790.

GEOMETRY (ANALYTICAL).

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THIS will be here treated as a method. The science is Geometry; and it would be possible, analytically, or by the method of coordinates, to develope the truths of geometry in a systematic course. But it is proposed not in any way to attempt this, but simply to explain the method, giving such examples, interesting (it may be) in themselves, as are suitable for showing how the method is employed in the demonstration and solution of theorems or problems.

Geometry is one-, two-, or three-dimensional, or, what is the same thing, it is lineal, plane, or solid, according as the space dealt with is the line, the plane, or ordinary (three-dimensional) space. No more general view of the subject need here be taken:—but in a certain sense one-dimensional geometry does not exist, inasmuch as the geometrical constructions for points in a line can only be performed by travelling out of the line into other parts of a plane which contains it, and conformably to the usual practice Analytical Geometry will be treated under the two divisions, Plane and Solid.

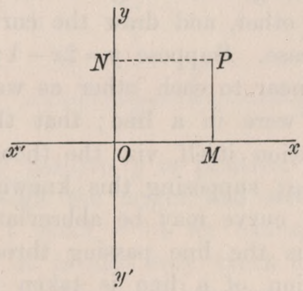
It is proposed to consider Cartesian coordinates almost exclusively; for the proper development of the science homogeneous coordinates (three and four in plane and solid geometry respectively) are required; and it is moreover necessary to have the correlative line- and plane-coordinates; and in solid geometry to have the *six* coordinates of the line. The most comprehensive English works are those of Dr Salmon, *Conic Sections* (5th edition, 1869), *Higher Plane Curves* (2nd edition, 1873), and *Geometry of Three Dimensions* (3rd edition, 1874); we have also, on plane geometry, Clebsch's *Vorlesungen über Geometrie*, posthumous, edited by Dr F. Lindemann, Leipsic, 1875, not yet complete.

I. PLANE ANALYTICAL GEOMETRY (§§ 1—25).

1. It is assumed that the points, lines, and figures considered exist in one and the same plane, which plane, therefore, need not be in any way referred to. The position of a point is determined by means of its (Cartesian) coordinates; i.e. as

explained under the article Curve, [785], we take the two lines $x'Ox$ and $y'Oy$, called the axes of x and y respectively, intersecting in a point O called the origin, and determine the position of any other point P by means of its coordinates $x = OM$ (or NP), and

Fig. 1.



$y = MP$ (or ON). The two axes are usually (as in fig. 1) at right angles to each other, and the lines PM, PN are then at right angles to the axes of x and y respectively. Assuming a scale at pleasure, the coordinates x, y of a point have numerical values.

It is necessary to attend to the signs: x has opposite signs according as the point is on one side or the other of the axis of y, and similarly y has opposite signs according as the point is on the one side or the other of the axis of x. Using the letters N., E., S., W. as in a map, and considering the plane as divided into four quadrants by the axes, the signs are usually taken to be—

x	y	for quadt.
+	+	N. E.
+	−	S. E.
−	+	N. W.
−	−	S. W.

A point is said to have the coordinates (a, b), and is referred to as the point (a, b), when its coordinates are $x = a$, $y = b$; the coordinates x, y of a variable point, or of a point which is for the time being regarded as variable, are said to be current coordinates.

2. It is sometimes convenient to use oblique coordinates; the only difference is that the axes are not at right angles to each other; the lines PM, PN are drawn parallel to the axes of y and x respectively, and the figure $OMPN$ is thus a parallelogram. But in all that follows, the Cartesian coordinates are taken to be rectangular; polar coordinates and other systems will be briefly referred to in the sequel.

3. If the coordinates (x, y) of a point are not given, but only a relation between them $f(x, y) = 0$, then we have a curve. For, if we consider x as a real quantity varying continuously from $-\infty$ to $+\infty$, then, for any given value of x, y has a value

or values. If these are all imaginary, there is not any real point; but if one or more of them be real, we have a real point or points, which (as the assumed value of x varies continuously) varies or vary continuously therewith; and the locus of all these real points is a curve. The equation completely defines the curve; to trace the curve directly from the equation, nothing else being known, we obtain as above a series of points sufficiently near to each other, and draw the curve through them. For instance, let this be done in a simple case. Suppose $y=2x-1$; it is quite easy to obtain and lay down a series of points as near to each other as we please, and the application of a ruler would show that these were in a line; that the curve is a line depends upon something more than the equation itself, viz. the theorem that every equation of the form $y=ax+b$ represents a line; supposing this known, it will be at once understood how the process of tracing the curve may be abbreviated; we have $x=0$, $y=-1$, and $x=\frac{1}{2}$, $y=0$; the curve is thus the line passing through these two points. But in the foregoing example the notion of a line is taken to be a known one, and such notion of a line does in fact precede the consideration of any equation of a curve whatever, since the notion of the coordinates themselves rests upon that of a line. In other cases it may very well be that the equation is the definition of the curve; the points laid down, although (as finite in number) they do not actually determine the curve, determine it to any degree of accuracy; and the equation thus enables us to construct the curve.

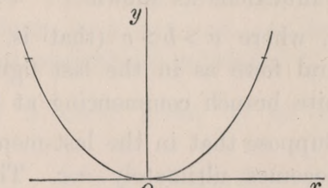
A curve may be determined in another way; viz. the coordinates x, y may be given each of them as a function of the same variable parameter θ; $x, y = f(\theta), \phi(\theta)$ respectively. Here, giving to θ any number of values in succession, these equations determine the values of x, y, that is, the positions of a series of points on the curve. The ordinary form $y=\phi(x)$, where y is given explicitly as a function of x, is a particular case of each of the other two forms: we have $f(x, y), = y-\phi(x), = 0$; and $x=\theta$, $y=\phi(\theta)$.

4. As remarked under Curve, [785], it is a useful exercise to trace a considerable number of curves, first taking equations which are purely numerical, and then equations which contain literal constants (representing numbers); the equations most easily dealt with are those wherein one coordinate is given as an explicit function of the other, say $y=\phi(x)$ as above. A few examples are here given, with such explanations as seem proper.

(i) $y=2x-1$, as before; it is at once seen that this is a line; and taking it to be so, any two points, for instance, $(0, -1)$ and $(\frac{1}{2}, 0)$, determine the line.

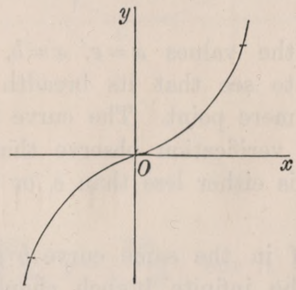
(ii) $y=x^2$. The equation shows that x may be positive or negative, but that y is always positive, and has the same values for equal positive and negative values of x: the curve passes through the origin, and through the points $(\pm 1, 1)$. It is already known that the curve lies wholly above the axis of x. To find its form in the neighbourhood of the origin, give x a small value, $x = \pm 0{\cdot}1$ or $\pm 0{\cdot}01$, then y is very much smaller, $= 0{\cdot}01$ and $0{\cdot}0001$ in the two cases respectively; this shows that the curve touches the axis of x at the origin. Moreover, x may be as large as we please, but when it is large, y is much larger; for instance, $x=10$, $y=100$. The curve is a parabola (fig. 2).

Fig. 2.



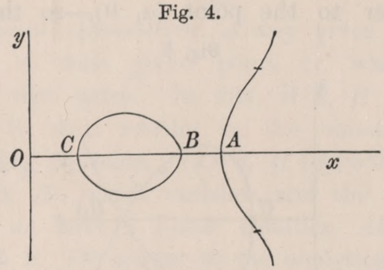
(iii) $y=x^3$. Here x being positive y is positive, but x being negative y is also negative: the curve passes through the origin, and also through the points (1, 1) and (−1, −1). Moreover, when x is small, $=0\cdot1$ for example, then not only is y, $=0\cdot001$, very much smaller than x, but it is also very much smaller than y was for the last-mentioned curve $y=x^2$, that is, in the neighbourhood of the origin the present curve approaches more closely the axis of x. The axis of x is a tangent at the origin, but it is a tangent of a peculiar kind (a stationary or inflexional tangent), cutting the curve at the origin, which is an inflexion. The curve is the cubical parabola (fig. 3).

Fig. 3.



(iv) $y^2=x-1\,.\,x-3\,.\,x-4$. Here $y=0$ for $x=1$, $=3$, $=4$. Whenever $x-1\,.\,x-3\,.\,x-4$ is positive, y has two equal and opposite values; but when $x-1\,.\,x-3\,.\,x-4$ is negative, then y is imaginary. In particular, for x less than 1, or between 3 and 4, y is imaginary, but for x between 1 and 3, or greater than 4, y has two values. It

Fig. 4.



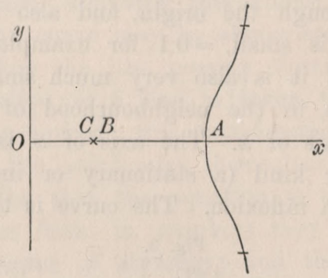
is clear that for x somewhere between 1 and 3, y will attain a maximum: the values of x and y may be found approximately by trial. The curve will consist of an oval and infinite branch, and it is easy to see that, as shown in fig. 4, the curve where it cuts the axis of x cuts it at right angles. It may be further remarked that, as

x increases from 4, the value of y will increase more and more rapidly; for instance, $x=5$, $y^2=8$, $x=10$, $y^2=378$, &c., and it is easy to see that this implies that the curve has on the infinite branch two inflexions as shown.

(v) $y^2 = x-c \,.\, x-b \,.\, x-a$, where $a > b > c$ (that is, a nearer to $+\infty$, c to $-\infty$). The curve has the same general form as in the last figure, the oval extending between the limits $x=c$, $x=b$, the infinite branch commencing at the point $x=a$.

(vi) $y^2=(x-c)^2(x-a)$. Suppose that in the last-mentioned curve, $y^2=x-c.x-b.x-a$, b gradually diminishes, and becomes ultimately $=c$. The infinite branch (see fig. 5) changes its form, but not in a very marked manner, and it retains the two inflexions.

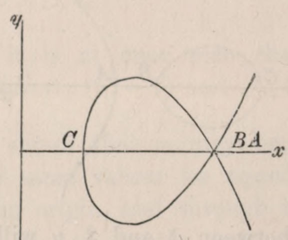
Fig. 5.



The oval lies always between the values $x=c$, $x=b$, and therefore its length continually diminishes; it is easy to see that its breadth will also continually diminish; ultimately it shrinks up into a mere point. The curve has thus a conjugate or isolated point, or acnode. For a direct verification observe that $x=c$, $y=0$, so that $(c, 0)$ is a point of the curve, but if x is either less than c, or between c and a, y^2 is negative, and y is imaginary.

(vii) $y^2=(x-c)(x-a)^2$. If in the same curve b gradually increases and becomes ultimately $=a$, the oval and the infinite branch change each of them its form, the oval extending always between the values $x=c$, $x=b$, and thus continually approaching the infinite branch, which begins at $x=a$. The consideration of a few numerical examples, *with careful drawing*, would show that the oval and the infinite branch as they approach sharpen out each towards the other, the two inflexions on the infinite branch coming always nearer to the point $(a, 0)$,—so that finally, when b becomes

Fig. 6.



$=a$, the curve has the form shown in fig. 6, there being now a double point or node (crunode) at A, and the inflexions on the infinite branch having disappeared.

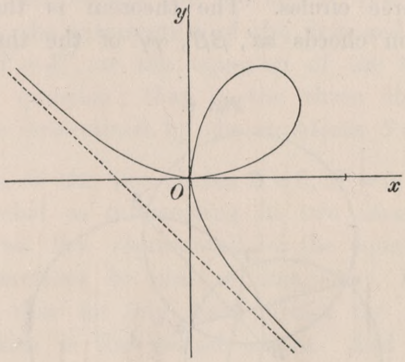
In the last four examples, the curve is one of the cubical curves called the divergent parabolas: (iv) is a mere numerical example of (v), and (vi), (vii), (viii) are in Newton's language the parabola *cum ovali*, *punctata*, and *nodata* respectively. When a, b, c are all equal, or the form is $y^2=(x-c)^3$, we have a cuspidal form, Newton's parabola *cuspidata*, otherwise the semicubical parabola.

(viii) As an example of a curve given by an implicit equation, suppose the equation is

$$x^3+y^3-3xy=0;$$

this is a nodal cubic curve, the node at the origin, and the axes touching the two branches respectively (fig. 7). An easy mode of tracing it is to express x, y each of them

Fig. 7.



in terms of a variable θ, $x=\dfrac{3\theta}{1+\theta^3}$, $y=\dfrac{3\theta^2}{1+\theta^3}$; but it is instructive to trace the curve directly from its equation.

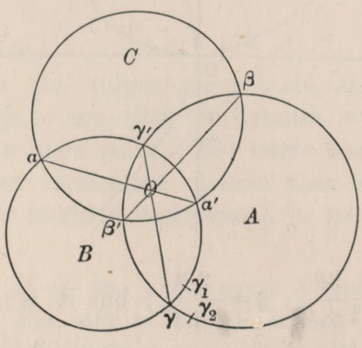
5. It may be remarked that the purely algebraical process, which is, in fact, that employed in finding a differential coefficient $\dfrac{dy}{dx}$, if applied directly to the equation of the curve, determines the point consecutive to any given point of the curve, that is, the direction of the curve at such given point, or, what is the same thing, the direction of the tangent at that point. In fact, if α, β are the coordinates of any point on a curve $f(x, y)=0$, then writing in the equation of the curve $x=\alpha+h$, $y=\beta+k$, and in the resulting equation $f(\alpha+h, \beta+k)=0$, developed in powers of h and k, omitting the term $f(\alpha, \beta)$, which vanishes, and the terms containing the second and higher powers of h, k, we have a linear equation $Ah+Bk=0$, which determines the ratio of the increments h, k. Of course, in the analytical development of the theory, we translate this into the notation of the differential calculus; but the question presents itself, and is thus seen to be solvable, as soon as it is attempted to trace a curve from its equation.

Geometry is Descriptive, or Metrical.

6. A geometrical proposition is either *descriptive* or *metrical*: in the former case it is altogether independent of the idea of magnitude (length, inclination, &c.); in the latter case it has reference to this idea. It is to be noticed that, although the method of coordinates seems to be by its inception essentially metrical, and we can hardly, except by metrical considerations, connect an equation with the curve which it represents (for instance, even assuming it to be known that an equation $Ax + By + C = 0$ represents a line, yet if it be asked what line, the only form of answer is, that it is the line cutting the axes at *distances* from the origin $-C \div A$, $-C \div B$ respectively), yet in dealing by this method with descriptive propositions, we are, in fact, eminently free from all metrical considerations.

7. It is worth while to illustrate this by the instance of the well-known theorem of the radical centre of three circles. The theorem is that, given any three circles A, B, C (fig. 8), the common chords $\alpha\alpha'$, $\beta\beta'$, $\gamma\gamma'$ of the three pairs of circles meet in a point.

Fig. 8.



The geometrical proof is metrical throughout:—

Take O the point of intersection of $\alpha\alpha'$, $\beta\beta'$, and joining this with γ', suppose that $\gamma' O$ does not pass through γ, but that it meets the circles A, B in two distinct points γ_1, γ_2 respectively. We have then the known metrical property of intersecting chords of a circle; viz. in the circle C, where $\alpha\alpha'$, $\beta\beta'$ are chords meeting at a point O,

$$O\alpha \,.\, O\alpha' = O\beta \,.\, O\beta',$$

where, as well as in what immediately follows $O\alpha$, &c., denote, of course, *lengths* or *distances*.

Similarly in the circle A,

$$O\beta \,.\, O\beta' = O\gamma_1 \,.\, O\gamma',$$

and in the circle B,

$$O\alpha \,.\, O\alpha' = O\gamma_2 \,.\, O\gamma'.$$

Consequently $O\gamma_1 \,.\, O\gamma' = O\gamma_2 \,.\, O\gamma'$, that is, $O\gamma_1 = O\gamma_2$, or the points γ_1 and γ_2 coincide; that is, they each coincide with γ.

We contrast this with the analytical method.

Here it only requires to be known that an equation $Ax + By + C = 0$ represents a line, and an equation $x^2 + y^2 + Ax + By + C = 0$ represents a circle. A, B, C have, in the two cases respectively, metrical significations; but these we are not concerned with. Using S to denote the function $x^2 + y^2 + Ax + By + C$, the equation of a circle is $S = 0$, where S stands for its value; more briefly, we say the equation is $S, = x^2+y^2+Ax+By+C, = 0$. Let the equation of any other circle be $S', = x^2 + y^2 + A'x + B'y + C' = 0$; the equation $S - S' = 0$ is a linear equation: $S - S'$ is, in fact, $= (A - A')x + (B - B')y + C - C'$: and it thus represents a line; this equation is satisfied by the coordinates of each of the points of intersection of the two circles (for at each of these points $S = 0$ and $S' = 0$, therefore also $S - S' = 0$); hence the equation $S - S' = 0$ is that of the line joining the two points of intersection of the two circles, or say it is the equation of the common chord of the two circles. Considering then a third circle $S'', = x^2+y^2+A''x+B''y + C'' = 0$, the equations of the common chords are $S - S' = 0$, $S - S'' = 0$, $S' - S'' = 0$ (each of these a linear equation); at the intersection of the first and second of these lines $S = S'$ and $S = S''$, therefore also $S' = S''$, or the equation of the third line is satisfied by the coordinates of the point in question; that is, the three chords intersect in a point O, the coordinates of which are determined by the equations $S = S' = S''$.

It further appears that, if the two circles $S = 0$, $S' = 0$ do not intersect in any real points, they must be regarded as intersecting in two imaginary points, such that the line joining them is the real line represented by the equation $S - S' = 0$; or that two circles, whether their intersections be real or imaginary, have always a real common chord (or radical axis), and that for *any* three circles the common chords intersect in a point (of course real) which is the radical centre. And by this very theorem, given two circles with imaginary intersections, we can, by drawing circles which meet each of them in real points, construct the radical axis of the first-mentioned two circles.

8. The principle employed in showing that the equation of the common chord of two circles is $S - S' = 0$ is one of very extensive application, and some more illustrations of it may be given.

Suppose $S = 0$, $S' = 0$ are lines, that is, let S, S' now denote linear functions $Ax + By + C$, $A'x + B'y + C'$, then $S - kS' = 0$ (k an arbitrary constant) is the equation of *any* line passing through the point of intersection of the two given lines. Such a line may be made to pass through any given point, say the point (x_0, y_0); i.e., if S_0, S_0' are what S, S' respectively become on writing for (x, y) the values (x_0, y_0), then the value of k is $k = S_0 \div S_0'$. The equation in fact is $SS_0' - S_0S' = 0$; and starting from this equation we at once verify it *à posteriori;* the equation is a linear equation satisfied by the values of (x, y) which make $S = 0$, $S' = 0$; and satisfied also by the values (x_0, y_0); and it is thus the equation of the line in question.

If, as before, $S = 0$, $S' = 0$ represent circles, then (k being arbitrary) $S - kS' = 0$ is the equation of any circle passing through the two points of intersection of the two circles; and to make this pass through a given point (x_0, y_0) we have again $k = S_0 \div S_0'$. In the particular case $k = 1$, the circle becomes the common chord; more accurately, it becomes the common chord together with the line infinity, but this is a question which is not here gone into.

If S denote the general quadric function,

$$S = ax^2 + 2hxy + by^2 + 2fy + 2gx + c, \ = (a,\ b,\ c,\ f,\ g,\ h)(x,\ y,\ 1)^2,$$

then the equation $S = 0$ represents a conic; assuming this, then, if $S' = 0$ represents another conic, the equation $S - kS' = 0$ represents *any* conic through the four points of intersection of the two conics.

Returning to the equation $Ax + By + C = 0$ of a line, if this pass through two given points $(x_1,\ y_1)$, $(x_2,\ y_2)$, then we must have $Ax_1 + By_1 + C = 0$, $Ax_2 + By_2 + C = 0$, equations which determine the ratios $A : B : C$, and it thus appears that the equation of the line through the two given points is

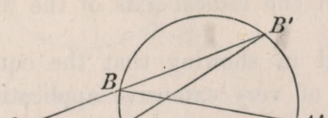
$$x(y_1 - y_2) - y(x_1 - x_2) + x_1y_2 - x_2y_1 = 0;$$

or, what is the same thing,

$$\begin{vmatrix} x, & y, & 1 \\ x_1, & y_1, & 1 \\ x_2, & y_2, & 1 \end{vmatrix} = 0.$$

9. The object still being to illustrate the mode of working with coordinates, we consider the theorem of the polar of a point in regard to a circle. Given a circle and a point O (fig. 9), we draw through O any two lines meeting the circle in the points

Fig. 9.



A, A' and B, B' respectively, and then taking Q as the intersection of the lines AB' and $A'B$, the theorem is that the locus of the point Q is a right line depending only upon O and the circle, but independent of the particular lines OAA' and OBB'.

Taking O as the origin, and for the axes any two lines through O at right angles to each other, the equation of the circle will be

$$x^2 + y^2 + 2Ax + 2By + C = 0;$$

and if the equation of the line OAA' is taken to be $y = mx$, then the points A, A' are found as the intersections of the straight line with the circle; or to determine x we have

$$x^2(1 + m^2) + 2x(A + Bm) + C = 0.$$

If (x_1, y_1) are the coordinates of A, and (x_2, y_2) of A', then the roots of this equation are x_1, x_2, whence easily

$$\frac{1}{x_1}+\frac{1}{x_2}=-2\,\frac{A+Bm}{C}.$$

And similarly, if the equation of the line OBB' is taken to be $y=m'x$, and the coordinates of B, B' to be (x_3, y_3) and (x_4, y_4) respectively, then

$$\frac{1}{x_3}+\frac{1}{x_4}=-2\,\frac{A+Bm'}{C}.$$

We have then

$$x(y_1-y_4)-y(x_1-x_4)+x_1y_4-x_4y_1=0,$$
$$x(y_2-y_3)-y(x_2-x_3)+x_2y_3-x_3y_2=0,$$

as the equations of the lines AB' and $A'B$ respectively; for the first of these equations, being satisfied if we write therein (x_1, y_1) or (x_4, y_4) for (x, y), is the equation of the line AB': and similarly the second equation is that of the line $A'B$. Reducing by means of the relations $y_1-mx_1=0$, $y_2-mx_2=0$, $y_3-m'x_3=0$, $y_4-m'x_4=0$, the two equations become

$$x(mx_1-m'x_4)-y(x_1-x_4)+(m'-m)\,x_1x_4=0,$$
$$x(mx_2-m'x_3)-y(x_2-x_3)+(m'-m)\,x_2x_3=0;$$

and if we divide the first of these equations by m_1m_4, and the second by m_2m_3, and then add, we obtain

$$x\left\{m\left(\frac{1}{x_3}+\frac{1}{x_4}\right)-m'\left(\frac{1}{x_1}+\frac{1}{x_2}\right)\right\}-y\left\{\frac{1}{x_3}+\frac{1}{x_4}-\left(\frac{1}{x_1}+\frac{1}{x_2}\right)\right\}+2m'-2m=0,$$

or, what is the same thing,

$$\left(\frac{1}{x_1}+\frac{1}{x_2}\right)(y-m'x)-\left(\frac{1}{x_3}+\frac{1}{x_4}\right)(y-mx)+2m'-2m=0,$$

which by what precedes is the equation of a line through the point Q. Substituting herein for $\frac{1}{x_1}+\frac{1}{x_2}$, $\frac{1}{x_3}+\frac{1}{x_4}$ their foregoing values, the equation becomes

$$-(A+Bm)(y-m'x)+(A+Bm')(y-mx)+m'-m=0;$$

that is,

$$(m-m')(Ax+By+C)=0;$$

or finally it is $Ax+By+C=0$, showing that the point Q lies in a line the position of which is independent of the particular lines OAA', OBB' used in the construction. It is proper to notice that there is no correspondence to each other of the points A, A' and B, B'; the grouping might as well have been A, A' and B', B; and it thence appears that the line $Ax+By+C=0$ just obtained is in fact the line joining the point Q with the point R which is the intersection of AB and $A'B'$.

10. The equation $Ax+By+C=0$ of a line contains in appearance 3, but really only 2 constants (for one of the constants can be divided out), and a line depends

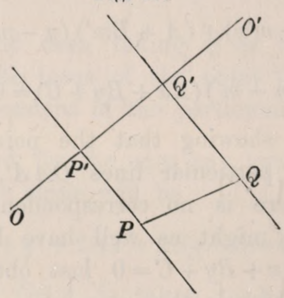
accordingly upon 2 parameters, or can be made to satisfy 2 conditions. Similarly, the equation $(a, b, c, f, g, h \between x, y, 1)^2 = 0$ of a conic contains really 5 constants, and the equation $(* \between x, y, 1)^3 = 0$ of a cubic contains really 9 constants. It thus appears that a cubic can be made to pass through 9 given points, and that the cubic so passing through 9 given points is completely determined. There is, however, a remarkable exception. Considering two given cubic curves $S = 0$, $S' = 0$, these intersect in 9 points, and through these 9 points we have the whole series of cubics $S - kS' = 0$, where k is an arbitrary constant: k may be determined so that the cubic shall pass through a given tenth point, viz. $k = S_0 \div S_0'$, if the coordinates are (x_0, y_0), and S_0, S_0' denote the corresponding values of S, S'. The resulting curve $SS_0' - S'S_0 = 0$ may be regarded as the cubic determined by the conditions of passing through 8 of the 9 points and through the given point (x_0, y_0); and from the equation it thence appears that the curve passes through the remaining one of the 9 points. In other words, we thus have the theorem, any cubic curve which passes through 8 of the 9 intersections of two given cubic curves passes through the 9th intersection.

The applications of this theorem are very numerous; for instance, we derive from it Pascal's theorem of the inscribed hexagon. Consider a hexagon inscribed in a conic. The three alternate sides constitute a cubic, and the other three alternate sides another cubic. The cubics intersect in 9 points, being the 6 vertices of the hexagon, and the 3 Pascalian points, or intersections of the pairs of opposite sides of the hexagon. Drawing a line through two of the Pascalian points, the conic and this line constitute a cubic passing through 8 of the 9 points of intersection, and it therefore passes through the remaining point of intersection—that is, the third Pascalian point; and since obviously this does not lie on the conic, it must lie on the line—that is, we have the theorem that the three Pascalian points (or points of intersection of the pairs of opposite sides) lie on a line.

Metrical Theory.

11. The foundation of the metrical theory consists in the simple theorem that if a finite line PQ (fig. 10) be projected upon any other line OO' by lines perpendicular

Fig. 10.

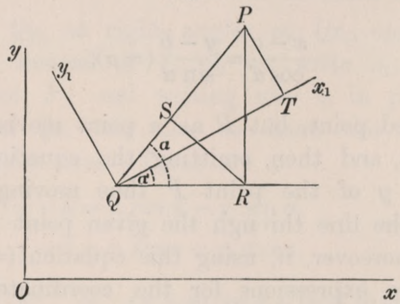


to OO', then the length of the projection $P'Q'$ is equal to the length of PQ multiplied by the cosine of its inclination to $P'Q'$; or, what is the same thing, that the perpendicular distance $P'Q'$ of any two parallel lines is equal to the inclined distance PQ

multiplied by the cosine of the inclination. It at once follows that the algebraical sum of the projections of the sides of a closed polygon upon any line is $=0$; or, reversing the signs of certain sides, and considering the polygon as consisting of two broken lines, each extending from the same initial to the same terminal point, the sum of the projections of the lines of the first set upon any line is equal to the sum of the projections of the lines of the second set. Observe that, if any line be perpendicular to the line on which the projection is made, then its projection is $=0$.

Thus, if we have a right-angled triangle PQR (fig. 11), where QR, RP, QP are

Fig. 11.



$=\xi$, η, ρ respectively, and whereof the base-angle is $=\alpha$, then projecting successively on the three sides, we have

$$\xi=\rho\cos\alpha,\quad \eta=\rho\sin\alpha,\quad \rho=\xi\cos\alpha+\eta\sin\alpha;$$

and we thence obtain

$$\rho^2=\xi^2+\eta^2;\ \cos^2\alpha+\sin^2\alpha=1.$$

And again, by projecting on a line Qx_1, inclined at the angle α' to QR, we have

$$\rho\cos(\alpha-\alpha')=\xi\cos\alpha'+\eta\sin\alpha';$$

and by substituting for ξ, η their foregoing values,

$$\cos(\alpha-\alpha')=\cos\alpha\cos\alpha'+\sin\alpha\sin\alpha'.$$

It is to be remarked that, assuming only the theory of similar triangles, we have herein a *proof* of Euclid, Book I., Prop. 47; in fact, the same as is given Book VI., Prop. 31; and also a *proof* of the trigonometrical formula for $\cos(\alpha-\alpha')$. The formulæ for $\cos(\alpha+\alpha')$ and $\sin(\alpha\pm\alpha')$ could be obtained in the same manner.

Draw PT at right angles to Qx_1, and suppose QT, $TP=\xi_1$, η_1 respectively, so that we have now the quadrilateral $QRPTQ$, or, what is the same thing, the two broken lines QRP and QTP, each extending from Q to P. Projecting on the four sides successively, we have

$$\begin{aligned}\xi &= \xi_1\cos\alpha'-\eta_1\sin\alpha',\\ \eta &= \xi_1\sin\alpha'+\eta_1\cos\alpha',\\ \xi_1 &= \xi\cos\alpha'+\eta\sin\alpha',\\ \eta_1 &= -\xi\sin\alpha'+\eta\cos\alpha',\end{aligned}$$

where the third equation is that previously written

$$\rho\cos(\alpha-\alpha')=\xi\cos\alpha+\eta\sin\alpha.$$

Equations of Right Line and Circle.—Transformation of Coordinates.

12. The required formulæ are really contained in the foregoing results. For, in fig. 11, supposing that the axis of x is parallel to QR, and taking a, b for the coordinates of Q, and (x, y) for those of P, then we have ξ, $\eta = x - a$, $y - b$ respectively; and therefore

$$x - a = \rho\cos\alpha, \quad y - b = \rho\sin\alpha,$$
$$\rho^2 \quad = (x-a)^2 + (y-b)^2.$$

Writing the first two of these in the form

$$\frac{x-a}{\cos\alpha} = \frac{y-b}{\sin\alpha} (= \rho),$$

we may regard Q as a fixed point, but P as a point moving in the direction Q to P, so that α remains constant, and then, omitting the equation $(= \rho)$, we have a relation between the coordinates x, y of the point P thus moving in a right line,—that is, we have the equation of the line through the given point (a, b) at a given inclination α to the axis of x. And, moreover, if, using this equation $(= \rho)$, we write $x = a + \rho\cos\alpha$, $y = b + \rho\sin\alpha$, then we have expressions for the coordinates x, y of a point of this line, in terms of the variable parameter ρ.

Again, take the point T to be fixed, but consider the point P as moving in the line TP at right angles to QT. If instead of ξ_1 we take p for the distance QT, then the equation $\xi_1 = \xi\cos\alpha' + \eta\sin\alpha'$ will be

$$(x-a)\cos\alpha' + (y-b)\sin\alpha' = p;$$

that is, this will be the equation of a line such that its perpendicular distance from the point (a, b) is $= p$, and that the inclination of this distance to the axis of x is $= \alpha'$.

From either form it appears that the equation of a line is, in fact, a linear equation of the form $Ax + By + C = 0$. It is important to notice that, starting from this equation, we can determine conversely the α but not the (a, b) of the form of equation which contains these quantities; and in like manner the α' but not the (a, b) or p of the other form of equation. The reason is obvious. In each case (a, b) denote the coordinates of a point, fixed indeed, but which is in the first form any point of the line, and in the second form any point whatever. Thus, in the second form the point from which the perpendicular is let fall may be the origin. Here $(a, b) = (0, 0)$, and the equation is $x\cos\alpha' + y\sin\alpha' - p = 0$. Comparing this with $Ax + By + C = 0$, we have the values of $\cos\alpha'$, $\sin\alpha'$, and p.

13. The equation

$$\rho^2 = (x-a)^2 + (y-b)^2$$

is an expression for the squared distance of the two points (a, b) and (x, y). Taking as before the point Q, coordinates (a, b), as a fixed point, and writing c in the place of ρ, the equation

$$(x-a)^2 + (y-b)^2 = c^2$$

expresses that the point (x, y) is always at a given distance c from the given point (a, b); viz. this is the equation of a circle, having (a, b) for the coordinates of its centre, and c for its radius.

The equation is of the form

$$x^2 + y^2 + 2Ax + 2By + C = 0,$$

and here, the number of constants being the same, we can identify the two equations; we find $a = -A$, $b = -B$, $c^2 = A^2 + B^2 - C$, or the last equation is that of a circle having $-A$, $-B$ for the coordinates of its centre, and $\sqrt{A^2 + B^2 - C}$ for its radius.

14. Drawing (fig. 11) Qy_1 at right angles at Qx_1, and taking Qx_1, Qy_1 as a new set of rectangular axes, if instead of ξ_1, η_1 we write x_1, y_1, we have x_1, y_1 as the new coordinates of the point P; and writing also α in place of α', α now denoting the inclination of the axes Qx_1 and Ox, we have the formulæ for transformation between two sets of rectangular axes. These are

$$x - a = x_1 \cos\alpha - y_1 \sin\alpha,$$
$$y - b = x_1 \sin\alpha + y_1 \cos\alpha,$$

and

$$x_1 = (x - a)\cos\alpha + (y - b)\sin\alpha,$$
$$y_1 = -(x - a)\sin\alpha + (y - b)\cos\alpha,$$

each set being obviously at once deducible from the other one. In these formulæ (a, b) are the xy-coordinates of the new origin Q_1, and α is the inclination of Qx_1 to Ox. It is to be noticed that Qx_1, Qy_1 are so placed that, by moving O to Q, and then turning the axes Ox_1, Oy_1 round Q (through an angle α measured in the sense Ox to Oy), the original axes Ox, Oy will come to coincide with Qx_1, Qy_1 respectively. This could not have been done if Qy_1 had been drawn (at right angles always to Qx_1) in the reverse direction: we should then have had in the formulæ $-y_1$ instead of y_1. The new formulæ which would be thus obtained are of an essentially distinct form: the analytical test is that in the formulæ as written down we can, by giving to α a proper value (in fact, $\alpha = 0$), make the $(x - a)$ and $(y - b)$ equal to x_1 and y_1 respectively; in the other system we could only make them equal to x_1, $-y_1$, or $-x_1$, y_1 respectively. But for the very reason that the second system can be so easily derived from the first, it is proper to attend exclusively to the first system,—that is, always to take the new axes so that the two sets admit of being brought into coincidence.

In the foregoing system of two pairs of equations, the first pair give the original coordinates x, y in terms of the new coordinates x_1, y_1; the second pair the new coordinates x_1, y_1 in terms of the original coordinates x, y. The formulæ involve (a, b), the original coordinates of the new origin; it would be easy, instead of these, to introduce (a_1, b_1), the new coordinates of the origin. Writing $(a, b) = (0, 0)$, we have, of course, the formulæ for transformation between two sets of rectangular axes *having the same origin*, and it is as well to write the formulæ in this more simple form; the subsequent transformation to a new origin, but with axes parallel to the original axes, can then be effected without any difficulty.

15. All questions in regard to the line may be solved by means of one or other of the foregoing forms—

$$Ax + By + C = 0,$$

$$y = Ax + B,$$

$$\frac{x-a}{\cos\alpha} = \frac{y-b}{\sin\alpha},$$

$$(x-a)\cos\alpha' + (y-b)\sin\alpha' - p = 0;$$

or it may be by a comparison of these different forms: thus, using the first form, it has been already shown that the equation of the line through two given points (x_1, y_1), (x_2, y_2) is

$$x(y_1 - y_2) - y(x_1 - x_2) + x_1y_2 - x_2y_1 = 0,$$

or, as this may be written,

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1).$$

A particular case is the equation

$$\frac{x}{a} + \frac{y}{b} = 1,$$

representing the line through the points $(a, 0)$ and $(0, b)$, or, what is the same thing, the line meeting the axes of x and y at the distances from the origin a and b respectively. It may be noticed that, in the form $Ax + By + C = 0$, $-\frac{A}{B}$ denotes the tangent of the inclination to the axis of x, or we may say that $B \div \sqrt{A^2 + B^2}$ and $-A \div \sqrt{A^2 + B^2}$ denote respectively the cosine and the sine of the inclination to the axis of x. A better form is this: $A \div \sqrt{A^2 + B^2}$ and $B \div \sqrt{A^2 + B^2}$ denote respectively the cosine and the sine of inclination to the axis of x of the perpendicular upon the line. So, of course, in regard to the form $y = Ax + B$, A is here the tangent of the inclination to the axis of x; $1 \div \sqrt{A^2 + 1}$ and $A \div \sqrt{A^2 + 1}$ are the cosine and sine of this inclination, &c. It thus appears that the condition, in order that the lines $Ax + By + C = 0$ and $A'x + B'y - C' = 0$ may meet at right angles, is $AA' + BB' = 0$; so when the equations are $y = Ax + B$, $y = A'x + B'$, the condition is $AA' + 1 = 0$, or say the value of A' is $= -1 \div A$.

The perpendicular distance of the point (a, b) from the line $Ax + By + C = 0$ is $(Aa + Bb + C) \div \sqrt{A^2 + B^2}$. In all the formulæ involving $\sqrt{A^2 + B^2}$ or $\sqrt{A^2 + 1}$, the radical should be written with the sign $\pm$, which is essentially indeterminate: the like indeterminateness of sign presents itself in the expression for the distance of two points $\rho = \pm\sqrt{(x-a)^2 + (y-b)^2}$; if, as before, the points are Q, P, and the indefinite line through these is $z'QPz$, then it is the same thing whether we measure off from Q along this line, considered as drawn from z' towards z, a positive distance k, or along the line considered as drawn reversely from z towards z', the equal negative distance $-k$, and the expression for the distance ρ is thus properly of the form $\pm k$. It is interesting to compare expressions which do not involve a radical: thus, in seeking for the expression for the perpendicular distance of the point (a, b) from a given line, let the equation of the given line be taken in the form, $x\cos\alpha + y\sin\alpha - p = 0$, p being the perpendicular distance from the origin, α its inclination to the axis of x: the equation of the line may also be written $(x-a)\cos\alpha + (y-b)\sin\alpha - p_1 = 0$, and we have thence $p_1 = p - a\cos\alpha - b\sin\alpha$, the required expression for the distance p_1: it is here assumed that p_1 is drawn from (a, b) in the same sense as p is drawn from the origin, and the indeterminateness of sign is thus removed.

16. As an instance of the mode of using the formulæ, take the problem of finding the locus of a point such that its distance from a given point is in a given ratio to its distance from a given line.

We take (a, b) as the coordinates of the given point, and it is convenient to take (x, y) as the coordinates of the variable point, the locus of which is required: it thus becomes necessary to use other letters, say (X, Y), for current coordinates in the equation of the given line. Suppose this is a line such that its perpendicular distance from the origin is $=p$, and that the inclination of p to the axis of x is $=\alpha$; the equation is $X\cos\alpha + Y\sin\alpha - p = 0$. In the result obtained in § 15, writing (x, y) in place of (a, b), it appears that the perpendicular distance of this line from the point (x, y) is

$$= p - x\cos\alpha - y\sin\alpha;$$

hence the equation of the locus is

$$\sqrt{(x-a)^2 + (y-b)^2} = e\,(p - x\cos\alpha - y\sin\alpha),$$

or say

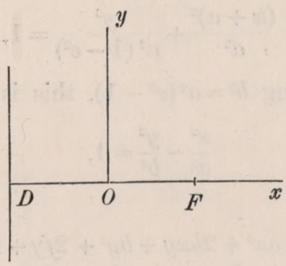
$$(x-a)^2 + (y-b)^2 - e^2(x\cos\alpha + y\sin\alpha - p)^2 = 0,$$

an equation of the second order.

The Conics (Parabola, Ellipse, Hyperbola).

17. The conics or, as they were called, conic sections were originally defined as the sections of a right circular cone; but Apollonius substituted a definition, which is, in fact, that of the last example: the curve is the locus of a point such that its

Fig. 12.



distance from a given point (called the focus) is in a given ratio to its distance from a given line (called the directrix); and taking the ratio as $e : 1$, then e is called the eccentricity.

Take FD for the perpendicular from the focus F upon the directrix, and the given ratio being that of $e : 1$ ($e >, =,$ or < 1, but positive), and let the distance FD

be divided at O in the given ratio, say we have $OD = m$, $OF = em$, where m is positive;—then the origin may be taken at O, the axis Ox being in the direction OF (that is, from O to F), and the axis Oy at right angles to it. The distance of the point (x, y) from F is $= \sqrt{(x - em)^2 + y^2}$, its distance from the directrix is $= x + m$; the equation therefore is

$$(x - em)^2 + y^2 = e^2 (x + m)^2;$$

or, what is the same thing, it is

$$(1 - e^2) x^2 - 2me (1 + e) x + y^2 = 0.$$

If $e^2 = 1$, or, since e is taken to be positive, if $e = 1$, this is

$$y^2 - 4mx = 0,$$

which is the parabola.

If e^2 not $= 1$, then the equation may be written

$$(1 - e^2)\left(x - \frac{me}{1 - e}\right)^2 + y^2 = \frac{m^2 e^2 (1 + e)}{1 - e}.$$

Supposing e positive and < 1, then, writing $m = \dfrac{a(1 - e)}{e}$, the equation becomes

$$(1 - e^2)(x - a)^2 + y^2 = a^2 (1 - e^2),$$

that is,

$$\frac{(x - a)^2}{a^2} + \frac{y^2}{a^2 (1 - e^2)} = 1;$$

or, changing the origin and writing $b^2 = a^2 (1 - e^2)$, this is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

which is the ellipse.

And similarly if e be positive and > 1, then writing $m = \dfrac{a(e - 1)}{e}$, the equation becomes

$$(1 - e^2)(x + a)^2 + y^2 = a^2 (1 - e^2),$$

that is,

$$\frac{(x + a)^2}{a^2} + \frac{y^2}{a^2 (1 - e^2)} = 1,$$

or changing the origin and writing $b^2 = a^2 (e^2 - 1)$, this is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1,$$

which is the hyperbola.

18. The general equation $ax^2 + 2hxy + by^2 + 2fy + 2gx + c = 0$, or as it is written $(a, b, c, f, g, h)(x, y, 1)^2 = 0$, may be such that the quadric function breaks up into factors, $= (\alpha x + \beta y + \gamma)(\alpha' x + \beta' y + \gamma')$; and in this case the equation represents a pair of lines, or (it may be) two coincident lines. When it does not so break up, the function can be put in the form $\lambda \{(x - a')^2 + (y - b')^2 - e^2 (x \cos \alpha + y \sin \alpha - p)^2\}$, or, equating the two expressions, there will be six equations for the determination of λ, a', b', e, p, α; and by what precedes, if a', b', e, p, α are real, the curve is either

a parabola, ellipse, or hyperbola. The original coefficients (a, b, c, f, g, h) may be such as not to give any system of real values for a', b', e, p, α; but when this is so the equation $(a, b, c, f, g, h)(x, y, 1)^2 = 0$ does not represent a real curve*; the imaginary curve which it represents is, however, regarded as a conic. Disregarding the special cases of the pair of lines and the twice repeated line, it thus appears that the only real curves represented by the general equation $(a, b, c, f, g, h)(x, y, 1)^2 = 0$ are the parabola, the ellipse, and the hyperbola. The circle is considered as a particular case of the ellipse.

The same result is obtained by transforming the equation $(a, b, c, f, g, h)(x, y, 1)^2 = 0$ to new axes. If in the first place the origin be unaltered, then the directions of the new (rectangular) axes Ox_1, Oy_1 can be found so that h_1 (the coefficient of the term x_1y_1) shall be $=0$; when this is done, then either one of the coefficients of x_1^2, y_1^2 is $=0$, and the curve is then a parabola, or neither of these coefficients is $=0$, and the curve is then an ellipse or hyperbola, according as the two coefficients are of the same sign or of opposite signs.

19. The curves can be at once traced from their equations:—

$$y^2 = 4mx, \text{ for the parabola (fig. 13)},$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \text{ for the ellipse (fig. 14)},$$

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1, \text{ for the hyperbola (fig. 15)};$$

Fig. 13.

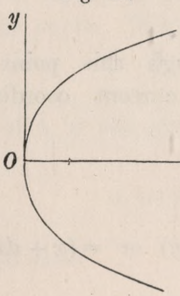


Fig. 14.

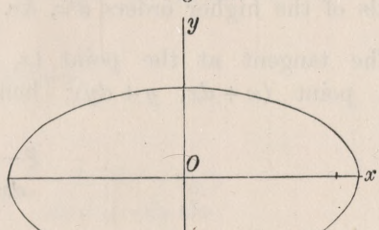
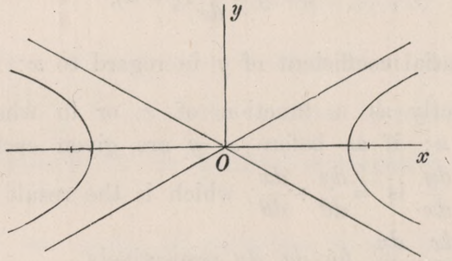


Fig. 15.



* It is proper to remark that, when $(a, b, c, f, g, h)(x, y, 1)^2 = 0$ *does* represent a real curve, there are, in fact, four systems of values of a', b', e, p, α, two real, the other two imaginary; we have thus two real equations and two imaginary equations, each of them of the form $(x-a')^2 + (y-b')^2 = e^2(x\cos\alpha + y\cos\beta - p)^2$, representing each of them one and the same real curve. This is consistent with the assertion of the text that the real curve is in every case represented by a real equation of this form.

and it will be noticed how the form of the last equation puts in evidence the two asymptotes $\frac{x}{a} = \pm \frac{y}{b}$ of the hyperbola. Referred to the asymptotes (as a set of oblique axes) the equation of the hyperbola takes the form $xy = c$; and in particular, if in this equation the axes are at right angles, then the equation represents the rectangular hyperbola referred to its asymptotes as axes.

Tangent, Normal, Circle and Radius of Curvature, &c.

20. There is great convenience in using the language and notation of the infinitesimal analysis; thus we consider on a curve a point with coordinates (x, y), and a consecutive point the coordinates of which are $(x + dx, y + dy)$, or again a second consecutive point with coordinates $(x + dx + \frac{1}{2}d^2x, y + dy + \frac{1}{2}d^2y)$, &c.; and in the final results the ratios of the infinitesimals must be replaced by differential coefficients in the proper manner; thus, if x, y are considered as given functions of a parameter θ, then dx, dy have in fact the values $\frac{dx}{d\theta}d\theta$, $\frac{dy}{d\theta}d\theta$, and (only the ratio being really material) they may in the result be replaced by $\frac{dx}{d\theta}$, $\frac{dy}{d\theta}$. This includes the case where the equation of the curve is given in the form $y = \phi(x)$; θ is here $= x$, and the increments dx, dy are in the result to be replaced by $1, \frac{dy}{dx}$. So also with the infinitesimals of the higher orders d^2x, &c.

21. The tangent at the point (x, y) is the line through this point and the consecutive point $(x + dx, y + dy)$; hence, taking ξ, η as current coordinates, the equation is

$$\frac{\xi - x}{dx} = \frac{\eta - y}{dy},$$

an equation which is satisfied on writing therein $\xi, \eta = (x, y)$ or $= (x + dx, y + dy)$. The equation may be written

$$\eta - y = \frac{dy}{dx}(\xi - x),$$

$\frac{dy}{dx}$ being now the differential coefficient of y in regard to x; and this form is applicable whether y is given directly as a function of x, or in whatever way y is in effect given as a function of x: if as before x, y are given each of them as a function of θ, then the value of $\frac{dy}{dx}$ is $= \frac{dy}{d\theta} \div \frac{dx}{d\theta}$, which is the result obtained from the original form on writing therein $\frac{dx}{d\theta}$, $\frac{dy}{d\theta}$, for dx, dy respectively.

So again, when the curve is given by an equation $u = 0$ between the coordinates (x, y), then $\frac{dy}{dx}$ is obtained from the equation $\frac{du}{dx} + \frac{du}{dy}\frac{dy}{dx} = 0$. But here it is more elegant, using the original form, to eliminate dx, dy by the formula $\frac{du}{dx}dx + \frac{du}{dy}dy$; we thus obtain the equation of the tangent in the form

$$\frac{du}{dx}(\xi - x) + \frac{du}{dy}(\eta - y) = 0.$$

For example, in the case of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, the equation is

$$\frac{x}{a^2}(\xi - x) + \frac{y}{b^2}(\eta - y) = 0;$$

or reducing by means of the equation of the curve the equation of the tangent is

$$\frac{\xi x}{a^2} + \frac{\eta y}{b^2} = 1.$$

The normal is a line through the point at right angles to the tangent; the equation therefore is

$$(\xi - x)\,dx + (\eta - y)\,dy = 0,$$

where dx, dy are to be replaced by their proportional values as before.

22. The circle of curvature is the circle through the point and two consecutive points of the curve. Taking the equation to be

$$(\xi - \alpha)^2 + (\eta - \beta)^2 = \gamma^2,$$

the values of α, β are given by

$$x - \alpha = \frac{dy\,(dx^2 + dy^2)}{dx\,d^2y - dy\,d^2x}, \quad y - \beta = \frac{-\,dx\,(dx^2 + dy^2)}{dx\,d^2y - dy\,d^2x},$$

and we then have

$$\gamma^2, = (x - \alpha)^2 + (y - \beta)^2, = \frac{(dx^2 + dy^2)^3}{(dx\,d^2y - dy\,d^2x)^2}.$$

In the case where y is given directly as a function of x, then, writing for shortness $p = \frac{dy}{dx}$, $q = \frac{d^2y}{dx^2}$, this is $\gamma^2 = \frac{(1 + p^2)^3}{q^2}$, or, as the equation is usually written, $\gamma = \frac{(1 + p^2)^{\frac{3}{2}}}{-q}$, the radius of curvature, considered to be positive or negative according as the curve is concave or convex to the axis of x.

It may be added that the centre of curvature is the intersection of the normal by the consecutive normal.

The locus of the centre of curvature is the evolute. If from the expressions of α, β regarded as functions of x we eliminate x, we have thus an equation between (α, β), which is the equation of the evolute.

Polar Coordinates.

23. The position of a point may be determined by means of its distance from a fixed point and the inclination of this distance to a fixed line through the fixed point. Say we have r the distance from the origin, and θ the inclination of r to the axis of x; r and θ are then the polar coordinates of the point, r the radius vector, and θ the inclination. These are immediately connected with the Cartesian coordinates x, y by the formulæ $x = r\cos\theta$, $y = r\sin\theta$; and the transition from either set of coordinates to the other can thus be made without difficulty. But the use of polar coordinates is very convenient, as well in reference to certain classes of questions relating to curves of any kind—for instance, in the dynamics of central forces—as in relation to curves having in regard to the origin the symmetry of the regular polygon (curves such as that represented by the equation $r = \cos m\theta$), and also in regard to the class of curves called spirals, where the radius vector r is given as an algebraical or exponential function of the inclination θ.

Trilinear Coordinates.

24. Consider a fixed triangle ABC, and (regarding the sides as indefinite lines) suppose for a moment that p, q, r denote the distances of a point P from the sides BC, CA, AB respectively,—these distances being measured either perpendicularly to the several sides, or each of them in a given direction. To fix the ideas each distance may be considered as positive for a point inside the triangle, and the sign is thus fixed for any point whatever. There is then an identical relation between p, q, r: if a, b, c are the lengths of the sides, and the distances are measured perpendicularly thereto, the relation is $ap + bq + cr =$ twice the area of triangle. But taking x, y, z proportional to p, q, r, or if we please proportional to given multiples of p, q, r, then only the ratios of x, y, z are determined; their absolute values remain arbitrary. But the ratios of p, q, r, and consequently also the ratios of x, y, z determine, and that uniquely, the point; and it being understood that only the ratios are attended to, we say that (x, y, z) are the coordinates of the point. The equation of a line has thus the form $ax + by + cz = 0$, and generally that of a curve of the nth order is a homogeneous equation of this order between the coordinates, $(*\between x, y, z)^n = 0$. The advantage over Cartesian coordinates is in the greater symmetry of the analytical forms, and in the more convenient treatment of the line infinity and of points at infinity. The method includes that of Cartesian coordinates, the homogeneous equation in x, y, z is, in fact, an equation in $\frac{x}{z}$, $\frac{y}{z}$, which two quantities may be regarded as denoting Cartesian coordinates; or, what is the same thing, we may in the equation write $z = 1$. It may be added that, if the trilinear coordinates (x, y, z) are regarded as the Cartesian coordinates of a point of space, then the equation is that of a cone having the origin for its vertex; and conversely that such equation of a cone may be regarded as the equation in trilinear coordinates of a plane curve.

General Point-Coordinates.—Line-Coordinates.

25. All the coordinates considered thus far are point-coordinates. More generally, any two quantities (or the ratios of three quantities) serving to determine the position

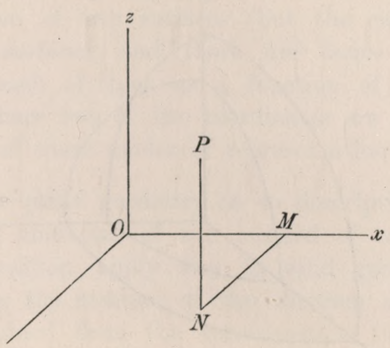
of a point in the plane may be regarded as the coordinates of the point; or, if instead of a single point they determine a system of two or more points, then as the coordinates of the system of points. But, as noticed under Curve, [785], there are also line-coordinates serving to determine the position of a line; the ordinary case is when the line is determined by means of the ratios of three quantities ξ, η, ζ (correlative to the trilinear coordinates x, y, z). A linear equation $a\xi + b\eta + c\zeta = 0$ represents then the system of lines such that the coordinates of each of them satisfy this relation, in fact, all the lines which pass through a given point; and it is thus regarded as the line-equation of this point; and generally a homogeneous equation $(*\between\xi, \eta, \zeta)^n = 0$ represents the curve which is the envelope of all the lines the coordinates of which satisfy this equation, and it is thus regarded as the line-equation of this curve.

II. Solid Analytical Geometry (§§ 26—40).

26. We are here concerned with points in space,—the position of a point being determined by its three coordinates x, y, z. We consider three coordinate planes, at right angles to each other, dividing the whole of space into eight portions called octants, the coordinates of a point being the perpendicular distances of the point from the three planes respectively, each distance being considered as positive or negative according as it lies on the one or the other side of the plane. Thus the coordinates in the eight octants have respectively the signs

x,	y,	z
+	+	+
+	−	+
−	+	+
−	−	+
+	+	−
+	−	−
−	+	−
−	−	−

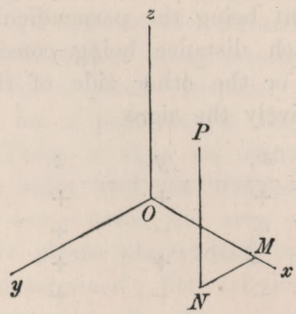
Fig. 16.



The positive parts of the axes are usually drawn as in fig. 16, which represents a point P, the coordinates of which have the positive values OM, MN, NP.

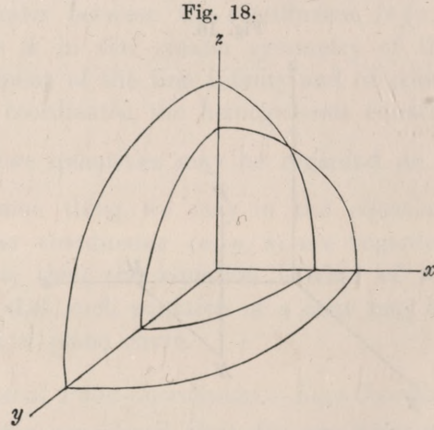
27. It may be remarked, as regards the delineation of such solid figures, that if we have in space three lines at right angles to each other, say Oa, Ob, Oc, of equal lengths, then it is possible to project these by parallel lines upon a plane in such wise that the projections Oa', Ob', Oc' shall be at given inclinations to each other, and that these lengths shall be to each other in given ratios: in particular, the two lines Oa', Oc' may be at right angles to each other, and their lengths equal, the direction of Ob', and its proportion to the two equal lengths Oa', Oc', being arbitrary. It thus appears that we may as in the figure draw Ox, Oz at right angles to each other, and Oy in an arbitrary direction; and moreover represent the coordinates x, z on equal scales, and the remaining coordinate y on an arbitrary scale (which may be that of the other two coordinates x, z, but is in practice usually smaller). The advantage, of course, is that a figure in *one* of the coordinate planes xz is represented in its proper form without distortion; but it may be in some cases preferable to employ the isometrical projection, wherein the three axes are represented by lines inclined to each other at angles of 120°, and the scales for the coordinates are equal (fig. 17).

Fig. 17.



For the delineation of a surface of a tolerably simple form, it is frequently sufficient to draw (according to the foregoing projection) the sections by the coordinate planes; and in particular, when the surface is symmetrical in regard to the

Fig. 18.



coordinate planes, it is sufficient to draw the quarter-sections belonging to a single octant of the surface; thus fig. 18 is a convenient representation of an octant of the wave surface. Or a surface may be delineated by means of a series of parallel sections, or (taking these to be the sections by a series of horizontal planes) say by a series of contour lines. Of course, other sections may be drawn or indicated, if necessary. For the delineation of a curve, a convenient method is to represent, as above, a series of the points P thereof, each point P being accompanied by the ordinate PN, which serves to refer the point to the plane of xy; this is in effect a representation of each point P of the curve, by means of two points P, N such that the line PN has a fixed direction. Both as regards curves and surfaces, the employment of stereographic representations is very interesting.

28. In plane geometry, reckoning the line as a curve of the first order, we have only the point and the curve. In solid geometry, reckoning a line as a curve of the first order, and the plane as a surface of the first order, we have the point, the curve, and the surface; but the increase of complexity is far greater than would hence at first sight appear. In plane geometry a curve is considered in connexion with lines (its tangents); but in solid geometry the curve is considered in connexion with lines and planes (its tangents and osculating planes), and the surface also in connexion with lines and planes (its tangent lines and tangent planes); there are surfaces arising out of the line—cones, skew surfaces, developables, doubly and triply infinite systems of lines, and whole classes of theories which have nothing analogous to them in plane geometry: it is thus a very small part indeed of the subject which can be even referred to in the present article.

In the case of a surface, we have between the coordinates (x, y, z) a single, or say a onefold relation, which can be represented by a single relation $f(x, y, z) = 0$; or we may consider the coordinates expressed each of them as a given function of two variable parameters p, q; the form $z = f(x, y)$ is a particular case of each of these modes of representation; in other words, we have in the first mode $f(x, y, z) = z - f(x, y)$, and in the second mode $x = p$, $y = q$ for the expression of two of the coordinates in terms of the parameters.

In the case of a curve, we have between the coordinates (x, y, z) a twofold relation: two equations $f(x, y, z) = 0$, $\phi(x, y, z) = 0$ give such a relation; i.e., the curve is here considered as the intersection of two surfaces (but the curve is not always the complete intersection of two surfaces, and there are hence difficulties); or, again, the coordinates may be given each of them as a function of a single variable parameter. The form $y = \phi x$, $z = \psi x$, where two of the coordinates are given in terms of the third, is a particular case of each of these modes of representation.

29. The remarks under plane geometry as to descriptive and metrical propositions, and as to the non-metrical character of the method of coordinates when used for the proof of a descriptive proposition, apply also to solid geometry; and they might be illustrated in like manner by the instance of the theorem of the radical centre of four spheres. The proof is obtained from the consideration that S and S' being each of them a function of the form $x^2 + y^2 + z^2 + ax + by + cz + d$, the difference $S - S'$ is a mere linear function of the coordinates, and consequently that $S - S' = 0$ is the equation of the plane containing the circle of intersection of the two spheres $S = 0$ and $S' = 0$.

Metrical Theory.

30. The foundation in solid geometry of the metrical theory is, in fact, the before-mentioned theorem that, if a finite right line PQ be projected upon any other line OO' by lines perpendicular to OO', then the length of the projection $P'Q'$ is equal to the length of PQ multiplied by the cosine of its inclination to $P'Q'$—or (in the form in which it is now convenient to state the theorem) the perpendicular distance $P'Q'$ of two parallel planes is equal to the inclined distance PQ into the cosine of the inclination. Hence also the algebraical sum of the projections of the sides of a closed polygon upon any line is $=0$; or, reversing the signs of certain sides and considering the polygon as made up of two broken lines each extending from the same initial to the same terminal point, the sum of the projections of the one set of lines upon any line is equal to the sum of the projections of the other set of lines upon the same line. When any of the lines are at right angles to the given line (or, what is the same thing, in a plane at right angles to the given line), the projections of these lines severally vanish.

31. Consider the skew quadrilateral $QMNP$, the sides QM, MN, NP being respectively parallel to the three rectangular axes Ox, Oy, Oz; let the lengths of these sides be ξ, η, ζ, and that of the side QP be $=\rho$; and let the cosines of the inclinations (or say the cosine-inclinations) of ρ to the three axes be α, β, γ; then projecting successively on the three sides and on QP, we have

$$\xi,\ \eta,\ \zeta = \rho\alpha,\ \rho\beta,\ \rho\gamma,$$

and

$$\rho = \alpha\xi + \beta\eta + \gamma\zeta,$$

whence $\rho^2 = \xi^2 + \eta^2 + \zeta^2$, which is the relation between a distance ρ and its projections ξ, η, ζ upon three rectangular axes. And from the same equations we obtain $\alpha^2 + \beta^2 + \gamma^2 = 1$, which is a relation connecting the cosine-inclinations of a line to three rectangular axes.

Suppose we have through Q any other line QT, and let the cosine-inclinations of this to the axes be α', β', γ', and δ be its cosine-inclination to QP; also let p be the length of the projection of QP upon QT; then projecting on QT, we have

$$p = \alpha'\xi + \beta'\eta + \gamma'\zeta,\ = \rho\delta.$$

And in the last equation substituting for ξ, η, ζ their values $\rho\alpha$, $\rho\beta$, $\rho\gamma$, we find

$$\delta = \alpha\alpha' + \beta\beta' + \gamma\gamma',$$

which is an expression for the mutual cosine-inclination of two lines, the cosine-inclinations of which to the axes are α, β, γ and α', β', γ' respectively. We have of course $\alpha^2 + \beta^2 + \gamma^2 = 1$, and $\alpha'^2 + \beta'^2 + \gamma'^2 = 1$, and hence also

$$\begin{aligned} 1 - \delta^2 &= (\alpha^2 + \beta^2 + \gamma^2)(\alpha'^2 + \beta'^2 + \gamma'^2) - (\alpha\alpha' + \beta\beta' + \gamma\gamma')^2 \\ &= (\beta\gamma' - \beta'\gamma)^2 + (\gamma\alpha' - \gamma'\alpha)^2 + (\alpha\beta' - \alpha'\beta)^2; \end{aligned}$$

so that the sine of the inclination can only be expressed as a square root. These formulæ are the foundation of spherical trigonometry.

The Line, Plane, and Sphere.

32. The foregoing formulæ give at once the equations of these loci.

For first, taking Q to be a fixed point, coordinates (a, b, c) and the cosine-inclinations (α, β, γ) to be constant, then P will be a point in the line through Q in the direction thus determined; or, taking (x, y, z) for its coordinates, these will be the current coordinates of a point in the line. The values of ξ, η, ζ then are $x-a$, $y-b$, $z-c$, and we thus have

$$\frac{x-a}{\alpha}=\frac{y-b}{\beta}=\frac{z-c}{\gamma}\,(=\rho),$$

which (omitting the last equation, $=\rho$) are the equations of the line through the point (a, b, c), the cosine-inclinations to the axes being α, β, γ, and these quantities being connected by the relation $\alpha^2+\beta^2+\gamma^2=1$. This equation may be omitted, and then α, β, γ, instead of being equal, will only be proportional to the cosine-inclinations.

Using the last equation, and writing

$$x,\ y,\ z=a+\alpha\rho,\ \ b+\beta\rho,\ \ c+\gamma\rho,$$

these are expressions for the current coordinates in terms of a parameter ρ, which is in fact the distance from the fixed point (a, b, c).

It is easy to see that, if the coordinates (x, y, z) are connected by any two linear equations, these equations can always be brought into the foregoing form, and hence that the two linear equations represent a line.

Secondly, taking for greater simplicity the point Q to be coincident with the origin, and $\alpha', \beta', \gamma', p$ to be constant, then p is the perpendicular distance of a plane from the origin, and α', β', γ' are the cosine-inclinations of this distance to the axes $(\alpha'^2+\beta'^2+\gamma'^2=1)$. Now P is any point in this plane; taking its coordinates to be (x, y, z), then (ξ, η, ζ) are $=(x, y, z)$, and the foregoing equation $p=\alpha'\xi+\beta'\eta+\gamma'\zeta$ becomes

$$\alpha'x+\beta'y+\gamma'z=p,$$

which is the equation of the plane in question.

If, more generally, Q is not coincident with the origin, then, taking its coordinates to be (a, b, c), and writing p_1 instead of p, the equation is

$$\alpha'(x-a)+\beta'(y-b)+\gamma'(z-c)=p_1;$$

and we thence have $p_1=p-(a\alpha'+b\beta'+c\gamma')$, which is an expression for the perpendicular distance of the point (a, b, c) from the plane in question.

It is obvious that any linear equation $Ax+By+Cz+D=0$ between the coordinates can always be brought into the foregoing form, and hence that such equation represents a plane.

Thirdly, supposing Q to be a fixed point, coordinates (a, b, c) and the distance QP, $=\rho$, to be constant, say this is $=d$, then, as before, the values of ξ, η, ζ are $x-a$, $y-b$, $z-c$, and the equation $\xi^2+\eta^2+\zeta^2=\rho^2$ becomes

$$(x-a)^2+(y-b)^2+(z-c)^2=d^2,$$

which is the equation of the sphere, coordinates of the centre $=(a, b, c)$ and radius $=d$.

A quadric equation wherein the terms of the second order are $x^2+y^2+z^2$, viz. an equation

$$x^2+y^2+z^2+Ax+By+Cz+D=0,$$

can always, it is clear, be brought into the foregoing form; and it thus appears that this is the equation of a sphere, coordinates of the centre $-\tfrac{1}{2}A$, $-\tfrac{1}{2}B$, $-\tfrac{1}{2}C$, and squared radius $=\tfrac{1}{4}(A^2+B^2+C^2)-D$.

Cylinders, Cones, Ruled Surfaces.

33. A singly infinite system of lines, or a system of lines depending upon one variable parameter, forms a surface; and the equation of the surface is obtained by eliminating the parameter between the two equations of the line.

If the lines all pass through a given point, then the surface is a cone; and, in particular, if the lines are all parallel to a given line, then the surface is a cylinder.

Beginning with this last case, suppose the lines are parallel to the line $x=mz$, $y=nz$, the equations of a line of the system are $x=mz+a$, $y=nz+b$,—where a, b are supposed to be functions of the variable parameter, or, what is the same thing, there is between them a relation $f(a, b)=0$: we have $a=x-mz$, $b=y-nz$, and the result of the elimination of the parameter therefore is $f(x-mz, y-nz)=0$, which is thus the general equation of the cylinder the generating lines whereof are parallel to the line $x=mz$, $y=nz$. The equation of the section by the plane $z=0$ is $f(x, y)=0$, and conversely if the cylinder be determined by means of its curve of intersection with the plane $z=0$, then, taking the equation of this curve to be $f(x, y)=0$, the equation of the cylinder is $f(x-mz, y-nz)=0$. Thus, if the curve of intersection be the circle $(x-\alpha)^2+(y-\beta)^2=\gamma^2$, we have $(x-mz-\alpha)^2+(y-nz-\beta)^2=\gamma^2$ as the equation of an oblique cylinder on this base, and thus also $(x-\alpha)^2+(y-\beta)^2=\gamma^2$ as the equation of the right cylinder.

If the lines all pass through a given point (a, b, c), then the equations of a line are $x-a=\alpha(z-c)$, $y-b=\beta(z-c)$, where α, β are functions of the variable parameter, or, what is the same thing, there exists between them an equation $f(\alpha, \beta)=0$; the elimination of the parameter gives, therefore, $f\left(\dfrac{x-a}{z-c}, \dfrac{y-b}{z-c}\right)=0$; and this equation, or, what is the same thing, any homogeneous equation $f(x-a, y-b, z-c)=0$, or, taking f to be a rational and integral function of the order n, say $(*)(x-a, y-b, z-c)^n=0$, is the general equation of the cone having the point (a, b, c) for its vertex. Taking the vertex to be at the origin, the equation is $(*)(x, y, z)^n=0$; and, in particular, $(*)(x, y, z)^2=0$ is the equation of a cone of the second order, or quadricone, having the origin for its vertex.

34. In the general case of a singly infinite system of lines, the locus is a ruled surface (or *regulus*). If the system be such that a line does not intersect the consecutive line, then the surface is a skew surface, or scroll; but if it be such that each line intersects the consecutive line, then it is a developable, or torse.

Suppose, for instance, that the equations of a line (depending on the variable parameter θ) are

$$\frac{x}{a}+\frac{z}{c}=\theta\left(1+\frac{y}{b}\right),\ \frac{x}{a}-\frac{z}{c}=\frac{1}{\theta}\left(1-\frac{y}{b}\right),$$

then, eliminating θ, we have $\frac{x^2}{a^2}-\frac{z^2}{c^2}=1-\frac{y^2}{b^2}$, or say $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$, the equation of a quadric surface, afterwards called the hyperboloid of one sheet; this surface is consequently a scroll. It is to be remarked that we have upon the surface a second singly infinite series of lines; the equations of a line of this second system (depending on the variable parameter ϕ) are

$$\frac{x}{a}+\frac{z}{c}=\phi\left(1-\frac{y}{b}\right),\ \frac{x}{a}-\frac{z}{c}=\frac{1}{\phi}\left(1+\frac{y}{b}\right).$$

It is easily shown that any line of the one system intersects every line of the other system.

Considering any curve (of double curvature) whatever, the tangent lines of the curve form a singly infinite system of lines, each line intersecting the consecutive line of the system,—that is, they form a developable, or torse; the curve and torse are thus inseparably connected together, forming a single geometrical figure. A plane through three consecutive points of the curve (or osculating plane of the curve) contains two consecutive tangents, that is, two consecutive lines of the torse, and is thus a tangent plane of the torse along a generating line.

Transformation of Coordinates.

35. There is no difficulty in changing the origin, and it is for brevity assumed that the origin remains unaltered. We have, then, two sets of rectangular axes, Ox, Oy, Oz, and Ox_1, Oy_1, Oz_1, the mutual cosine-inclinations being shown by the diagram—

	x	y	z
x_1	α	β	γ
y_1	α'	β'	γ'
z_1	α''	β''	γ''

that is, α, β, γ are the cosine-inclinations of Ox_1 to Ox, Oy, Oz; α', β', γ' those of Oy_1, &c.

And this diagram gives also the linear expressions of the coordinates (x_1, y_1, z_1) or (x, y, z) of either set in terms of those of the other set; we thus have

$$\begin{aligned} x_1 &= \alpha\, x + \beta\, y + \gamma\, z, & x &= \alpha x_1 + \alpha' y_1 + \alpha'' z_1, \\ y_1 &= \alpha' x + \beta' y + \gamma' z, & y &= \beta x_1 + \beta' y_1 + \beta'' z_1, \\ z_1 &= \alpha'' x + \beta'' y + \gamma'' z, & z &= \gamma x_1 + \gamma' y_1 + \gamma'' z_1, \end{aligned}$$

which are obtained by projection, as above explained. Each of these equations is, in fact, nothing else than the before-mentioned equation $p = \alpha'\xi + \beta'\eta + \gamma'\zeta$, adapted to the problem in hand.

But we have to consider the relations between the nine coefficients. By what precedes, or by the consideration that we must have identically $x^2 + y^2 + z^2 = x_1{}^2 + y_1{}^2 + z_1{}^2$, it appears that these satisfy the relations—

$$\begin{aligned} \alpha^2 + \beta^2 + \gamma^2 &= 1, & \alpha^2 + \alpha'^2 + \alpha''^2 &= 1, \\ \alpha'^2 + \beta'^2 + \gamma'^2 &= 1, & \beta^2 + \beta'^2 + \beta''^2 &= 1, \\ \alpha''^2 + \beta''^2 + \gamma''^2 &= 1, & \gamma^2 + \gamma'^2 + \gamma''^2 &= 1, \\ \alpha'\alpha'' + \beta'\beta'' + \gamma'\gamma'' &= 0, & \beta\gamma + \beta'\gamma' + \beta''\gamma'' &= 0, \\ \alpha''\alpha + \beta''\beta + \gamma''\gamma &= 0, & \gamma\alpha + \gamma'\alpha' + \gamma''\alpha'' &= 0, \\ \alpha\,\alpha' + \beta\,\beta' + \gamma\,\gamma' &= 0, & \alpha\beta + \alpha'\beta' + \alpha''\beta'' &= 0, \end{aligned}$$

either set of six equations being implied in the other set.

It follows that the square of the determinant

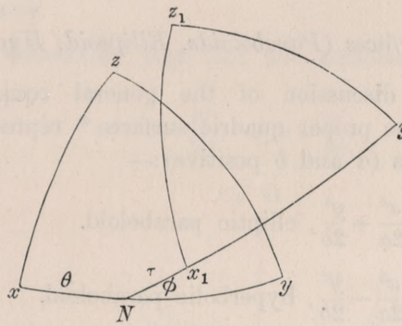
$$\begin{vmatrix} \alpha, & \beta, & \gamma \\ \alpha', & \beta', & \gamma' \\ \alpha'', & \beta'', & \gamma'' \end{vmatrix}$$

is $=1$; and hence that the determinant itself is $=\pm 1$. The distinction of the two cases is an important one: if the determinant is $=+1$, then the axes Ox_1, Oy_1, Oz_1 are such that they can by a rotation about O be brought to coincide with Ox, Oy, Oz respectively; if it is $=-1$, then they cannot. But in the latter case, by measuring x_1, y_1, z_1 in the opposite directions we change the signs of all the coefficients and so make the determinant to be $=+1$; hence this case need alone be considered, and it is accordingly assumed that the determinant is $=+1$. This being so, it is found that we have a further set of nine equations, $\alpha = \beta'\gamma'' - \beta''\gamma'$, &c.; that is, the coefficients arranged as in the diagram have the values

$\beta'\gamma'' - \beta''\gamma'$	$\gamma'\alpha'' - \gamma''\alpha'$	$\alpha'\beta'' - \alpha''\beta'$
$\beta''\gamma - \beta\gamma''$	$\gamma''\alpha - \gamma\alpha''$	$\alpha''\beta - \alpha\beta''$
$\beta\gamma' - \beta'\gamma$	$\gamma\alpha' - \gamma'\alpha$	$\alpha\beta' - \alpha'\beta$

36. It is important to express the nine coefficients in terms of three independent quantities. A solution which, although unsymmetrical, is very convenient in Astronomy and Dynamics is to use for the purpose the three angles θ, ϕ, τ of fig. 19; say $\theta =$ longitude of the node; $\phi =$ inclination; and $\tau =$ longitude of x_1 from node.

Fig. 19.



The diagram of transformation then is

	x	y	z
x_1	$\cos\tau\cos\theta - \sin\tau\sin\theta\cos\phi$	$\cos\tau\sin\theta + \sin\tau\cos\theta\cos\phi$	$\sin\tau\sin\phi$
y_1	$-\sin\tau\cos\theta - \cos\tau\sin\theta\cos\phi$	$-\sin\tau\sin\theta + \cos\tau\cos\theta\cos\phi$	$\cos\tau\sin\phi$
z_1	$\sin\theta\sin\phi$	$-\cos\theta\sin\phi$	$\cos\phi$

But a more elegant solution (due to Rodrigues) is that contained in the diagram

	x	y	z
x_1	$1+\lambda^2-\mu^2-\nu^2$	$2(\lambda\mu-\nu)$	$2(\lambda\nu+\mu)$
y_1	$2(\lambda\mu+\nu)$	$1-\lambda^2+\mu^2-\nu^2$	$2(\mu\nu-\lambda)$
z_1	$2(\nu\lambda-\mu)$	$2(\mu\nu+\lambda)$	$1-\lambda^2-\mu^2+\nu^2$

$$\div (1+\lambda^2+\mu^2+\nu^2).$$

The nine coefficients of transformation are the nine functions of the diagram, each divided by $1+\lambda^2+\mu^2+\nu^2$; the expressions contain as they should do the three arbitrary quantities λ, μ, ν; and the identity $x_1^2+y_1^2+z_1^2=x^2+y^2+z^2$ can be at once verified. It may be added that the transformation can be expressed in the quaternion form

$$ix_1+jy_1+kz_1=(1+\Lambda)(ix+jy+kz)(1+\Lambda)^{-1},$$

where Λ denotes the vector $i\lambda+j\mu+k\nu$.

Quadric Surfaces (Paraboloids, Ellipsoid, Hyperboloids).

37. It appears, by a discussion of the general equation of the second order $(a, \ldots \between x, y, z, 1)^2=0$, that the proper quadric surfaces * represented by such an equation are the following five surfaces (a and b positive):—

(1) $z = \dfrac{x^2}{2a}+\dfrac{y^2}{2b}$, elliptic paraboloid.

(2) $z = \dfrac{x^2}{2a}-\dfrac{y^2}{2b}$, hyperbolic paraboloid.

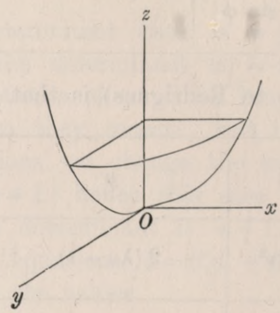
(3) $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2}= 1$, ellipsoid.

(4) $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}-\dfrac{z^2}{c^2}= 1$, hyperboloid of one sheet.

(5) $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}-\dfrac{z^2}{c^2}=-1$, hyperboloid of two sheets.

It is at once seen that these are distinct surfaces; and the equations also show very readily the general form and mode of generation of the several surfaces.

Fig. 20.



In the elliptic paraboloid (fig. 20), the sections by the planes of zx and zy are the parabolas

$$z=\frac{x^2}{2a},\quad z=\frac{y^2}{2b},$$

* The improper quadric surfaces represented by the general equation of the second order are (1) the pair of planes or plane-pair, including as a special case the twice repeated plane, and (2) the cone, including as a special case the cylinder. There is but one form of cone; but the cylinder may be parabolic, elliptic, or hyperbolic.

having the common axis Oz; and the section by any plane $z=\gamma$ parallel to that of xy is the ellipse

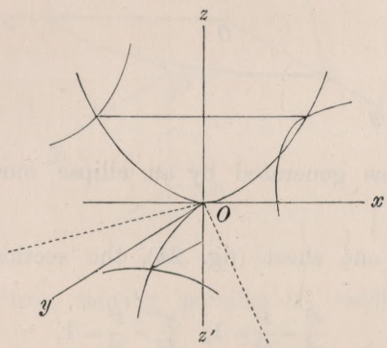
$$\gamma=\frac{x^2}{2a}+\frac{y^2}{2b};$$

so that the surface is generated by a variable ellipse moving parallel to itself along the parabolas as directrices.

In the hyperbolic paraboloid (fig. 21), the sections by the planes of zx, zy are the parabolas

$$z=\frac{x^2}{2a},\quad z=-\frac{y^2}{2b},$$

Fig. 21.

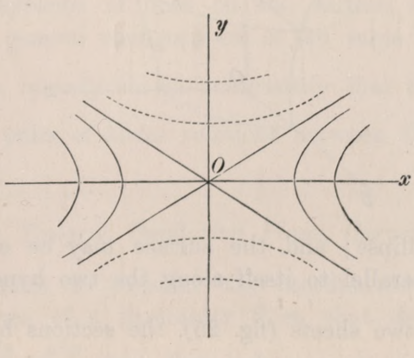


having the opposite axes Oz, Oz'; and the section by a plane $z=\gamma$ parallel to that of xy is the hyperbola

$$\gamma=\frac{x^2}{2a}-\frac{y^2}{2b},$$

which has its transverse axis parallel to Ox or Oy according as γ is positive or negative. The surface is thus generated by a variable hyperbola moving parallel to

Fig. 22.

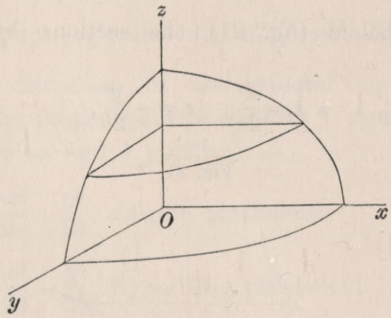


itself along the parabolas as directrices. The form is best seen from fig. 22, which represents the sections by planes parallel to the plane of xy, or say the contour lines;

the continuous lines are the sections above the plane of xy, and the dotted lines the sections below this plane. The form is, in fact, that of a saddle.

In the ellipsoid (fig. 23), the sections by the planes of zx, zy, and xy are each of them an ellipse, and the section by any parallel plane is also an ellipse. The

Fig. 23.



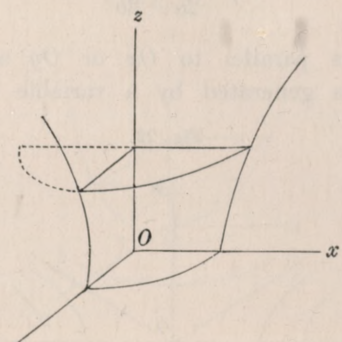
surface may be considered as generated by an ellipse moving parallel to itself along two ellipses as directrices.

In the hyperboloid of one sheet (fig. 24), the sections by the planes of zx, zy are the hyperbolas

$$\frac{x^2}{a^2}-\frac{z^2}{c^2}=1, \quad \frac{y^2}{b^2}-\frac{z^2}{c^2}=1,$$

having a common conjugate axis zOz'; the section by the plane of xy, and that by

Fig. 24.



any parallel plane, is an ellipse; and the surface may be considered as generated by a variable ellipse moving parallel to itself along the two hyperbolas as directrices.

In the hyperboloid of two sheets (fig. 25), the sections by the planes of zx and zy are the hyperbolas

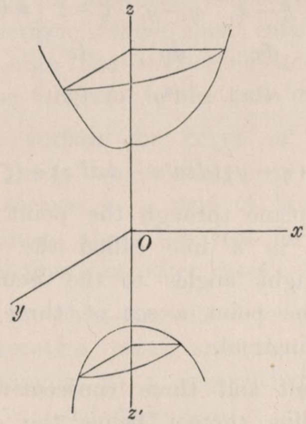
$$\frac{z^2}{c^2}-\frac{x^2}{a^2}=1, \quad \frac{z^2}{c^2}-\frac{y^2}{b^2}=1,$$

having the common transverse axis zOz'; the section by any plane $z = \pm\gamma$ parallel to that of xy, γ being in absolute magnitude $> c$, is the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{\gamma^2}{c^2} - 1\,;$$

and the surface, consisting of two distinct portions or sheets, may be considered as

Fig. 25.



generated by a variable ellipse moving parallel to itself along the hyperbolas as directrices.

The hyperbolic paraboloid is such (and it is easy from the figure to understand how this may be the case) that there exist upon it two singly infinite series of right lines. The same is the case with the hyperboloid of one sheet (ruled or skew hyperboloid, as with reference to this property it is termed). If we imagine two equal and parallel circular disks, their points connected by strings of equal length, so that these are the generating lines of a right circular cylinder, then by turning one of the disks about its centre through the same angle in one or the other direction, the strings will in each case generate one and the same hyperboloid, and will in regard to it be the two systems of lines on the surface, or say the two systems of generating lines; and the general configuration is the same when instead of circles we have ellipses. It has been already shown analytically that the equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$ is satisfied by each of two pairs of linear relations between the coordinates.

Curves; Tangent, Osculating Plane, Curvature, &c.

38. It will be convenient to consider the coordinates (x, y, z) of the point on the curve as given in terms of a parameter θ, so that dx, dy, dz, d^2x, &c., will be proportional to $\frac{dx}{d\theta}$, $\frac{dy}{d\theta}$, $\frac{dz}{d\theta}$, $\frac{d^2x}{d\theta^2}$, &c. But only a part of the analytical formulæ will be given; in them ξ, η, ζ are used as current coordinates.

The tangent is the line through the point (x, y, z) and the consecutive point $(x+dx,\ y+dy,\ z+dz)$; its equations therefore are

$$\frac{\xi - x}{dx} = \frac{\eta - y}{dy} = \frac{\zeta - z}{dz}.$$

The osculating plane is the plane through the point and two consecutive points, and contains therefore the tangent; its equation is

$$\begin{vmatrix} \xi - x, & \eta - y, & \zeta - z \\ dx\ , & dy\ , & dz \\ d^2x\ , & d^2y\ , & d^2z \end{vmatrix} = 0,$$

or, what is the same thing,

$$(\xi - x)(dyd^2z - dzd^2y) + (\eta - y)(dzd^2x - dxd^2z) + (\zeta - z)(dxd^2y - dyd^2x) = 0.$$

The normal plane is the plane through the point at right angles to the tangent. It meets the osculating plane in a line called the principal normal; and drawing through the point a line at right angles to the osculating plane, this is called the binormal. We have thus at the point a set of three rectangular axes—the tangent, the principal normal, and the binormal.

We have through the point and three consecutive points a sphere of spherical curvature,—the centre and radius thereof being the centre, and radius, of spherical curvature. The sphere is met by the osculating plane in the circle of absolute curvature,—the centre and radius thereof being the centre, and radius, of absolute curvature. The centre of absolute curvature is also the intersection of the principal normal by the normal plane at the consecutive point.

Surfaces; Tangent Lines and Plane, Curvature, &c.

39. It will be convenient to consider the surface as given by an equation $f(x, y, z) = 0$ between the coordinates; taking (x, y, z) for the coordinates of a given point, and $(x+dx,\ y+dy,\ z+dz)$ for those of a consecutive point, the increments dx, dy, dz satisfy the condition

$$\frac{df}{dx}dx + \frac{df}{dy}dy + \frac{df}{dz}dz = 0,$$

but the ratio of two of the increments, suppose $dx : dy$, may be regarded as arbitrary. Only a part of the analytical formulæ will be given; in them ξ, η, ζ are used as current coordinates.

We have through the point a singly infinite series of right lines, each meeting the surface in a consecutive point, or say having each of them two-point intersection with the surface. These lines lie all of them in a plane which is the tangent plane; its equation is

$$\frac{df}{dx}(\xi - x) + \frac{df}{dy}(\eta - y) + \frac{df}{dz}(\zeta - z) = 0,$$

as is at once verified by observing that this equation is satisfied (irrespectively of the value of $dx : dy$) on writing therein $\xi, \eta, \zeta = x + dx,\ y + dy,\ z + dz$.

The line through the point at right angles to the tangent plane is called the normal; its equations are

$$\frac{\xi - x}{\dfrac{df}{dx}} = \frac{\eta - y}{\dfrac{df}{dy}} = \frac{\zeta - z}{\dfrac{df}{dz}}.$$

In the series of tangent lines there are in general two (real or imaginary) lines, each of which meets the surface in a second consecutive point, or say it has three-point intersection with the surface; these are called the chief-tangents (Haupt-tangenten). The tangent-plane cuts the surface in a curve, having at the point of contact a node (double point), the tangents to the two branches being the chief-tangents.

In the case of a quadric surface the curve of intersection, *qua* curve of the second order, can only have a node by breaking up into a pair of lines; that is, every tangent-plane meets the surface in a pair of lines, or we have on the surface two singly infinite systems of lines; these are real for the hyperbolic paraboloid and the hyperboloid of one sheet, imaginary in other cases.

At each point of a surface the chief-tangents determine two directions; and passing along one of them to a consecutive point, and thence (without abrupt change of direction) along the new chief-tangent to a consecutive point, and so on, we have on the surface a chief-tangent curve; and there are, it is clear, two singly infinite series of such curves. In the case of a quadric surface, the curves are the right lines on the surface.

40. If at the point we draw in the tangent-plane two lines bisecting the angles between the chief-tangents, these lines (which are at right angles to each other) are called the principal tangents*. We have thus at each point of the surface a set of rectangular axes, the normal and the two principal tangents.

Proceeding from the point along a principal tangent to a consecutive point on the surface, and thence (without abrupt change of direction) along the new principal tangent to a consecutive point, and so on, we have on the surface a curve of curvature; there are, it is clear, two singly infinite series of such curves, cutting each other at right angles at each point of the surface.

Passing from the given point in an arbitrary direction to a consecutive point on the surface, the normal at the given point is not intersected by the normal at the consecutive point; but passing to the consecutive point along a curve of curvature (or, what is the same thing, along a principal tangent) the normal at the given point is intersected by the normal at the consecutive point; we have thus on the normal two centres of curvature, and the distances of these from the point on the surface are the two principal radii of curvature of the surface at that point; these are also the radii of curvature of the sections of the surface by planes through the normal and the two principal tangents respectively; or say they are the radii of curvature of the

* The point on the surface may be such that the directions of the principal tangents become arbitrary; the point is then an umbilicus. It is in the text assumed that the point on the surface is not an umbilicus.

normal sections through the two principal tangents respectively. Take at the point the axis of z in the direction of the normal, and those of x and y in the directions of the principal tangents respectively, then, if the radii of curvature be a, b (the signs being such that the coordinates of the two centres of curvature are $z=a$ and $z=b$ respectively), the surface has in the neighbourhood of the point the form of the paraboloid

$$z=\frac{x^2}{2a}+\frac{y^2}{2b},$$

and the chief-tangents are determined by the equation $0=\frac{x^2}{2a}+\frac{y^2}{2b}$. The two centres of curvature may be on the same side of the point or on opposite sides; in the former case a and b have the same sign, the paraboloid is elliptic, and the chief-tangents are imaginary; in the latter case a and b have opposite signs, the paraboloid is hyperbolic, and the chief-tangents are real.

The normal sections of the surface and the paraboloid by the same plane have the same radius of curvature; and it thence readily follows that the radius of curvature of a normal section of the surface by a plane inclined at an angle θ to that of zx is given by the equation

$$\frac{1}{\rho}=\frac{\cos^2\theta}{a}+\frac{\sin^2\theta}{b}.$$

The section in question is that by a plane through the normal and a line in the tangent plane inclined at an angle θ to the principal tangent along the axis of x. To complete the theory, consider the section by a plane having the same trace upon the tangent plane, but inclined to the normal at an angle ϕ; then it is shown without difficulty (Meunier's theorem) that the radius of curvature of this inclined section of the surface is $=\rho\cos\phi$.