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FUNCTION.

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FUNCTIONALITY, in Analysis, is dependence on a variable or variables; in the case of a single variable u , it is the same thing to say that v depends upon u , or to say that v is a function of u , only in the latter form of expression the mode of dependence is embodied in the term "function." We have given or known functions such as u^2 or $\sin u$, and the general notation of the form ϕu , where the letter ϕ is used as a functional symbol to denote a function of u , known or unknown as the case may be: in each case u is the independent variable or argument of the function, but it is to be observed that, if v be a function of u , then v like u is a variable, the values of v regarded as known serve to determine those of u ; that is, we may conversely regard u as a function of v . In the case of two or more independent variables, say when w depends on or is a function of $u, v, \&c.$, or $w = \phi(u, v, \dots)$, then u, v, \dots are the independent variables or arguments of the function; frequently when one of these variables, say u , is principally or alone attended to, it is regarded as the independent variable or argument of the function, and the other variables $v, \&c.$, are regarded as parameters, the values of which serve to complete the definition of the function. We may have a set of quantities w, t, \dots each of them a function of the same variables u, v, \dots ; and this relation may be expressed by means of a single functional symbol ϕ , $(w, t, \dots) = \phi(u, v, \dots)$; but, as to this, more hereafter.

The notion of a function is applicable in geometry and mechanics as well as in analysis; for instance, a point Q , the position of which depends upon that of a variable point P , may be regarded as a function of the point P ; but here, substituting for the points themselves the coordinates (of any kind whatever) which determine their positions, we may say that the coordinates of Q are each of them a function of the coordinates of P , and we thus return to the analytical notion of a function. And in what follows a function is regarded exclusively in this point of view,

viz. the variables are regarded as numbers; and we attend to the case of a function of one variable $v = fu$. But it has been remarked (see Equation) that it is not allowable to confine the attention to *real* numbers; a number u must in general be taken to be a complex number $u = x + iy$, x and y being real numbers, each susceptible of continuous variation between the limits $-\infty$, $+\infty$, and i denoting $\sqrt{-1}$. In regard to any particular function, fu , although it may for some purposes be sufficient to know the value of the function for any real value whatever of u , yet to attend only to the real values of u is an essentially incomplete view of the question; to properly know the function, it is necessary to consider u under the aforesaid imaginary or complex form $u = x + iy$.

To a given value $x + iy$ of u there corresponds in general for v a value or values of the like form $v = x' + iy'$, and we obtain a geometrical notion of the meaning of the functional relation $v = fu$ by regarding x, y as rectangular coordinates of a point P in a plane Π , and x', y' as rectangular coordinates of a point P' in a plane (for greater convenience a different plane) Π' ; P, P' are thus the geometrical representations, or representative points, of the variables $u = x + iy$ and $u' = x' + iy'$ respectively; and, according to a locution above referred to, the point P' might be regarded as a function of the point P ; a given value of $u = x + iy$ is thus represented by a point P in the plane Π , and corresponding hereto we have a point or points P' in the plane Π' , representing (if more than one, each of them) a value of the variable $v = x' + iy'$. And, if we attend only to the values of u as corresponding to a series of positions of the representative point P , we have the notion of the "path" of a complex variable $u = x + iy$.

Known Functions.

1. The most simple kind of function is the rational and integral function. We have the series of powers u^2, u^3, \dots each calculable not only for a real but also for a complex value of u , $(x + iy)^2 = x^2 - iy^2 + 2ixy$, $(x + iy)^3 = x^3 - 3xy^2 + i(3x^2y - y^3)$, &c., and thence, if a, b, \dots be real or complex numbers, the general form $a + bu + cu^2 + \dots + ku^m$, of a rational and integral function of the order m . And taking two such functions, say of the orders m and n respectively, the quotient of one of these by the other represents the general form of a rational function of u .

The function which next presents itself is the algebraical function, and in particular the algebraical function expressible by radicals. To take the most simple case, suppose (m being a positive integer) that $v^m = u$; v is here the irrational function $= u^{\frac{1}{m}}$. Obviously, if u is real and positive, there is always a real and positive value of v , calculable to any extent of approximation from the equation $v^m = u$, which serves as the definition of $u^{\frac{1}{m}}$; but it is known (see Equation) that, as well in this case as in the general case where u is a complex number, there are in fact m values of the function $u^{\frac{1}{m}}$; and that for their determination we require the theory of the so-called

circular functions sine and cosine; and these depend on the exponential function $\exp u$, or, as it is commonly written, e^u , which has for its inverse the logarithmic function $\log u$; these are all of them transcendental functions.

2. In a rational and integral function $a + bu + cu^2 + \dots + ku^m$, the number of terms is finite, and the coefficients $a, b, \dots k$ may have any values whatever, but if we imagine a like series $a + bu + cu^2 + \dots$ extending to infinity, *non constat* that such an expression has any calculable value,—that is, any meaning at all; the coefficients a, b, c, \dots must be such as, either for every value whatever of u (that is, for every finite value) or for values included within certain limits, to make the series *convergent*. It is easy to see that the values of a, b, c, \dots may be such as to make the series always convergent; for instance, this is the case for the exponential function,

$$\exp u = 1 + \frac{u}{1} + \frac{u^2}{1 \cdot 2} + \frac{u^3}{1 \cdot 2 \cdot 3} + \&c.;$$

taking for the moment u to be real and positive, then it is evident that however large u may be, the successive terms will become ultimately smaller and smaller, and the series will have a determinate calculable value. A function thus expressed by means of a convergent infinite series is not in general algebraical, and when it is not so, it is said to be transcendental (but observe that it is in nowise true that we have thus the most general form of a transcendental function); in particular, the exponential function above written down is not an algebraical function.

By forming the expression of $\exp v$, and multiplying together the two series, we derive the fundamental property

$$\exp u \exp v = \exp (u + v);$$

whence also

$$\exp x \exp iy = \exp (x + iy),$$

so that $\exp(x + iy)$ is given as the product of the two series $\exp x$ and $\exp iy$. As regards this last, if in place of u we actually write the value iy , we find

$$\exp iy = \left(1 - \frac{y^2}{1 \cdot 2} + \frac{y^4}{1 \cdot 2 \cdot 3 \cdot 4} - \dots\right) + i \left(y - \frac{y^3}{1 \cdot 2 \cdot 3} + \dots\right),$$

where obviously each series is convergent and actually calculable for any real value whatever of y . Calling the two series cosine y and sine y respectively, or in the ordinary abbreviated notation $\cos y$ and $\sin y$, the equation is

$$\exp iy = \cos y + i \sin y;$$

and if we herein for y write z , and multiply the two expressions together, observing that the product will be $= \exp i(y + z)$, we obtain the fundamental equations

$$\cos (y + z) = \cos y \cos z - \sin y \sin z,$$

$$\sin (y + z) = \sin y \cos z + \sin z \cos y,$$

for the functions sine and cosine.

Taking y as an angle, and defining as usual the sine and cosine as the ratios of the perpendicular and base respectively to the radius, the sine and cosine will be functions of y ; and we obtain geometrically the foregoing fundamental equations for the sine and cosine; but in order to the truth of the foregoing equation $\exp iy = \cos y + i \sin y$, it is further necessary that the angle should be measured in circular measure, that is, by the ratio of the arc to the radius; so that π denoting as usual the number 3.14159..., the measure of a right angle is $=\frac{1}{2}\pi$. And this being so, the functions sine and cosine, obtained as above by consideration of the exponential function, have their ordinary geometrical significations.

3. The foregoing investigation was given in detail in order to the completion of the theory of the irrational function $u^{\frac{1}{m}}$. We henceforth take the theory of the circular functions as known, and speak of $\tan x$, &c., as the occasion may arise.

We have

$$x + iy = r(\cos \theta + i \sin \theta),$$

where, writing $\sqrt{x^2 + y^2}$ to denote the positive value of the square root, we have

$$r = \sqrt{x^2 + y^2}, \quad \cos \theta = \frac{x}{\sqrt{x^2 + y^2}}, \quad \sin \theta = \frac{y}{\sqrt{x^2 + y^2}},$$

and therefore also

$$\tan \theta = \frac{y}{x}.$$

Treating x, y as the rectangular coordinates of a point P , r is the distance (regarded as positive) of this point from the origin, and θ is the inclination of r to the positive part of the axis of x ; to fix the ideas θ may be regarded as lying within the limits $0, \pi$, or $0, -\pi$, according as y is positive or negative; θ is thus completely determinate, except in the case, x negative, $y=0$, for which θ is $=\pi$ or $-\pi$ indifferently.

And if $u = x + iy$, we hence have

$$u^{\frac{1}{m}} = (x + iy)^{\frac{1}{m}} = r^{\frac{1}{m}} \left(\cos \frac{\theta + 2s\pi}{m} + i \sin \frac{\theta + 2s\pi}{m} \right),$$

where $r^{\frac{1}{m}}$ is real and positive and s has any positive or negative integer value whatever: but we thus obtain for $u^{\frac{1}{m}}$ only the m values corresponding to the values $0, 1, 2, \dots, m-1$ of s . More generally we may, instead of the index $\frac{1}{m}$, take the index

to be any rational fraction $\frac{n}{m}$. Supposing this to be in its least terms, and m to

be positive, the number of distinct values is always $=m$. If instead of $\frac{n}{m}$ we take the index to be the general real or complex quantity m , we have u^m , no longer an algebraical function of u , and having in general an infinity of values.

4. The foregoing equation $\exp(x+y) = \exp x \cdot \exp y$ is, in fact, the equation of indices, $a^{x+y} = a^x \cdot a^y$; $\exp x$ is thus the same thing as e^x , where e denotes a properly determined number, and putting e^x equal to the series, and then writing $x=1$, we have $e = 1 + \frac{1}{1} + \frac{1}{1 \cdot 2} + \frac{1}{1 \cdot 2 \cdot 3} + \&c.$, that is, $e = 2.7128 \dots$ But as well theoretically as for convenience of printing, there is considerable advantage in the use of the notation $\exp u$.

From the equation, $\exp iy = \cos y + i \sin y$, we deduce $\exp(-iy) = \cos y - i \sin y$, and thence

$$\cos y = \frac{1}{2} \{ \exp(iy) + \exp(-iy) \},$$

$$\sin y = \frac{1}{2i} \{ \exp(iy) - \exp(-iy) \};$$

if we write herein ix instead of y we have

$$\cos ix = \frac{1}{2} \{ \exp x + \exp(-x) \},$$

$$\sin ix = \frac{i}{2} \{ \exp x - \exp(-x) \},$$

viz. these values are

$$\cos ix = 1 + \frac{x^2}{1 \cdot 2} + \frac{x^4}{1 \cdot 2 \cdot 3 \cdot 4} + \dots$$

$$\frac{1}{i} \sin ix = x + \frac{x^3}{1 \cdot 2 \cdot 3} + \dots$$

each of them real when x is real. The functions in question $1 + \frac{x^2}{1 \cdot 2} + \frac{x^4}{1 \cdot 2 \cdot 3 \cdot 4} + \dots$ and $x + \frac{x^3}{1 \cdot 2 \cdot 3} + \dots$, regarded as functions of x , are termed the hyperbolic cosine and sine, and are represented by the notations $\cosh x$ and $\sinh x$ respectively; and similarly we have the hyperbolic tangent $\tanh x$, &c.: although it is easy to remember that $\cos ix$, $\frac{1}{i} \sin ix$, are, in fact, real functions of x , and to understand accordingly the formulæ wherein they occur, yet the use of these notations of the hyperbolic functions is often convenient.

5. Writing $u = \exp v$, then v is conversely a function of u which is called the logarithm (hyperbolic logarithm, to distinguish it from the tabular or Briggian logarithm), and we write $v = \log u$, or what is the same thing, we have $u = \exp(\log u)$: and it is clear that if u be real and positive there is always a real and positive value of $\log u$, in particular the real logarithm of e is $=1$; it is however to be observed that the logarithm is not a one-valued function, but has an infinity of values corresponding to the different integer values of a constant s ; in fact, if $\log u$ be any one of its values, then $\log u + 2s\pi i$ is also a value, for we have $\exp(\log u + 2s\pi i) = \exp \log u \exp 2s\pi i$, or since $\exp 2s\pi i$ is $=1$, this is $=u$; that is, $\log u + 2s\pi i$ is a value of the logarithm of u .

We have

$$uv = \exp(\log uv) = \exp \log u \cdot \exp \log v,$$

and hence the equation which is commonly written

$$\log uv = \log u + \log v,$$

but which requires the addition on one side of a term $2s\pi i$. And reverting to the equation $x + iy = r(\cos \theta + i \sin \theta)$, or as it is convenient to write it, $x + iy = r \exp i\theta$, we hence have

$$\log(x + iy) = \log r + i(\theta + 2s\pi),$$

where $\log r$ may be taken to denote the real logarithm of the real positive quantity r , and θ the completely determinate angle defined as already mentioned.

Reverting to the function u^m , we have $u = \exp \log u$, and thence $u^m = \exp(m \log u)$, which, on account of the infinity of values of $\log u$, has in general (as before remarked) an infinity of values; if $u = e$, then $e^m = \exp(m \log e)$, has in general in like manner an infinity of values, but in regarding e^m as identical with the one-valued function $\exp m$, we take $\log e$ to be = its real value, 1.

The inverse functions $\cos^{-1} x$, $\sin^{-1} x$, $\tan^{-1} x$, are in fact logarithmic functions; thus in the equation $\exp ix = \cos x + i \sin x$, writing first $\cos x = u$, the equation becomes $\exp i \cos^{-1} u = u + i \sqrt{1 - u^2}$, or we have $\cos^{-1} u = \frac{1}{i} \log(u + i \sqrt{1 - u^2})$, and from the same equation, writing secondly $\sin x = u$, we have $\sin^{-1} u = \frac{1}{i} \log(\sqrt{1 - u^2} + iu)$. But the formula for $\tan^{-1} u$ is a more elegant one, as not involving the radical $\sqrt{1 - u^2}$; we have

$$i \tan x = \frac{\exp ix - \exp(-ix)}{\exp ix + \exp(-ix)}, = \frac{\exp 2ix - 1}{\exp 2ix + 1},$$

and thence

$$\exp 2ix = \frac{1 + i \tan x}{1 - i \tan x},$$

that is,

$$x = \frac{1}{2i} \log \frac{1 + i \tan x}{1 - i \tan x},$$

or, if $\tan x = u$, then

$$\tan^{-1} u = \frac{1}{2i} \log \frac{1 + iu}{1 - iu}.$$

The logarithm (or inverse exponential function) and the inverse circular functions present themselves as the integrals of algebraic functions

$$\int \frac{dx}{x} = \log x,$$

whence also

$$\int \frac{dx}{1 + x^2} = \frac{1}{2i} \log \frac{1 + ix}{1 - ix} = \tan^{-1} x,$$

and

$$\int \frac{dx}{\sqrt{1 - x^2}} = \sin^{-1} x.$$

6. Each of the functions $\exp u$, $\sin u$, $\cos u$, $\tan u$, &c., as a one-valued function of u , is in this respect analogous to a rational function of u ; and there are further analogies of $\exp u$, $\sin u$, $\cos u$, to a rational and integral function; and of $\tan u$, $\sec u$, &c., to a rational non-integral function.

A rational and integral function has a certain number of roots, or zeros, each of a given multiplicity, and is completely determined (except as to a constant factor) when the several roots and the multiplicity of each of them is given; i.e., if a, b, c, \dots be the roots, p, q, r, \dots their multiplicities, then the form is $A \left(1 - \frac{u}{a}\right)^p \left(1 - \frac{u}{b}\right)^q \dots$; a rational (non-integral) function has a certain number of infinities, or poles, each of them of a given multiplicity, viz. the infinities are the roots or zeros of the rational and integral function which is its denominator.

The function $\exp u$ has no finite roots, but an infinity of roots each $= -\infty$; this appears from the equation $\exp u = \left(1 + \frac{u}{n}\right)^n$, where n is indefinitely large and positive.

The function $\sin u$ has the roots $s\pi$ where s is any positive or negative integer, zero included; or, what is the same thing, its roots are 0 and $\pm s\pi$, s now denoting any positive integer from 1 to ∞ ; each of these is a simple root, and we in fact have $\sin u = u \prod \left(1 - \frac{u^2}{s^2\pi^2}\right)$. Similarly the roots of $\cos u$ are $(s + \frac{1}{2})\pi$, s denoting any positive or negative integer, zero included, or, what is the same thing, they are $\pm(s + \frac{1}{2})\pi$, s now denoting any positive integer from 0 to ∞ ; each root is simple, and we have $\cos u = \prod \left(1 + \frac{u^2}{(s + \frac{1}{2})^2\pi^2}\right)$. Obviously $\tan u$, as the quotient $\sin u \div \cos u$, has both roots and infinities, its roots being the roots of $\sin u$, its infinities the roots of $\cos u$; $\sec u$ as the reciprocal of $\cos u$ has infinities only, these being the roots of $\cos u$, &c.

In the foregoing expression $\sin u = u \prod \left(1 - \frac{u^2}{s^2\pi^2}\right)$, the product must be understood to mean the limit of $\prod_1^n \left(1 - \frac{u^2}{s^2\pi^2}\right)$ for an indefinitely large positive integer value of n , viz. the product is first to be formed for the values $s = 1, 2, 3, \dots$ up to a determinate number n , and then n is to be taken indefinitely large. If, separating the positive and the negative values of s , we consider the product $u \prod_1^n \left(1 + \frac{u}{s\pi}\right) \prod_1^m \left(1 - \frac{u}{s\pi}\right)$, (where in the first product s has all the positive integer values from 1 to n , and in the second product s has all the positive integer values from 1 to m), then by making m and n each of them indefinitely large, the function does *not* approximate to $\sin u$, unless $m : n$ be a ratio of equality*. And similarly as regards $\cos u$, the product $\prod_0^n \left(1 + \frac{u}{(s + \frac{1}{2})\pi}\right) \prod \left(\frac{u}{(s + \frac{1}{2})\pi}\right)$, m and n indefinitely large, does not approximate to $\cos u$, unless $m : n$ be a ratio of equality.

* The value of the function in question $u \prod_1^n \left(1 + \frac{u}{s\pi}\right) \prod_1^m \left(1 - \frac{u}{s\pi}\right)$, when m, n are each indefinitely large, but $\frac{m}{n}$ not $= 1$, is $= \left(\frac{n}{m}\right)^{\frac{u}{\pi}} \sin u$.

7. The functions $\sin u$, $\cos u$, are *periodic*, having the period 2π , $\frac{\sin}{\cos}(u + 2\pi) = \frac{\sin}{\cos}(u)$; and the half-period π , $\frac{\sin}{\cos}(u + \pi) = -\frac{\sin}{\cos} u$; the periodicity may be verified by means of the foregoing fractional forms, but some attention is required; thus writing, as we may do, $\sin u = \frac{\prod (u + s\pi)}{\prod s\pi}$, where s extends from $-n$ to n , n ultimately infinite, if for u we write $u + \pi$, each factor of the numerator is changed into the following one and the numerator is unaltered, save only that there is an introduced factor $u + (n+1)\pi$ at the superior limit, and an omitted factor $u - n\pi$ at the inferior limit; the ratio of these, $(u + \overline{n+1}\pi) \div (u - n\pi)$, for n infinite is $= -1$, and we thus have, as we should have, $\sin(u + \pi) = -\sin u$.

The most general periodic function having no infinities, and each root a simple root, and having a given period a , has the form $A \sin \frac{2\pi u}{a} + B \cos \frac{2\pi u}{a}$, or, what is the same thing, $L \frac{\sin \left(\frac{2\pi u}{a} + \lambda \right)}{\cos \left(\frac{2\pi u}{a} + \lambda \right)}$.

8. We come now to the Elliptic Functions. These arose from the consideration of the integral $\int \frac{Rdx}{\sqrt{X}}$, where R is a rational function of x , and X is the general rational and integral quartic function

$$ax^4 + \beta x^3 + \gamma x^2 + \delta x + \epsilon;$$

a form arrived at was

$$\int \frac{dx}{\sqrt{1-x^2} \cdot \sqrt{1-k^2x^2}}, = \int \frac{d\phi}{\sqrt{1-k^2 \sin^2 \phi}},$$

on putting therein $x = \sin \phi$, and this last integral was represented by $F\phi$, and called the elliptic integral of the first kind. In the particular case $k=0$, the integral is

$\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x$, and it thus appears that $F\phi$ is of the nature of an inverse function;

for passing to the direct functions we write $F\phi = u$, and consider ϕ as hereby determined as a function of u , $\phi =$ amplitude of u , or for shortness $\text{am } u$. And the functions $\sin \phi$, $\cos \phi$, and $\sqrt{1-k^2 \sin^2 \phi}$ were then considered as functions of the amplitude, and written $\text{sn } \text{am } u$, $\text{cn } \text{am } u$, $\Delta \text{ am } u$; these were afterwards written $\text{sn } u$, $\text{cn } u$, $\text{dn } u$, which may be regarded either as mere abbreviations of the former functional symbols, or (in a different point of view) as functions, no longer of $\text{am } u$, but of u itself as the argument of the functions; sn is thus a function in some respects analogous to a sine, and cn and dn functions analogous to a cosine; they have the corresponding property that the three functions of $u+v$ are expressible in terms of the functions of u and of v . The following formulæ may be mentioned:

$$\text{cn}^2 u = 1 - \text{sn}^2 u, \quad \text{dn}^2 u = 1 - k^2 \text{sn}^2 u,$$

$$\text{sn}' u = \text{cn } u \text{ dn } u, \quad \text{cn}' u = -\text{sn } u \text{ dn } u, \quad \text{dn}' u = -k^2 \text{sn } u \text{ cn } u,$$

where the accent denotes differentiation in regard to u ; and the addition-formulæ:

$$\operatorname{sn}(u+v) = \operatorname{sn} u \operatorname{cn} v \operatorname{dn} v + \operatorname{sn} v \operatorname{cn} u \operatorname{dn} u, \quad (\div),$$

$$\operatorname{cn}(u+v) = \operatorname{cn} u \operatorname{cn} v - \operatorname{sn} u \operatorname{dn} u \operatorname{sn} v \operatorname{dn} v, \quad (\div),$$

$$\operatorname{dn}(u+v) = \operatorname{dn} u \operatorname{dn} v - k^2 \operatorname{sn} u \operatorname{cn} u \operatorname{sn} v \operatorname{cn} v, \quad (\div),$$

each of the expressions on the right-hand side being the numerator of a fraction of which

$$\text{Denom.} = 1 - k^2 \operatorname{sn}^2 u \operatorname{sn}^2 v.$$

It may be remarked that any one of the fractional expressions, differentiated in regard to u and to v respectively, gives the same result; such expression is therefore a function of $u+v$, and the addition-formulæ can be thus directly verified.

9. The existence of a denominator in the addition-formulæ suggests that sn , cn , dn are not, like the sine and cosine, functions having zeros only, without infinities; they are in fact functions, having each its own zeros, but having a common set of infinities; moreover, the zeros and the infinities are simple zeros and infinities respectively. And this further suggests, what in fact is the case, that the three functions are quotients having each its own numerator but a common denominator, say they are the quotients of four θ -functions, each of them having zeros only (and these simple zeros) but no infinities.

The functions sn , cn , dn , but not the θ -functions, are moreover *doubly periodic*; that is, there exist values 2ω , 2ν , $=4K$ and $4(K+iK')$ in the ordinary notation, such that the sn , cn , or dn of $u+2\omega$, and the sn , cn , and dn of $u+2\nu$ are equal to the sn , cn , and dn respectively of u ; or say that $\phi(u+2\omega) = \phi(u+2\nu) = \phi u$, where ϕ is any one of the three functions.

As regards this double periodicity, it is to be observed that the equations $\phi(u+2\omega) = \phi u$, $\phi(u+2\nu) = \phi u$, imply $\phi(u+2m\omega+2n\nu) = \phi u$, and hence it easily follows that if ω , ν were commensurable, say if they were multiples of some quantity α , we should have $\phi(u+2\alpha) = \phi u$, an equation which would replace the original two equations $\phi(u+2\omega) = \phi u$, $\phi(u+2\nu) = \phi u$, or there would in this case be only the single period α ; ω and ν must therefore be incommensurable. And this being so, they cannot have a real ratio, for if they had, the integer values m , n could be taken such as to make $2m\omega+2n\nu = k$ times a given real or imaginary value, k as small as we please; the ratio $\omega : \nu$ must be therefore imaginary, as is in fact the case when the values are $4K$ and $4(K+iK')$.

10. The function $\operatorname{sn} u$ has the zero 0 and the zeros $m\omega+n\nu$, m and n any positive or negative integers whatever; and this suggests that the numerator of $\operatorname{sn} u$ is equal to a doubly infinite product, (Cayley, "On the Inverse Elliptic Functions," *Camb. Math. Jour.* t. IV., 1845, [24]; and "Mémoire sur les fonctions doublement périodiques," *Liouville*, t. x., 1845, [25]). The numerator is equal to

$$u \prod \left(1 + \frac{u}{m\omega + n\nu} \right),$$

m and n having any positive or negative integer values whatever, including zero, except that m, n must not be simultaneously $= 0$, these values being taken account of in the factor u outside the product. But until further defined, such a product has no definite value, and consequently no meaning whatever. Imagine m, n to be coordinates, and suppose that we have, surrounding the origin, a closed curve having the origin for its centre, i.e. the curve is such that, if α, β be the coordinates of a point thereof, then $-\alpha, -\beta$ are also the coordinates of a point thereof; suppose further that the form of the curve is given, but that its magnitude depends upon a parameter h , and that the curve is such that, when h is indefinitely large, each point of the curve is at an indefinitely large distance from the origin; for instance, the curve might be a circle or ellipse, or a parallelogram, the origin being in each case the centre. Then if in the double product, taking the value of h as given, we first give to m, n all the positive or negative integer values (the simultaneous values $0, 0$ excluded) which correspond to points within the curve, and then make h indefinitely large, the product will thus have a definite value; *but this value will still be dependent on the form of the curve.* Moreover, varying in any manner the form of the curve, the ratio of the two values of the double product will be $= \exp \beta u^2$, where β is a determinate value depending only on the forms of the two curves; or, what is the same thing, if we first give to the curve a certain form, say we take it to be a circle, and then give it any other form, the product in the latter case is equal to its former value multiplied by $\exp \beta u^2$, where β depends only upon the form of the curve in the latter case.

Considering the form of the bounding curve as given, and writing the double product in the form

$$\text{III} \left(\frac{u + m\omega + nv}{m\omega + nv} \right),$$

the simultaneous values $m=0, n=0$ being now admitted in the numerator, although still excluded from the denominator, then if we write for instance $u + 2\omega$ instead of u , each factor in the numerator is changed into a contiguous factor, and the numerator remains unaltered, except that we introduce certain factors which lie outside the bounding curve, and omit certain factors which lie inside the bounding curve; we, in fact, affect the result by a singly infinite series of factors belonging to points adjacent to the bounding curve; and it appears on investigation that we thus introduce a constant factor $\exp \gamma(u + \omega)$. The final result thus is that the product

$$u \text{III} \left(1 + \frac{u}{m\omega + nv} \right)$$

does not remain unaltered when u is changed into $u + 2\omega$, but that it becomes therefore affected with a constant factor, $\exp \gamma(u + \omega)$. And similarly the function does not remain unaltered when u is changed into $u + 2\nu$, but it becomes affected with a factor, $\exp \delta(u + \nu)$. The bounding curve may however be taken such that the function is unaltered when u is changed into $u + 2\omega$: this will be the case if the curve is a rectangle such that the length in the direction of the axis of m is infinitely great in comparison of that in the direction of the axis of n ; or it may be taken such that the function is unaltered when u is changed into $u + 2\nu$: this will be so if the curve

be a rectangle such that the length in the direction of the axis of n is indefinitely great in comparison with that in the direction of the axis of m ; but the two conditions cannot be satisfied simultaneously.

11. We have three other like functions, viz. writing for shortness \bar{m} , \bar{n} to denote $m + \frac{1}{2}$, $n + \frac{1}{2}$ respectively, and (m, n) to denote $m\omega + n\nu$, then the four functions are

$$u\Pi\Pi\left(1 + \frac{u}{(m, n)}\right), \quad \Pi\Pi\left(1 + \frac{u}{(\bar{m}, n)}\right), \quad \Pi\Pi\left(1 + \frac{u}{(m, \bar{n})}\right), \quad \Pi\Pi\left(1 + \frac{u}{(\bar{m}, \bar{n})}\right),$$

the bounding curve being in each case the same; and, dividing the first three of these each by the last, we have (except as to constant factors) the three functions $\text{sn } u$, $\text{cn } u$, $\text{dn } u$; writing in each of the four functions $u + 2\omega$ or $u + 2\nu$ in place of u , the functions acquire each of them the *same* exponential factor $\exp \gamma(u + \omega)$, or $\exp \delta(u + \nu)$, and the quotient of any two of them, and therefore each of the functions $\text{sn } u$, $\text{cn } u$, $\text{dn } u$, remains unaltered.

It is easily seen that, disregarding constant factors, the four θ -functions are in fact one and the same function with different arguments, or they may be written θu , $\theta(u + \frac{1}{2}\omega)$, $\theta(u + \frac{1}{2}\nu)$, $\theta(u + \frac{1}{2}\omega + \frac{1}{2}\nu)$; by what precedes, the functions may be so determined that they shall remain unaltered when u is changed into $u + 2\omega$, that is, be singly periodic, but that the change u into $u + 2\nu$ shall affect them each with the same exponential factor $\exp \delta(u + \nu)$.

12. Taking the last-mentioned property as a definition of the function θ , it appears that θu may be expressed as a sum of exponentials

$$\theta u = A \Sigma \exp \frac{\pi i}{\omega} (\nu m^2 + um),$$

where the summation extends to all positive and negative integer values of m , including zero. In fact, if we first write herein $u + 2\omega$ instead of u , then in each term the index of the exponential is altered by $\frac{\pi i}{\omega} 2\omega m$, $= 2m\pi i$, and the term itself thus remains unaltered; that is, $\theta(u + 2\omega) = \theta u$. But writing $u + 2\nu$ in place of u , each term is changed into the succeeding term, multiplied by the factor $\exp \frac{\pi i}{\omega} (u + \nu)$; in fact, making the change in question u into $u + 2\nu$, and writing also $m - 1$ in place of m , $\nu m^2 + um$ becomes $\nu(m - 1)^2 + (u + 2\nu)(m - 1)$, $= \nu m^2 + um - u - \nu$, and we thus have $\theta(u + 2\nu) = \exp \left\{ -\frac{\pi i}{\omega} (u + \nu) \right\} \cdot \theta u$. In order to the convergency of the series it is necessary that $\exp \frac{\pi i \nu m^2}{\omega}$ should vanish for indefinitely large values of m , and this will be so if $\frac{i\nu}{\omega}$ be a complex quantity of the form $\alpha + \beta i$, α negative; for instance, this will be the case if ω be real and positive and ν be $= i$ multiplied by a real and positive quantity. The original definition of θ as a double product seems to put more clearly in evidence the real nature of this function, but the new definition has the advantage that it admits of extension to the θ -functions of two or more variables.

The elliptic functions $\operatorname{sn} u$, $\operatorname{cn} u$, $\operatorname{dn} u$, have thus been expressed each of them as the quotient of two θ -functions, but the question arises to express conversely a θ -function by means of the elliptic functions; the form is found to be

$$\theta u = C \exp \left(Au^2 + B \int_0^u \int_0^u \operatorname{sn}^2 u \, du^2 \right),$$

viz. θu is expressible as an exponential, the index of which depends on the double integral

$$\int_0^u \int_0^u \operatorname{sn}^2 u \, du^2.$$

The object has been to explain the general nature of the elliptic functions $\operatorname{sn} u$, $\operatorname{cn} u$, $\operatorname{dn} u$, and of the θ -functions with which they are thus intimately connected; it would be out of place to go into the theories of the multiplication, division, and transformation of the elliptic functions, or into the theory of the elliptic integrals, and of the application of the θ -functions to the representation of the elliptic integrals of the second and third kinds.

13. The reasoning which shows that for a doubly periodic function the ratio of the two periods 2ω , 2ν is imaginary shows that we cannot have a function of a single variable, which shall be triply periodic, or of any higher order of periodicity. For if the periods of a triply periodic function $\phi(u)$ were 2ω , 2ν , 2χ , then m , n , p being any positive or negative integer values, we should have $\phi(u + 2m\omega + 2n\nu + 2p\chi) = \phi u$; ω , ν , χ must be incommensurable, for if not, the three periods would really reduce themselves to two periods, or to a single period; and being incommensurable, it would be possible to determine the integers m , n , p in such manner that the real part and also the coefficient of i of the expression $m\omega + n\nu + p\chi$ shall be each of them as small as we please, say $\phi(u + \epsilon) = \phi u$, and thence $\phi(u + k\epsilon) = \phi u$ (k an integer), and $k\epsilon$ as near as we please to any given real or imaginary value whatever. We have thus the nugatory result $\phi u = a$ constant, or at least the function if not a constant is a function of an infinitely and perpetually discontinuous kind, a conception of which can hardly be formed. But a function of two variables may be triply or quadruply periodic—viz. we may have a function $\phi(u, v)$ having for u, v the simultaneous periods $2\omega, 2\omega'$; $2\nu, 2\nu'$; $2\chi, 2\chi'$; $2\psi, 2\psi'$; or, what is the same thing, it may be such that, m, n, p, q being any integers whatever, we have

$$\phi(u + 2m\omega + 2n\nu + 2p\chi + 2q\psi, v + 2m\omega' + 2n\nu' + 2p\chi' + 2q\psi') = \phi(u, v);$$

and similarly a function of $2n$ variables may be $2n$ -tuply periodic.

It is, in fact, in this manner that we pass from the elliptic functions and the single θ -functions to the hyperelliptic or Abelian functions and the multiple θ -functions; the case next succeeding the elliptic functions is when we have X, Y the same rational and integral sextic functions of x, y respectively, and then writing

$$\frac{dx}{\sqrt{X}} + \frac{dy}{\sqrt{Y}} = du, \quad \frac{x \, dx}{\sqrt{X}} + \frac{y \, dy}{\sqrt{Y}} = dv,$$

we regard certain symmetrical functions of x, y , in fact, the ratios of ($2^4 =$) 16 such symmetrical functions, as functions of (u, v) ; say we thus have 15 hyperelliptic functions $f(u, v)$, analogous to the 3 elliptic functions $\text{sn } u, \text{cn } u, \text{dn } u$, and being quadruply periodic. And these are the quotients of 16 double θ -functions $\theta(u, v)$, the general form being

$$\theta(u, v) = A \Sigma \Sigma \exp \left\{ \frac{1}{2} (a, h, b) (m, n)^2 + mu + nv \right\},$$

where the summations extend to all positive and negative integer values of (m, n) ; and we thus see the form of the θ -function for any number of variables whatever. The epithet "hyperelliptic" is used in the case where the differentials are of the form just mentioned $\frac{dx}{\sqrt{X}}$, where X is a rational and integral function of x ; the epithet "Abelian" extends to the more general case where the differential involves the irrational function of x , determined by any rational and integral equation $\phi(x, y) = 0$ whatever.

As regards the literature of the subject, it may be noticed that the various memoirs by Riemann, 1851—1866, are republished in the collected edition of his works, Leipsic, 1876; and shortly after his death we have the *Theorie der Abel'schen Functionen*, by Clebsch and Gordan, Leipsic, 1866. Preceding this, we have by MM. Briot and Bouquet, the *Théorie des Fonctions doublement périodiques et en particulier des Fonctions Elliptiques*, Paris, 1859, the results of which are reproduced and developed in their larger work, *Théorie des Fonctions Elliptiques*, 2nd ed., Paris, 1875.

14. It is proper to mention the gamma (Γ) or Π function, $\Gamma(n+1) = \Pi n, = 1.2.3\dots n$, when n is a positive integer. In the case just referred to, n a positive integer, this presents itself almost everywhere in analysis,—for instance, the binomial coefficients, and the coefficients of the exponential series are expressible by means of such functions of a number n . The definition for any real positive value of n is taken to be

$$\Gamma n = \int_0^{\infty} x^{n-1} e^{-x} dx;$$

it is then shown that, n being real and positive, $\Gamma(n+1) = n\Gamma n$, and by assuming that this equation holds good for positive or negative real values of n , the definition is extended to real negative values; the equation gives $\Gamma 1 = 0\Gamma 0$, that is, $\Gamma 0 = \infty$, and similarly $\Gamma(-n) = \infty$, where $-n$ is any negative integer. The definition by the definite integral has been or may be extended to imaginary values of n , but the theory is not an established one. A definition extending to all values of n is that of Gauss

$$\Pi n = \text{limit} \frac{1 \cdot 2 \cdot 3 \dots k}{n+1 \cdot n+2 \cdot n+3 \dots n+k} k^n,$$

the ultimate value of k being $= \infty$; but the function is chiefly considered for real values of the variable.

A formula for the calculation, when x has a large real and positive value, is

$$\Pi x = \sqrt{2\pi} x^{x+\frac{1}{2}} \exp \left(-x + \frac{1}{12x} + \dots \right),$$

or as this may also be written, neglecting the negative powers of x ,

$$\Pi x = \sqrt{2\pi} \exp \left\{ \left(x + \frac{1}{2} \right) \log x - x \right\}.$$

Another formula is $\Gamma x \Gamma (1-x) = \frac{\pi}{\sin \pi x}$: or, as this may also be written,

$$\Pi (x-1) \Pi (-x) = \frac{\pi}{\sin \pi x}.$$

It is to be observed that the function Π serves to express the product of a set of factors in arithmetical progression; we have

$$(x+a)(x+2a) \dots (x+ma) = a^m \left(\frac{x}{a} + 1 \right) \left(\frac{x}{a} + 2 \right) \dots \left(\frac{x}{a} + m \right) = a^m \Pi \left(\frac{x}{a} + m \right) \div \Pi \frac{x}{a}.$$

We can consequently express by means of it the product of any number of the factors which present themselves in the factorial expression of $\sin u$. Starting from the form

$$u \Pi_1^m \left(1 + \frac{u}{s\pi} \right) \Pi_1^n \left(1 - \frac{u}{s\pi} \right),$$

where Π is here as before the sign of a product of factors corresponding to the different integer values of s , this is thus converted into

$$u \Pi \left(\frac{u}{\pi} + m \right) \Pi \left(-\frac{u}{\pi} + n \right) \div \Pi \left(\frac{u}{\pi} \right) \Pi \left(-\frac{u}{\pi} \right) \Pi m \Pi n,$$

or as this may also be written,

$$\pi \Pi \left(\frac{u}{\pi} + m \right) \Pi \left(-\frac{u}{\pi} + n \right) \div \Pi \left(\frac{u}{\pi} - 1 \right) \Pi \left(-\frac{u}{\pi} \right) \Pi m \Pi n,$$

which, in virtue of

$$\Pi \left(\frac{u}{\pi} - 1 \right) \Pi \left(-\frac{u}{\pi} \right) = \frac{\pi}{\sin u},$$

becomes

$$= \sin u \Pi \left(\frac{u}{\pi} + m \right) \Pi \left(-\frac{u}{\pi} + n \right) \div \Pi m \Pi n.$$

Here m and n are large and positive; calculating the second factor by means of the formula for Πx , in this case we have the before-mentioned formula

$$u \Pi_1^m \left(1 + \frac{u}{s\pi} \right) \Pi_1^n \left(1 - \frac{u}{s\pi} \right) = \left(\frac{n}{m} \right)^{\frac{u}{\pi}} \sin u.$$

The gamma or Π function is the so-called second Eulerian integral; the first Eulerian integral

$$\int_0^1 x^{p-1} (1-x)^{q-1} dx, = \Gamma p \Gamma q \div \Gamma (p+q),$$

is at once expressible in terms of Γ , and is therefore not a new function to be considered.

15. We have the function defined by its expression as a hypergeometric series

$$F(\alpha, \beta, \gamma, u) = 1 + \frac{\alpha \cdot \beta}{1 \cdot \gamma} u + \frac{\alpha \cdot \alpha + 1 \cdot \beta \cdot \beta + 1}{1 \cdot 2 \cdot \gamma \cdot \gamma + 1} u^2 + \&c.,$$

i.e., this expression of the function serves as a definition, if the series be finite or if, being infinite, it is convergent. The function may also be defined as a definite integral; in other words, if, in the integral

$$\int_0^1 x^{\alpha-1} (1-x)^{\beta-1} (1-ux)^{-\gamma} dx,$$

we expand the factor $(1-ux)^{-\gamma}$ in powers of ux , and then integrate each term separately by the formula for the second Eulerian integral, the result is

$$= \frac{\Gamma \alpha' \cdot \Gamma \beta'}{\Gamma(\alpha' + \beta')} + \frac{\Gamma(\alpha' + 1) \cdot \Gamma \beta'}{\Gamma(\alpha' + \beta' + 1)} \frac{\gamma'}{1} u + \&c.,$$

which is

$$= \frac{\Gamma \alpha' \cdot \Gamma \beta'}{\Gamma(\alpha' + \beta')} \left\{ 1 + \frac{\alpha' \cdot \gamma'}{\alpha' + \beta' \cdot 1} u + \frac{\alpha' \cdot \alpha' + 1 \cdot \gamma' \cdot \gamma' + 1}{\alpha' + \beta' \cdot \alpha' + \beta' + 1 \cdot 1 \cdot 2} u^2 + \dots \right\}$$

or writing $\alpha', \beta', \gamma' = \alpha, \gamma - \alpha, \beta$ respectively, this is

$$= \frac{\Gamma \alpha \Gamma(\gamma - \alpha)}{\Gamma \gamma} F(\alpha, \beta, \gamma, u),$$

so that the new definition is

$$F(\alpha, \beta, \gamma, u) = \frac{\Gamma \alpha \Gamma(\gamma - \alpha)}{\Gamma \gamma} \int_0^1 x^{\alpha-1} (1-x)^{\beta-1} (1-ux)^{-\gamma} dx;$$

but this is in like manner only a definition under the proper limitations as to the values of α, β, γ, u . It is not here considered how the definition is to be extended so as to give a meaning to the function $F(\alpha, \beta, \gamma, u)$ for all values, say of the parameters α, β, γ , and of the variable u . There are included a large number of special forms which are either algebraic or circular or exponential, for instance $F(\alpha, \beta, \beta, u) = (1-u)^{-\alpha}$, &c.; or which are special transcendents which have been separately studied, for instance, Bessel's functions, the Legendrian functions X_n presently referred to, series occurring in the development of the reciprocal of the distance between two planets, &c.

16. There is a class of functions depending upon a variable or variables x, y, \dots and a parameter n , say the function for the parameter n is X_n such that the product of two functions having the same variables, multiplied it may be by a given function of the variables, and integrated between given limits, gives a result $= 0$ or not $= 0$, according as the parameters are unequal or equal; $\int UX_m X_n dx = 0$, but $\int UX_n^2 dx$ not $= 0$; the admissible values of the parameters being either any integer values, or it may be the roots of a determinate algebraical or transcendental equation; and the functions X_n may be either algebraical or transcendental. For instance, such a function is $\cos nx$; m and n being integers, we have $\int_0^\pi \cos mx \cdot \cos nx dx = 0$, but $\int_0^\pi \cos^2 nx dx = \frac{1}{2}\pi$.

Assuming the existence of the expansion of a function fx , in a series of multiple cosines, we thus obtain at once the well-known Fourier series, wherein the coefficient of $\cos mx$ is $=\frac{1}{2}\pi\int_0^\pi \cos mx \cdot fxdx$. The question whether the process is applicable is elaborately discussed in Riemann's memoir (1854), *Ueber die Darstellbarkeit einer Function durch eine trigonometrische Reihe*, No. XII. in the collected works. And again we have the Legendrian functions, which present themselves as the coefficients of the successive powers of α in the development of $(1-2\alpha x+\alpha^2)^{-\frac{1}{2}}$, $X_0=1$, $X_1=x$, $X_2=\frac{3}{2}(x^2-\frac{1}{3})$, &c.: here m, n being any positive integers, $\int_{-1}^1 X_m X_n dx = 0$, but $\int_{-1}^1 X_n^2 dx = \frac{2}{2n+1}$. And we have also Laplace's functions, &c.

Functions in General.

17. In what precedes, a review has been given, not by any means an exhaustive one, but embracing the most important kinds of known functions; but there are questions to be considered in regard to functions in general.

A function of $x+iy$ has been built up by means of analytical operations performed upon $x+iy$, $(x+iy)^2 = x^2 - y^2 + i \cdot 2xy$, &c., and the question next referred to has not arisen. But observe that, knowing $x+iy$, we know x and y , and therefore any two given functions $\phi(x, y)$, $\psi(x, y)$ of x and y : we therefore also know $\phi(x, y) + i\psi(x, y)$, and the question is, whether such a function of x, y (being known when $x+iy$ is known) is to be regarded as a function of $x+iy$; and if not, what is the condition to be satisfied in order that $\phi(x, y) + i\psi(x, y)$ may be a function of $x+iy$. Cauchy at one time considered that the general form was to be regarded as a function of $x+iy$, and he introduced the expression "fonction monogène," monogenous function, to denote the more restricted form which is the proper function of $x+iy$.

Consider for a moment the above general form, say $x'+iy' = \phi(x, y) + i\psi(x, y)$, where ϕ, ψ are any real functions of the real variables (x, y) ; or what is the same thing, assume $x' = \phi(x, y)$, $y' = \psi(x, y)$; if these functions have each or either of them more than one value, we attend only to one value of each of them. We may then as before take x, y to be the coordinates of a point P in a plane Π , and x', y' to be the coordinates of a point P' in a plane Π' . If, for any given values of x, y , the increments of $\phi(x, y)$, $\psi(x, y)$ corresponding to the indefinitely small real increments h, k of x, y be $Ah+Bk, Ch+Dk$, A, B, C, D being functions of x, y , then if the new coordinates of P are $x+h, y+k$, the new coordinates of P' will be $x'+Ah+Bk, y'+Ch+Dk$; or P, P' will respectively describe the indefinitely small straight paths at the inclinations $\tan^{-1}\frac{k}{h}, \tan^{-1}\frac{Ch+Dk}{Ah+Bk}$ to the axes of x, x' respectively; calling

these angles θ, θ' , we have therefore $\tan \theta' = \frac{C+D \tan \theta}{A+B \tan \theta}$. Now in order that $x'+iy'$

may be $=\phi(x+iy)$, a function of $x+iy$, the condition to be satisfied is that the increment of $x'+iy'$ shall be proportional to the increment $h+ik$ of $x+iy$, or say that it shall be $=(\lambda+i\mu)(h+ik)$, λ, μ being functions of x, y , but independent of

h, k ; we must therefore have $Ah + Bk, Ch + Dk = \lambda h - \mu k, \mu h + \lambda k$ respectively, that is $A, B, C, D = \lambda, -\mu, \mu, \lambda$ respectively, and the equation for $\tan \theta'$ thus becomes $\tan \theta' = \frac{\mu + \lambda \tan \theta}{\lambda - \mu \tan \theta}$; hence writing $\frac{\mu}{\lambda} = \tan \alpha$, where α is a function of x, y , but independent of h, k , we have $\tan \theta' = \frac{\tan \alpha + \tan \theta}{1 - \tan \alpha \tan \theta}$, that is, $\theta' = \alpha + \theta$; or for the given points $(x, y), (x', y')$, the path of P being at any inclination whatever θ to the axis of x , the path of P' is at the inclination $\theta + \text{constant angle } \alpha$ to the axis of x' ; also $(\lambda h - \mu k)^2 + (\mu h + \lambda k)^2 = (\lambda^2 + \mu^2)(h^2 + k^2)$, i.e., the lengths of the paths are in a constant ratio.

The condition may be written $\delta(x' + iy') = (\lambda + i\mu)(\delta x + i\delta y)$; or what is the same thing

$$\left(\frac{dx'}{dx} + i\frac{dy'}{dx}\right)\delta x + \left(\frac{dx'}{dy} + i\frac{dy'}{dy}\right)\delta y = (\lambda + i\mu)(\delta x + i\delta y),$$

that is,

$$\frac{dx'}{dx} + i\frac{dy'}{dx} = (\lambda + i\mu), \quad \frac{dx'}{dy} + i\frac{dy'}{dy} = i(\lambda + i\mu);$$

consequently

$$\frac{dx'}{dy} + i\frac{dy'}{dy} = i\left(\frac{dx'}{dx} + i\frac{dy'}{dx}\right);$$

that is,

$$\frac{dx'}{dy} = -\frac{dy'}{dx}, \quad \frac{dy'}{dy} = \frac{dx'}{dx},$$

as the analytical conditions in order that $x' + iy'$ may be a function of $x + iy$. They obviously imply

$$\frac{d^2x'}{dx^2} + \frac{d^2x'}{dy^2} = 0, \quad \frac{d^2y'}{dx^2} + \frac{d^2y'}{dy^2} = 0;$$

and if x' be a function of x, y , satisfying the first of these conditions, then

$$-\frac{dx'}{dy} dx + \frac{dx'}{dx} dy$$

is a complete differential, and

$$y' = \int \left(-\frac{dx'}{dy} dx + \frac{dx'}{dx} dy \right).$$

18. We have, in what just precedes, the ordinary behaviour of a function $\phi(x + iy)$ in the neighbourhood of the value $x + iy$ of the argument or point $x + iy$; or say the behaviour in regard to a point $x + iy$ such that the function is in the neighbourhood of this point a *continuous* function of $x + iy$ (or that the point is not a point of *discontinuity*): the correlative definition of continuity will be that the function $\phi(x + iy)$, assumed to have at the given point $x + iy$ a single finite value, is continuous in the neighbourhood of this point, when the point $x + iy$ describing continuously a straight infinitesimal element $h + ik$, the point $\phi(x + iy)$ describes continuously a straight infinitesimal element $(\lambda + i\mu)(h + ik)$; or what is really the same thing, when the function $(x + iy)$ has at the point $x + iy$ a differential coefficient.

19. It would doubtless be possible to give for the continuity of a function $\phi(x + iy)$ a less stringent definition not implying the existence of a differential coefficient; but we have this theory only in regard to the functions ϕx of a real variable in memoirs by Riemann, Hankel, du Bois Reymond, Schwarz, Gilbert, Klein, and Darboux. The last-mentioned geometer, in his "Mémoire sur les fonctions discontinues," *Jour. de l'École Normale*, t. iv. (1875), pp. 57—112, gives (after Bonnet) the following definition of a continuous function (observe that we are now dealing with real quantities only):—the function $f(x)$ is continuous for the value $x = x_0$ when, h and ϵ being positive quantities as small as we please and θ any positive quantity at pleasure between 0 and 1, we have, for all the values of θ , $f(x_0 \pm \theta h) - f(x)$ less in absolute magnitude than ϵ ; and moreover $f(x)$ is continuous through the interval x_0, x_1 ($x_1 > x_0$, that is, nearer $+\infty$) when $f(x)$ is continuous for every value of x between x_0 and x_1 , and, h tending to zero through positive values, $f(x_0 + h)$ and $f(x_0 - h)$ tend to the limits $f(x_0), f(x_1)$ respectively. It is possible, consistently with this definition, to form continuous functions not having in any proper sense a differential coefficient, and having other anomalous properties; thus if a_1, a_2, a_3, \dots be an infinite series of real positive or negative quantities, such that the series Σa_n is absolutely convergent (i.e. the sum $\Sigma \pm a_n$, each term being made positive, is convergent), then the function $\Sigma a_n (\sin n\pi x)^{\frac{2}{3}}$ is a continuous function actually calculable for any assumed value of x ; but it is shown in the memoir that, taking $x =$ any commensurable value $\frac{p}{q}$ whatever, and then writing $x = \frac{p}{q} + h$, h indefinitely small, the increment of the function is of the form $(k + \epsilon)h^{\frac{2}{3}}$, k finite, ϵ an indefinitely small quantity vanishing with h ; there is thus no term varying with h , nor consequently any differential coefficient. See also Riemann's Memoir, *Ueber die Darstellbarkeit*, &c. (No. XII. in the collected works), already referred to.

20. It was necessary to allude to the foregoing theory of (as they may be termed) infinitely discontinuous functions; but the ordinary and most important functions of analysis are those which are continuous, except for a finite number (or it may be an infinite number) of points of discontinuity. It is to be observed that a point at which the function becomes infinite is *ipso facto* a point of discontinuity; a value of the variable for which the function becomes infinite is, as already mentioned, said to be an "infinity" (or a "pole") of the function; thus, in the case of a rational function expressed as a fraction in its least terms, if the denominator contains a factor $(x - a)^m$, a a real or imaginary value, m a positive integer, then a is said to be an infinity of the m th order (and in the particular case $m = 1$, it is said to be a simple infinity). The circular functions $\tan x, \sec x$ are instances of a function having an infinite number of simple infinities.

A rational function is a one-valued function, and in regard to a rational function the infinities are the only points of discontinuity; but a one-valued function may have points of discontinuity of a character quite distinct from an infinity: for instance, in the exponential function $\exp\left(\frac{1}{u - a}\right)$ where a is real or imaginary, the value $u (= x + iy) = a$, is a point of discontinuity but not an infinity; taking $u = a + \rho e^{ai}$, where ρ is an

indefinitely small real positive quantity, the value of the function is $\exp\left(\frac{1}{\rho}e^{-\alpha i}\right)$,
 $= \exp\frac{1}{\rho}(\cos \alpha - i \sin \alpha)$, which is indefinitely large or indefinitely small according as $\cos \alpha$
 is positive or negative, and in the separating case $\cos \alpha = 0$, and therefore $\sin \alpha = \pm 1$,
 it is $= \cos\frac{1}{\rho} \pm i \sin\frac{1}{\rho}$ which is indeterminate. If, instead of $\exp\frac{1}{u-a}$, we consider a
 linear function

$$\left\{A + B \exp\frac{1}{u-a}\right\} \div \left\{C + D \exp\frac{1}{u-a}\right\},$$

then writing as before $u = a + \rho e^{\alpha i}$, the value is $= A \div C$, or $= B \div D$, according as $\cos \alpha$
 is negative or positive. As regards the theory of one-valued functions in general, the
 memoir by Weierstrass, "Zur Theorie der eindeutigen analytischen Functionen," *Berl.*
Abh. 1876, pp. 11—60, may be referred to.

21. A one-valued function *ex vi termini* cannot have a point of discontinuity of
 the kind next referred to; if the representative point P , moving in any manner
 whatever, returns to its original position, the corresponding point P' cannot but return
 to its original position. But consider a many-valued function, say an n -valued function
 $x' + iy'$, of $x + iy$; to each position of P there correspond n positions, in general all
 of them different, of P' . But the point P may be such that (to take the most simple
 case) two of the corresponding points P' coincide with each other, say such a position
 of P is at V , then (using for greater distinctness a different letter W' instead of
 V') corresponding thereto we have two coincident points (W'), and $n-2$ other points
 W' ; V is then a branch-point (Verzweigungspunkt). Taking for P any point which
 is not a branch-point, then in the neighbourhood of this value each of the n functions
 $x' + iy'$ is a continuous function of $x + iy$, and by what precedes, if P describing an
 infinitely small closed curve (or oval) return to its original position, then each of the
 corresponding points P' describing a corresponding indefinitely small oval will return
 to its original position. But imagine the oval described by P to be gradually enlarged,
 so that it comes to pass through a branch-point V ; the ovals described by two of
 the corresponding points P' will gradually approach each other, and will come to unite
 at the point (W'), each oval then sharpening itself out so that the two form together
 a figure of eight. And if we imagine the oval described by P to be still further
 enlarged so as to include within it the point V , then the figure of eight, losing
 the crossing, will be at first an hour-glass form, or twice-indented oval, and ultimately
 in form an ordinary oval, but having the character of a twofold oval; i.e. to the
 oval described by P (and which surrounds the branch-point V) there will correspond
 this twofold oval, and $n-2$ onefold ovals, in such wise that to a given position of
 P on its oval there correspond two points, say P'_1, P'_2 , on the twofold oval, and
 $n-2$ points P'_3, \dots, P'_n , each on its own onefold oval. And then as P describing its
 oval returns to its original position, the point P'_1 describing a portion only of the
 twofold oval, will pass to the original position of P'_2 , while the point P'_2 describing
 the remaining portion of the twofold oval will pass to the original position of P'_1 ;
 the other points P'_3, \dots, P'_n , describing each of them its own onefold oval, will

return each of them to its original position. And it is easy to understand how, when the oval described by P surrounds two or more of the branch-points V , the corresponding curves for P' may be a system of manifold ovals, such that the sum of the manifoldness is always $=n$, and to conceive in a general way the behaviour of the corresponding points P and P' .

Writing for a moment $x + iy = u$, $x' + iy' = v$, the branch-points are the points of contact of parallel tangents to the curve $\phi(u, v) = 0$, a line through a cusp (but not a line through a node), being reckoned as a tangent; that is, if this be a curve of the order n and class m , with δ nodes and κ cusps, the number of branch-points is $=m + \kappa$, that is, it is $=n^2 - n - 2\delta - 2\kappa$, or if $p = \frac{1}{2}(n-1)(n-2) - \delta - \kappa$, be the deficiency, then the number is $=2n - 2 + 2p$.

To illustrate the theory of the n -valued algebraical function $x' + iy'$ of the complex variable $x + iy$, Riemann introduces the notion of a surface composed of n coincident planes or sheets, such that the transition from one sheet to another is made at the branch-points, and that the n sheets form together a multiply-connected surface, which can be by cross-cuts (Querschnitte) converted into a simply-connected surface; the n -valued function $x' + iy'$ becomes thus a one-valued function of $x + iy$, considered as belonging to a point on some determinate sheet of the surface: and upon such considerations he finds the whole theory of the functions which arise from the *integration* of the differential expressions depending on the n -valued algebraical function (that is, any irrational algebraical function whatever) of the independent variable, establishing as part of the investigation the theory of the multiple θ -functions. But it would be difficult to give a further account of these investigations.

The Calculus of Functions.

22. The so-called Calculus of Functions, as considered chiefly by Herschel and Babbage and De Morgan, is not so much a theory of functions as a theory of the solution of functional equations; or, as perhaps should rather be said, the solution of functional equations by means of known functions, or symbols,—the epithet known being here used in reference to the actual state of analysis. Thus for a functional equation $\phi x + \phi y = \phi(xy)$, taking the logarithm as a known function, the solution is $\phi x = c \log x$; or if the logarithm is not taken to be a known function, then a solution

may be obtained by means of the sign of integration $\phi x = c \int \frac{dx}{x}$; but the establishment of the properties of the function logarithm (assumed to be previously unknown) would not be considered as coming within the theory. A class of equations specially considered is where $\alpha x, \beta x, \dots$ being given functions of x , the unknown function ϕ is to be determined by means of a given relation between $x, \phi x, \phi \alpha x, \phi \beta x, \dots$; in particular the given relation may be between $x, \phi x, \phi \alpha x$; this can be at once reduced to equations of finite differences; for writing $x = u_n, \alpha x = u_{n+1}$, we have $u_{n+1} = \alpha u_n$, giving u_n , and therefore also x , each of them as a function of n ; and then writing $\phi x = v_n, \phi \alpha x$ will be the same function of $n+1, = v_{n+1}$, and the given relation is again an equation of finite differences in v_{n+1}, v_n , and n ; we have thus $v_n = \phi x$,

as a function of n , that is, of x . As regards the equation $u_{n+1} = \alpha u_n$, considered in itself apart from what precedes, observe that this is satisfied by writing $u_n = \alpha^n(x)$, or the question of solving this equation of finite differences is, in fact, identical with that of finding the n th function $\alpha^n(x)$, where $\alpha(x)$ is a given function of x . It of course depends on the form of $\alpha(x)$ whether this question admits of solution in any proper sense; thus, for a function such as $\log x$, the n th logarithm is expressible in its original function $\log^n x$, ($= \log \log \dots x$), and not in any other form. But there are forms, for instance $\alpha x = \frac{a + bx}{c + dx}$, where the n th function $\alpha^n x$ is a function of the

like form $\alpha^n x = \frac{A + Bx}{C + Dx}$, in which the actual value can be expressed as a function of n ; if α be such a form, then $\phi \alpha \phi^{-1}$, whatever ϕ may be, is a like form, for we obviously have $(\phi \alpha \phi^{-1})^n = \phi \alpha^n \phi^{-1}$. The determination of the n th function is, in fact, a leading question in the calculus of functions.

It is to be observed that considering the case of two variables, if for instance $\alpha(x, y)$ denote a given function of x, y , the notation $\alpha^2(x, y)$ is altogether meaningless; in order to generalize the question, we require an extended notation wherein a single functional symbol is used to denote two functions of the two variables. Thus $\phi(x, y) = \alpha(x, y), \beta(x, y), \alpha$ and β given functions; writing for shortness $x_1 = \alpha(x, y), y_1 = \beta(x, y)$, then $\phi^2(x, y)$ will denote $\phi(x_1, y_1)$, that is, two functions $\alpha(x_1, y_1), \beta(x_1, y_1)$, say these are x_2, y_2 ; $\phi^3(x, y)$ will denote $\phi(x_2, y_2)$, and so on, so that $\phi^n(x, y)$ will have a determinate meaning. And the like is obviously the case in regard to any number of variables, the single functional symbol denoting in each case a set of functions equal in number to the variables.