the definition; the terms in any other order may have a different sum, or may have no sum. A series having a sum is said to be "convergent"; a series which has no sum is "divergent."

If a series of positive terms be convergent, the terms cannot, it is clear, continually increase, nor can they tend to a fixed limit: the series $1 + 1 + 1 + \ldots$ is divergent. For the convergency of the series it is necessary (but, as has been shown, not sufficient) that the terms shall decrease to zero. So, if a series with alternately positive and negative terms be convergent, the absolute magnitudes cannot, it is clear, continually increase. In reference to such a series Abel remarks, "Peut-on imaginer rien de plus horrible que de débiter $0 = 1^n - 2^n + 3^n - 4^n +$, &c., où n est un nombre entier positif?" Neither is it allowable that the absolute magnitudes shall tend to a fixed limit. The so-called "neutral" series $1 - 1 + 1 - 1 \ldots$ is divergent: the successive sums do not tend to a determinate limit, but are alternately $+1$ and 0; it is necessary (and also sufficient) that the absolute magnitudes shall decrease to zero.

In the so-called semi-convergent series, we have an equation of the form

$$S = U_0 - U_1 + U_2 - \ldots,$$

where the positive values $U_0, U_1, U_2, \ldots$ decrease to a minimum value, suppose U_p, and afterwards increase; the series is divergent and has no sum, and thus S is not the sum of the series. S is only a number or function calculable approximately by means of the series regarded as a finite series terminating with the term $\pm U_p$. The successive sums $U_0, U_0 - U_1, U_0 - U_1 + U_2, \ldots$ up to that containing $\pm U_p$, give alternately superior and inferior limits of the number or function S.

12. The condition of convergency may be presented under a different form: let the series $u_0 + u_1 + u_2 + \ldots$ be convergent, then, taking m sufficiently large, the sum is the limit not only of $u_0 + u_1 + \ldots + u_m$ but also of $u_0 + u_1 + \ldots + u_{m+r}$, where r is any number as large as we please. The difference of these two expressions must therefore be indefinitely small; by taking m sufficiently large the sum $u_{m+1} + u_{m+2} + \ldots + u_{m+r}$. (where r is any number however large) can be made as small as we please; or, as this may also be stated, the sum of the infinite series $u_{m+1} + u_{m+2} + \ldots$ can be made as small as we please. If the terms are all positive (but not otherwise), we may take, instead of the entire series $u_{m+1} + u_{m+2} + \ldots$, any set of terms (not of necessity consecutive terms) subsequent to u_m; that is, for a convergent series of positive terms the sum of any set of terms subsequent to u_m can, by taking m sufficiently large, be made as small as we please.

13. It follows that, in a convergent series of positive terms, the terms may be grouped together in any manner so as to form a finite number of partial series which will be each of them convergent, and such that the sum of their sums will be the sum of the given series. For instance, if the given series be $u_0 + u_1 + u_2 + \ldots$, then the two series $u_0 + u_2 + u_4 + \ldots$ and $u_1 + u_3 + \ldots$ will each be convergent and the sum of their sums will be the sum of the original series.

14. Obviously the conclusion does not hold good in general for series of positive and negative terms: for instance, the series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots$ is convergent, but the two series $1 + \frac{1}{3} + \frac{1}{5} + \ldots$ and $-\frac{1}{2} - \frac{1}{4} - \ldots$ are each divergent, and thus without a sum. In order that the conclusion may be applicable to a series of positive and negative terms the series must be "absolutely convergent," that is, it must be convergent when all the terms are made positive. This implies that the positive terms taken by themselves are a convergent series, and also that the negative terms taken by themselves are a convergent series. It is hardly necessary to remark that a convergent series of positive terms is absolutely convergent. The question of the convergency or divergency of a series of positive and negative terms is of less importance than the question whether it is or is not absolutely convergent. But in this latter question we regard the terms as all positive, and the question in effect relates to series containing positive terms only.

15. Consider, then, a series of positive terms $u_0 + u_1 + u_2 + \ldots$; if they are increasing—that is, if in the limit u_{n+1}/u_n be greater than 1—the series is divergent, but if less than 1 the series is convergent. This may be called a first criterion; but there is the doubtful case where the limit $= 1$. A second criterion was given by Cauchy and Raabe; but there is here again a doubtful case when the limit considered $= 1$. A succession of criteria was established by De Morgan, which it seems proper to give in the original form; but the equivalent criteria established by Bertrand are somewhat more convenient. In what follows lx is for shortness written to denote the logarithm of x, no matter to what base. De Morgan's form is as follows:—Writing $u_n = \dfrac{1}{\phi(n)}$, put $p_0 = \dfrac{x\phi' x}{\phi x}$; if for $x = \infty$ the limit a_0 of p_0 be greater than 1 the series is convergent, but if less than 1 it is divergent. If the limit $a_0 = 1$, seek for the limit of $p_1, = (p_0 - 1)\, lx$; if this limit a_1 be greater than 1 the series is convergent, but if less than 1 it is divergent. If the limit $a_1 = 1$, seek for the limit $p_2, = (p_1 - 1)\, llx$; if this limit a_2 be greater than 1 the series is convergent, but if less than 1 it is divergent. And so on indefinitely.

16. Bertrand's form is:—If, in the limit for $n = \infty$, $l\dfrac{1}{u_n}\Big/ ln$ be negative or less than 1 the series is divergent, but if greater than 1 it is convergent. If it $= 1$, then if $l\dfrac{1}{nu_n}\Big/ lln$ be negative or less than 1 the series is divergent, but if greater than 1 it is convergent. If it $= 1$, then if $l\dfrac{1}{nu_n ln}\Big/ llln$ be negative or less than 1 the series is divergent, but if greater than 1 it is convergent. And so on indefinitely.

The last-mentioned criteria follow at once from the theorem that the several series having the general terms $\frac{1}{n^\alpha}$, $\frac{1}{n\,(ln)^\alpha}$, $\frac{1}{nln\,(lln)^\alpha}$, $\frac{1}{nlnlln\,(llln)^\alpha}$, ... respectively are each of them convergent if α be greater than 1, but divergent if α be negative or less than 1 or $=1$. In the simplest case, the series having the general term $\frac{1}{n^\alpha}$, the theorem may be proved nearly in the manner in which it is shown above (cf. § 9) that the harmonic series is divergent.

17. Two or more absolutely convergent series may be added together,

$$\{u_0 + u_1 + u_2 + \dots\} + \{v_0 + v_1 + v_2 + \dots\} = (u_0 + v_0) + (u_1 + v_1) + \dots;$$

that is, the resulting series is absolutely convergent and has for its sum the sum of the two sums. And similarly two or more absolutely convergent series may be multiplied together

$$\{u_0 + u_1 + u_2 + \dots\} \times \{v_0 + v_1 + v_2 + \dots\} = u_0v_0 + (u_0v_1 + u_1v_0) + (u_0v_2 + u_1v_1 + u_2v_0) + \dots;$$

that is, the resulting series is absolutely convergent and has for its sum the product of the two sums. But more properly the multiplication gives rise to a doubly infinite series—

$$\begin{array}{lll} u_0v_0, & u_0v_1, & u_0v_2 \dots \\ u_1v_0, & u_1v_1, & u_1v_2 \\ \vdots & & \end{array}$$

—which is a kind of series which will be presently considered.

18. But it is, in the first instance, proper to consider a single series extending backwards and forwards to infinity, or say a back-and-forwards infinite series $\dots + u_{-2} + u_{-1} + u_0 + u_1 + u_2 + \dots$; such a series may be absolutely convergent, and the sum is then independent of the order of the terms, and in fact equal to the sum of the sums of the two series $u_0 + u_1 + u_2 + \dots$ and $u_{-1} + u_{-2} + u_{-3} + \dots$ respectively. But, if not absolutely convergent, the expression has no definite meaning until it is explained in what manner the terms are intended to be grouped together; for instance, the expression may be used to denote the foregoing sum of two series, or to denote the series $u_0 + (u_1 + u_{-1}) + (u_2 + u_{-2}) + \dots$ and the sum may have different values, or there may be no sum, accordingly. Thus, if the series be $\dots - \frac{1}{2} - \frac{1}{1} + 0 + \frac{1}{1} + \frac{1}{2} + \dots$, in the former meaning the two series $0 + \frac{1}{1} + \frac{1}{2} + \dots$ and $-\frac{1}{1} - \frac{1}{2} - \dots$ are each divergent, and there is not any sum. But in the latter meaning the series is $0 + 0 + 0 + \dots$, which has a sum $= 0$. So, if the series be taken to denote the limit of

$$(u_0 + u_1 + u_2 + \dots + u_m) + (u_{-1} + u_{-2} + \dots + u_{-m'}),$$

where m, m' are each of them ultimately infinite, there may be a sum depending on the ratio $m : m'$, which sum consequently acquires a determinate value only when this ratio is given.

19. In a singly infinite series we have a general term u_n, where n is an integer positive in the case of an ordinary series, and positive or negative in the case of a

back-and-forwards series. Similarly for a doubly infinite series, we have a general term $u_{m,n}$, where m, n are integers which may be each of them positive, and the form of the series is then

$$\begin{matrix} u_{0,0}, & u_{0,1}, & u_{0,2} \cdots \\ u_{1,0}, & u_{1,1}, & u_{1,2} \\ \vdots & & \end{matrix} ;$$

or m, n may be each of them positive or negative. The latter is the more general supposition, and includes the former, since $u_{m,n}$ may $=0$ for m or n each or either of them negative. To put a definite meaning on the notion of a sum, we may regard m, n as the rectangular coordinates of a point in a plane; that is, if m, n are each of them positive, we attend only to the positive quadrant of the plane, but otherwise to the whole plane; and we have thus a doubly infinite system or lattice-work of points. We may imagine a boundary depending on a parameter T which for $T=\infty$ is at every point thereof at an infinite distance from the origin; for instance, the boundary may be the circle $x^2+y^2=T$, or the four sides of a rectangle, $x=\pm\alpha T$, $y=\pm\beta T$. Suppose the form is given and the value of T, and let the sum $\Sigma u_{m,n}$ be understood to denote the sum of those terms $u_{m,n}$ which correspond to points within the boundary, then, if as T increases without limit the sum in question continually approaches a determinate limit (dependent, it may be, on the form of the boundary), *for such form of boundary* the series is said to be convergent, and the sum of the doubly infinite series is the aforesaid limit of the sum $\Sigma u_{m,n}$. The condition of convergency may be otherwise stated: it must be possible to take T so large that the sum $\Sigma u_{m,n}$ for all terms $u_{m,n}$ which correspond to points outside the boundary shall be as small as we please.

It is easy to see that, if the terms $u_{m,n}$ be all of them positive, and the series be convergent for any particular form of boundary, it will be convergent for any other form of boundary, and the sum will be the same in each case. Thus, let the boundary be in the first instance the circle $x^2+y^2=T$; by taking T sufficiently large the sum $\Sigma u_{m,n}$ for points outside the circle may be made as small as we please. Consider any other form of boundary—for instance, an ellipse of given eccentricity,—and let such an ellipse be drawn including within it the circle $x^2+y^2=T$. Then the sum $\Sigma u_{m,n}$ for terms $u_{m,n}$ corresponding to points outside the ellipse will be smaller than the sum for points outside the circle, and the difference of the two sums—that is, the sum for points outside the circle and inside the ellipse—will also be less than that for points outside the circle, and can thus be made as small as we please. Hence finally the sum $\Sigma u_{m,n}$, whether restricted to terms $u_{m,n}$ corresponding to points inside the circle or to terms corresponding to points inside the ellipse, will have the same value, or the sum of the series is independent of the form of the boundary. Such a series, viz. a doubly infinite convergent series of positive terms, is said to be absolutely convergent; and similarly a doubly infinite series of positive and negative terms which is convergent when the terms are all taken as positive is absolutely convergent.

20. We have in the preceding theory the foundation of the theorem (§ 17) as to the product of two absolutely convergent series. The product is in the first instance

expressed as a doubly infinite series; and, if we sum this for the boundary $x+y=T$, this is in effect a summation of the series $u_0v_0+(u_0v_1+u_1v_0)+\ldots$, which is the product of the two series. It may be further remarked that, starting with the doubly infinite series and summing for the rectangular boundary $x=\alpha T$, $y=\beta T$, we obtain the sum as the product of the sums of the two single series. For series not absolutely convergent, the theorem is not true. A striking instance is given by Cauchy: the series $1-\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}-\frac{1}{\sqrt{4}}+\ldots$ is convergent and has a calculable sum, but it can be shown without difficulty that its square, viz. the series $1-\frac{2}{\sqrt{2}}+\left(\frac{2}{\sqrt{3}}+\frac{1}{2}\right)-\ldots$, is divergent.

21. The case where the terms of a series are imaginary comes under that where they are real. Suppose the general term is p_n+q_ni, then the series will have a sum, or will be convergent, if and only if the series having for its general term p_n and the series having for its general term q_n be each convergent; then the sum = sum of first series + i multiplied by sum of second series. The notion of absolute convergence will of course apply to each of the series separately; further, if the series having for its general term the modulus $\sqrt{p^2_n+q^2_n}$ be convergent (that is, absolutely convergent, since the terms are all positive), each of the component series will be absolutely convergent; but the condition is not necessary for the convergence, or the absolute convergence, of the two component series respectively.

22. In the series thus far considered, the terms are actual numbers, or are at least regarded as constant; but we may have a series $u_0+u_1+u_2+\ldots$, where the successive terms are functions of a parameter z; in particular, we may have a series $a_0+a_1z+a_2z^2+\ldots$ arranged in powers of z. It is in view of a complete theory *necessary* to consider z as having the imaginary value $x+iy=r(\cos\phi+i\sin\phi)$. The two component series will then have the general terms $a_nr^n\cos n\phi$ and $a_nr^n\sin n\phi$ respectively; accordingly each of these series will be absolutely convergent for any value whatever of ϕ, provided the series with the general term a_nr^n be absolutely convergent. Moreover, the series, if thus absolutely convergent for any particular value R of r, will be absolutely convergent for any smaller value of r, that is, for any value of $x+iy$ having a modulus not exceeding R; or, representing as usual $x+iy$ by the point whose rectangular coordinates are x, y, the series will be absolutely convergent for any point whatever inside or on the circumference of the circle having the origin for centre and its radius $=R$. The origin is of course an arbitrary point: or, what is the same thing, instead of a series in powers of z, we may consider a series in powers of $z-c$ (where c is a given imaginary value $=\alpha+\beta i$). Starting from the series, we may within the aforesaid limit of absolute convergency consider the series as the definition of a function of the variable z; in particular, the series may be absolutely convergent for every finite value of the modulus, and we have then a function defined for every finite value whatever $x+iy$ of the variable. Conversely, starting from a given function of the variable, we may inquire under what conditions it admits of expansion in a series of powers of z (or $z-c$), and seek to determine the expansion of the function in a series of this form. But in all this, however, we are travelling out of the theory of series into the general theory of functions.

23. Considering the modulus r as a given quantity and the several powers of r as included in the coefficients, the component series are of the forms $a_0+a_1\cos\phi+a_2\cos 2\phi+\ldots$ and $a_1\sin\phi+a_2\sin 2\phi+\ldots$ respectively. The theory of these trigonometrical or multiple sine and cosine series, and of the development, under proper conditions, of an arbitrary function in series of these forms, constitutes an important and interesting branch of analysis.

24. In the case of a real variable z, we may have a series $a_0+a_1z+a_2z^2+\ldots$, where the series $a_0+a_1+a_2+\ldots$ is a divergent series of decreasing positive terms (or as a limiting case where this series is $1+1+1+\ldots$). For a value of z inferior but indefinitely near to ± 1, say $z=\pm(1-\epsilon)$, where ϵ is indefinitely small and positive, the series will be convergent and have a determinate sum $\phi(z)$, and we may write $\phi(\pm 1)$ to denote the limit of $\phi(\pm(1-\epsilon))$ as ϵ diminishes to zero; but unless the series be convergent for the value $z=\pm 1$, it cannot for this value have a sum, nor consequently a sum $=\phi(\pm 1)$. For instance, let the series be $z+\frac{z^2}{2}+\frac{z^3}{3}+\ldots$, which for values of z between the limits ± 1 (both limits excluded) $=-\log(1-z)$. For $z=+1$ the series is divergent and has no sum; but for $z=1-\epsilon$, as ϵ diminishes to zero, we have $-\log\epsilon$ and $(1-\epsilon)+\frac{1}{2}(1-\epsilon)^2+\ldots$, each positive and increasing without limit; for $z=-1$, the series $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\ldots$ is convergent, and we have *at the limit* $\log 2=1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\ldots$. As a second example, consider the series $1+z+z^2+\ldots$, which for values of z between the limits ± 1 (both limits excluded) $=\frac{1}{1-z}$. For $z=+1$, the series is divergent and has no sum; but for $z=1-\epsilon$, as ϵ diminishes to zero, we have $\frac{1}{\epsilon}$ and $1+(1-\epsilon)+(1-\epsilon)^2+\ldots$, each positive and increasing without limit; for $z=-1$ the series is divergent and has no sum; the equation $\frac{1}{2-\epsilon}=1-(1-\epsilon)+(1-\epsilon)^2-\ldots$ is true for any positive value of ϵ however small, but *not* for the value $\epsilon=0$.

The following memoirs and works may be consulted:—Cauchy, *Cours d'Analyse de l'École Polytechnique*—part I., *Analyse Algébrique*, 8vo. Paris, 1821; Abel, "Untersuchungen über die Reihe $1+\frac{m}{1}x+\frac{m(m-1)}{1\,.\,2}x^2+\ldots$," in Crelle's *Journ. de Math.*, vol. I. (1826), pp. 211—239, and *Œuvres* (French trans.), vol. I.; De Morgan, *Treatise on the Differential and Integral Calculus*, 8vo. London, 1842; Id., "On Divergent Series and various Points of Analysis connected with them" (1844), in *Camb. Phil. Trans.*, vol. VIII. (1849), and other memoirs in *Camb. Phil. Trans.*; Bertrand, "Règles sur la Convergence des Séries," in *Liouv. Journ. de Math.*, vol. VII. (1842), pp. 35—54; Cayley, "On the Inverse Elliptic Functions," *Camb. Math. Journ.*, vol. IV. (1845), pp. 257—277, [24], and "Mémoire sur les Fonctions doublement périodiques," in *Liouv. Journ. de Math.*, vol. X. (1845), pp. 385—420, [25], (as to the boundary for a doubly infinite series): Riemann, "Ueber die Darstellbarkeit einer Function durch eine trigonometrische Reihe," in *Gött. Abh.*, vol. XIII. (1854), and *Werke*, Leipsic, 1876, pp. 213—253 (contains an account of preceding researches by Euler, D'Alembert, Fourier, Lejeune-Dirichlet, &c.); Catalan, *Traité Élémentaire des Séries*, 8vo. Paris, 1860; Boole, *Treatise on the Calculus of Finite Differences*, 2nd ed. by Moulton, 8vo. London, 1872.