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[ADDITION TO MR HUDSON'S PAPER "ON EQUAL ROOTS OF EQUATIONS."]

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It seems desirable to present in a more developed form some of the results of the foregoing paper.

Thus, if the equation $(a_0, a_1, \ldots, a_n \between x, 1)^n = 0$ of the order n has $n - v$ equal roots, where v is not $> \frac{1}{2}n - 1$, then we have $\psi(r, v+1, m) = 0$, where m has any one of the values $0, 1, \ldots, n - 2v - 2$, and r any one of the values

$$2v + 2,\ 2v + 3,\ \ldots,\ n - m.$$

The signification is

$$\begin{aligned}
\psi(r, v+1, m) = \quad & r && . \quad 1 && . \frac{1}{[r]^{v+2}} && a_m && a_{r+m} \\
& -(r-2) && . \frac{v+1}{1} && . \frac{1}{[r-1]^{v+2}} && a_{m+1} && a_{r+m-1} \\
& +(r-4) && . \frac{v+1\,.\,v+2}{1\,.\,2} && . \frac{1}{[r-2]^{v+2}} && a_{m+2} && a_{r+m-2} \\
& \quad \vdots \\
& +(-)^s (r-2s) && . \frac{[v+1]^s}{[s]^s} && . \frac{1}{[r-s]^{v+2}} && a_{m+s} && a_{r+m-s} \\
& \quad \vdots \\
& +(-)^{v+1}(r-2v-2) && . && . \frac{1}{[r-v-1]^{v+2}} && a_{m+v+1} && a_{r+m-v-1}.
\end{aligned}$$

Thus, when $v = 0$, the condition is

$$\left.\begin{aligned}
& r \quad \frac{1}{r\,.\,r-1} \quad a_m \quad a_{r+m} \\
& -(r-2)\,\frac{1}{r-1\,.\,r-2}\, a_{m+1} a_{r+m-1}
\end{aligned}\right\} = 0,$$

that is,

$$a_m a_{r+m} - a_{m+1} a_{r+m-1} = 0,$$

satisfied when the equation has all its roots equal.

The values of m are 0, 1, 2, ..., $n-2$, and those of r are $2v+2$, $2v+3$, ..., $n-m$; in particular, if $m=0$, the values of r are 2, 3, ..., n, and the corresponding conditions are

$$\begin{aligned} a_0 a_2 - a_1^2 \quad &= 0, \\ a_0 a_3 - a_1 a_2 \quad &= 0, \\ &\vdots \\ a_0 a_n - a_1 a_{n-1} &= 0, \end{aligned}$$

and so for the different values of m up to the final value $n-2$, for which $r=2$, and the condition is

$$a_{n-2} a_n - a^2{}_{n-1} = 0\,;$$

we have thus, it is clear, the whole series of conditions included in

$$\left\| \begin{matrix} a_0, & a_1, & a_2, & \ldots, & a_{n-2}, & a_{n-1} \\ a_1, & a_2, & a_3, & \ldots, & a_{n-1}, & a_n \end{matrix} \right\| = 0,$$

which are obviously satisfied in the case in question of the roots being all equal.

Again, when $v=1$, the condition for $n-1$ equal roots is

$$\left.\begin{aligned} r \qquad . \,1\,. \frac{1}{r\,.\,r-1\,.\,r-2} \quad & a_m \;\; a_{r+m} \\ -(r-2)\,.\,2\,. \frac{1}{r-1\,.\,r-2\,.\,r-3} & a_{m+1} a_{r+m-1} \\ +(r-4)\,.\,1\,. \frac{1}{r-2\,.\,r-3\,.\,r-4} & a_{m+2} a_{r+m-2} \end{aligned}\right\} = 0,$$

that is,

$$\frac{a_m a_{r+m}}{r-1\,.\,r-2} - \frac{2a_{m+1} a_{r+m-1}}{r-1\,.\,r-3} + \frac{a_{m+2} a_{r+m-2}}{r-2\,.\,r-3} = 0\,;$$

or, what is the same thing,

$$(r-3)\, a_m a_{r+m} - 2\,(r-2)\, a_{m+1} a_{r+m-1} + (r-1)\, a_{m+2} a_{r+m-2} = 0,$$

where $n=4$ at least, and m, r have the values

$$\begin{array}{r|l} m = & 0,\ 1,\ 2,\ \ldots,\ n-4 \\ \hline r = & 4,\ 4, \qquad\qquad\quad 4 \\ & 5,\ 5 \\ & \vdots\ \ \vdots \\ & \vdots\ \ n-1 \\ & n \end{array}$$

thus, when $n=4$, the only values are $m=0$, $r=4$, and the condition is

$$a_0 a_4 - 4a_1 a_3 + 3a_2^2 = 0.$$

Similarly, when $v=2$, the condition for $n-2$ equal roots is found to be

$$\frac{a_m a_{r+m}}{r-1\,.\,r-2\,.\,r-3}-\frac{3a_{m+1}a_{r+m-1}}{r-1\,.\,r-3\,.\,r-4}+\frac{3a_{m+2}a_{r+m-2}}{r-2\,.\,r-3\,.\,r-5}-\frac{a_{m+3}a_{r+m-3}}{r-3\,.\,r-4\,.\,r-5}=0\,;$$

or, what is the same thing,

$$\begin{aligned} &r-4\,.\,r-5\,.\,a_m\ \ a_{r+m}\\ -3\,.\,&r-2\,.\,r-5\,.\,a_{m+1}a_{r+m-1}\\ +3\,.\,&r-1\,.\,r-4\,.\,a_{m+2}a_{r+m-2}\\ -\ \ \,.\,&r-1\,.\,r-2\,.\,a_{m+3}a_{r+m-3}=0, \end{aligned}$$

where $n=6$ at least, and m, r have the values

$m=$	$0,\ 1,\ \ldots,\ n-6$
$r=$	$6,\ 6,\qquad 6$
	$7,\ 7$
	$\vdots\ \ \vdots$
	$\vdots\ \ n-1$
	n

Observe that the sum of the coefficients is $=0$, viz.

$$(r-4)(r-5)-3(r-2)(r-5)+3(r-1)(r-4)-(r-1)(r-2)=0,$$

this should obviously be the case, since the conditions for $n-2$ equal roots must be satisfied when the roots are all of them equal; and the property serves as a verification.

It is to be remarked that the equation $\psi(r,\ v+1,\ m)=0$ does not in all cases give all the conditions for the existence of $n-v$ equal roots in an equation of the order n; thus when $n=3$ and $v=1$, we cannot by means of it obtain the condition that a cubic equation may have 2 equal roots. The problem really considered is that of the determination of those *quadric* functions of the coefficients which vanish in the case of $n-v$ equal roots; and in the case in question ($n=3$, $v=1$) there is no quadric function which vanishes, but the condition depends on a cubic function.

The question of the quadric functions which vanish in the case of $n-v$ equal roots, and to a small extent that of the *cubic* functions which thus vanish, is considered in Dr Salmon's "Note on the conditions that an equation may have equal roots," *Camb. and Dublin Math. Jour.*, t. v. (1850), pp. 159—165, and in particular the equation there obtained p. 161 is the equation $\psi(0,\ v+1,\ n)=0$.