773.

ON THE 8-SQUARE IMAGINARIES.

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I WRITE throughout **0** to denote positive unity, and uniting with it the seven imaginaries **1**, ..., **7**, form an octavic system **0**, **1**, **2**, **3**, **4**, **5**, **6**, **7**, the laws of combination being

$$\mathbf{0}^2=\mathbf{0},\quad \mathbf{1}^2=\mathbf{2}^2=\mathbf{3}^2=\mathbf{4}^2=\mathbf{5}^2=\mathbf{6}^2=\mathbf{7}^2=-\mathbf{0},$$
$$\mathbf{123}=\epsilon_1,\quad \mathbf{145}=\epsilon_2,\quad \mathbf{167}=\epsilon_3,$$
$$\mathbf{246}=\epsilon_4,\quad \mathbf{257}=\epsilon_5,$$
$$\mathbf{347}=\epsilon_6,\quad \mathbf{356}=\epsilon_7,$$

where $\epsilon=\pm$, viz. each ϵ has a determinate value + or − as the case may be; and where the formula, $\mathbf{123}=\epsilon_1$, denotes the six equations

$$\mathbf{23}=\ \epsilon_1\mathbf{1},\quad \mathbf{31}=\ \epsilon_1\mathbf{2},\quad \mathbf{12}=\ \epsilon_1\mathbf{3},$$
$$\mathbf{32}=-\epsilon_1\mathbf{1},\quad \mathbf{13}=-\epsilon_1\mathbf{2},\quad \mathbf{21}=-\epsilon_1\mathbf{3},$$

and so for the other formulæ. The multiplication table of the eight symbols thus is

	0	1	2	3	4	5	6	7
0	0	1	2	3	4	5	6	7
1	1	-0	$\epsilon_1 3$	$-\epsilon_1 2$	$\epsilon_2 5$	$-\epsilon_2 4$	$\epsilon_3 7$	$-\epsilon_3 6$
2	2	$-\epsilon_1 3$	-0	$\epsilon_1 1$	$\epsilon_4 6$	$\epsilon_5 7$	$-\epsilon_4 4$	$-\epsilon_5 5$
3	3	$\epsilon_1 2$	$-\epsilon_1 1$	-0	$\epsilon_6 7$	$\epsilon_7 6$	$-\epsilon_7 5$	$-\epsilon_6 4$
4	4	$-\epsilon_2 5$	$-\epsilon_4 6$	$-\epsilon_6 7$	-0	$\epsilon_2 1$	$\epsilon_4 2$	$\epsilon_6 3$
5	5	$\epsilon_2 4$	$-\epsilon_5 7$	$-\epsilon_7 6$	$-\epsilon_2 1$	-0	$\epsilon_7 3$	$\epsilon_5 2$
6	6	$-\epsilon_3 7$	$\epsilon_4 4$	$\epsilon_7 5$	$-\epsilon_4 2$	$-\epsilon_7 3$	-0	$\epsilon_3 1$
7	7	$\epsilon_3 6$	$\epsilon_5 5$	$\epsilon_6 4$	$-\epsilon_6 3$	$-\epsilon_5 2$	$-\epsilon_3 1$	-0

Hence if 0, 1, 2, 3, 4, 5, 6, 7 and 0′, 1′, 2′, 3′, 4′, 5′, 6′, 7′ denote ordinary algebraical magnitudes, and we form the product

$$(\mathbf{0}0+\mathbf{1}1+\mathbf{2}2+\mathbf{3}3+\mathbf{4}4+\mathbf{5}5+\mathbf{6}6+\mathbf{7}7)(0'\mathbf{0}+1'\mathbf{1}+2'\mathbf{2}+3'\mathbf{3}+4'\mathbf{4}+5'\mathbf{5}+6'\mathbf{6}+7'\mathbf{7}),$$

this is at once found to be =

$$\begin{aligned}
&(00'-11'-22'-33'-44'-55'-66'-77')\,\mathbf{0}\\
&+(01'+0'1+\epsilon_1 23+\epsilon_2 45+\epsilon_3 67)\,\mathbf{1}\\
&+(02'+0'2+\epsilon_1 31+\epsilon_4 46+\epsilon_5 57)\,\mathbf{2}\\
&+(03'+0'3+\epsilon_1 12+\epsilon_6 47+\epsilon_7 56)\,\mathbf{3}\\
&+(04'+0'4+\epsilon_1 51+\epsilon_4 62+\epsilon_6 73)\,\mathbf{4}\\
&+(05'+0'5+\epsilon_2 14+\epsilon_5 72+\epsilon_7 63)\,\mathbf{5}\\
&+(06'+0'6+\epsilon_3 71+\epsilon_4 24+\epsilon_7 35)\,\mathbf{6}\\
&+(07'+0'7+\epsilon_3 16+\epsilon_5 25+\epsilon_6 34)\,\mathbf{7},
\end{aligned}$$

where 12 is written to denote $12'-1'2$, and so in other cases.

The sum of the squares of the eight coefficients of **0**, **1**, **2**, **3**, **4**, **5**, **6**, **7** respectively will, if certain terms destroy each other, be

$$=(0^2+1^2+2^2+3^2+4^2+5^2+6^2+7^2)(0'^2+1'^2+2'^2+3'^2+4'^2+5'^2+6'^2+7'^2);$$

viz. the sum of the squares contains the several terms

$$\begin{array}{lllllll}
\epsilon_1\epsilon_2 23.45, & \epsilon_1\epsilon_3 23.67, & \epsilon_1\epsilon_4 31.46, & \epsilon_1\epsilon_5 31.57, & \epsilon_1\epsilon_6 12.47, & \epsilon_1\epsilon_7 12.56, & \epsilon_2\epsilon_3 45.67,\\
\epsilon_4\epsilon_7 24.35, & \epsilon_4\epsilon_6 62.73, & \epsilon_2\epsilon_7 14.63, & \epsilon_2\epsilon_6 51.73, & \epsilon_2\epsilon_5 14.72, & \epsilon_2\epsilon_4 51.62, & \epsilon_4\epsilon_5 46.57,\\
\epsilon_5\epsilon_6 25.34, & \epsilon_5\epsilon_7 72.63, & \epsilon_3\epsilon_6 16.34, & \epsilon_3\epsilon_7 71.35, & \epsilon_3\epsilon_4 71.24, & \epsilon_3\epsilon_5 16.25, & \epsilon_6\epsilon_7 47.56,
\end{array}$$

and observing that $\mathbf{21}=-\mathbf{12}$, etc., and that we have identically

$$\mathbf{23}.\mathbf{45}+\mathbf{24}.\mathbf{53}+\mathbf{25}.\mathbf{34}=\text{zero, etc.,}$$

then the three terms of each column will vanish, provided a proper relation exists between the ϵ's: viz. the conditions which we thus obtain are

$$\begin{aligned}
\epsilon_1\epsilon_2 &= -\epsilon_4\epsilon_7 = \epsilon_5\epsilon_6,\\
\epsilon_1\epsilon_3 &= -\epsilon_4\epsilon_6 = \epsilon_5\epsilon_7,\\
\epsilon_1\epsilon_4 &= -\epsilon_3\epsilon_6 = -\epsilon_2\epsilon_7,\\
\epsilon_1\epsilon_5 &= \epsilon_3\epsilon_7 = \epsilon_2\epsilon_6,\\
\epsilon_1\epsilon_6 &= \epsilon_2\epsilon_5 = -\epsilon_3\epsilon_4,\\
\epsilon_1\epsilon_7 &= -\epsilon_2\epsilon_4 = \epsilon_3\epsilon_5,\\
\epsilon_2\epsilon_3 &= -\epsilon_4\epsilon_5 = \epsilon_6\epsilon_7.
\end{aligned}$$

We may without loss of generality assume $\epsilon_1=\epsilon_2=\epsilon_3=+$; the equations then become

$$\begin{aligned}
+ &= -\epsilon_4\epsilon_7 = \epsilon_5\epsilon_6,\\
+ &= -\epsilon_4\epsilon_6 = \epsilon_5\epsilon_7,\\
+ &= -\epsilon_4\epsilon_5 = \epsilon_6\epsilon_7,\\
\epsilon_4 &= -\epsilon_6 = -\epsilon_7,\\
\epsilon_5 &= \epsilon_7 = \epsilon_6,\\
\epsilon_6 &= \epsilon_5 = -\epsilon_4,\\
\epsilon_7 &= -\epsilon_4 = \epsilon_5;
\end{aligned}$$

and writing $\theta = \pm$ at pleasure, these are all satisfied if $-\epsilon_4 = \epsilon_5 = \epsilon_6 = \epsilon_7 = \theta$. The terms written down all disappear, and the sum of the squares of the eight coefficients thus becomes equal to the product of two sums each of them of eight squares, viz. this is the case if $\epsilon_1 = \epsilon_2 = \epsilon_3 = +$, $-\epsilon_4 = \epsilon_5 = \epsilon_6 = \epsilon_7 = \theta$, θ being $= \pm$ at pleasure: the resulting system of imaginaries may be said to be an 8-square system.

We may inquire whether the system is associative; for this purpose, supposing in the first instance that the ϵ's remain arbitrary, we form the complete system of the values of the triplets **12 . 3**, **1 . 23**, etc., (read the top line **12 . 3** $= -\epsilon_1$**0**, **1 . 23** $= -\epsilon_1$**0**, the next line **12 . 4** $= \epsilon_1\epsilon_6$**7**, **1 . 24** $= \epsilon_3\epsilon_4$**7**, and so in other cases):

12 . 3 =	1 . 23 =	$-\epsilon_1$	$-\epsilon_1$	0
12 . 4 =	1 . 24 =	$\epsilon_1\epsilon_6$	$\epsilon_3\epsilon_4$	7
12 . 5 =	1 . 25 =	$\epsilon_1\epsilon_7$	$-\epsilon_3\epsilon_5$	6
12 . 6 =	1 . 26 =	$-\epsilon_1\epsilon_7$	$-\epsilon_2\epsilon_4$	5
12 . 7 =	1 . 27 =	$-\epsilon_1\epsilon_6$	$\epsilon_2\epsilon_5$	4
13 . 4 =	1 . 34 =	$-\epsilon_1\epsilon_4$	$-\epsilon_3\epsilon_6$	6
13 . 5 =	1 . 35 =	$-\epsilon_1\epsilon_5$	$\epsilon_3\epsilon_7$	7
13 . 6 =	1 . 36 =	$\epsilon_1\epsilon_4$	$\epsilon_2\epsilon_7$	4
13 . 7 =	1 . 37 =	$\epsilon_1\epsilon_5$	$-\epsilon_2\epsilon_6$	5
14 . 5 =	1 . 45 =	$-\epsilon_2$	$-\epsilon_2$	0
14 . 6 =	1 . 46 =	$\epsilon_2\epsilon_7$	$\epsilon_1\epsilon_4$	3
14 . 7 =	1 . 47 =	$\epsilon_2\epsilon_5$	$-\epsilon_1\epsilon_6$	2
15 . 6 =	1 . 56 =	$-\epsilon_2\epsilon_4$	$-\epsilon_1\epsilon_7$	2
15 . 7 =	1 . 57 =	$-\epsilon_2\epsilon_6$	$\epsilon_1\epsilon_5$	3
16 . 7 =	1 . 67 =	$-\epsilon_3$	$-\epsilon_3$	0
23 . 4 =	2 . 34 =	$\epsilon_1\epsilon_2$	$-\epsilon_5\epsilon_6$	5
23 . 5 =	2 . 35 =	$-\epsilon_1\epsilon_2$	$-\epsilon_4\epsilon_7$	4
23 . 6 =	2 . 36 =	$\epsilon_1\epsilon_3$	$-\epsilon_5\epsilon_7$	7
23 . 7 =	2 . 37 =	$-\epsilon_1\epsilon_3$	$-\epsilon_4\epsilon_6$	6
24 . 5 =	2 . 45 =	$-\epsilon_4\epsilon_7$	$-\epsilon_1\epsilon_2$	3
24 . 6 =	2 . 46 =	$-\epsilon_4$	$-\epsilon_4$	0
24 . 7 =	2 . 47 =	$\epsilon_3\epsilon_4$	$\epsilon_1\epsilon_6$	1
25 . 6 =	2 . 56 =	$-\epsilon_3\epsilon_5$	$\epsilon_1\epsilon_7$	1
25 . 7 =	2 . 57 =	$-\epsilon_5$	$-\epsilon_5$	0
26 . 7 =	2 . 67 =	$-\epsilon_4\epsilon_6$	$-\epsilon_1\epsilon_3$	3
34 . 5 =	3 . 45 =	$-\epsilon_5\epsilon_6$	$\epsilon_1\epsilon_2$	2
34 . 6 =	3 . 46 =	$-\epsilon_3\epsilon_6$	$-\epsilon_1\epsilon_4$	1
34 . 7 =	3 . 47 =	$-\epsilon_6$	$-\epsilon_6$	0
35 . 6 =	3 . 56 =	$-\epsilon_7$	$-\epsilon_7$	0
35 . 7 =	3 . 57 =	$\epsilon_3\epsilon_7$	$-\epsilon_1\epsilon_5$	1
36 . 7 =	3 . 67 =	$-\epsilon_5\epsilon_7$	$\epsilon_1\epsilon_3$	2
45 . 6 =	4 . 56 =	$\epsilon_2\epsilon_3$	$-\epsilon_6\epsilon_7$	7
45 . 7 =	4 . 57 =	$-\epsilon_2\epsilon_3$	$-\epsilon_4\epsilon_5$	6
46 . 7 =	4 . 67 =	$-\epsilon_4\epsilon_5$	$-\epsilon_2\epsilon_3$	5
56 . 7 =	5 . 67 =	$-\epsilon_6\epsilon_7$	$\epsilon_2\epsilon_3$	4.

Write as before $\epsilon_1 = \epsilon_2 = \epsilon_3 = +$; then, disregarding the lines (such as the first line) which contain the symbol **0**, and writing down only the signs as given in the third and fourth columns, these are

ϵ_6	ϵ_4
ϵ_7	$-\epsilon_5$
$-\epsilon_7$	$-\epsilon_4$
$-\epsilon_6$	ϵ_5
$-\epsilon_4$	$-\epsilon_6$
$-\epsilon_5$	ϵ_7
ϵ_4	ϵ_7
ϵ_5	$-\epsilon_6$
ϵ_7	ϵ_4
ϵ_5	$-\epsilon_6$
$-\epsilon_4$	$-\epsilon_7$
$-\epsilon_6$	ϵ_5
$+$	$-\epsilon_5\epsilon_6$
$-$	$-\epsilon_4\epsilon_7$
$+$	$-\epsilon_5\epsilon_7$
$-$	$-\epsilon_4\epsilon_6$
$-\epsilon_4\epsilon_7$	$-$
ϵ_4	ϵ_6
$-\epsilon_5$	ϵ_7
$-\epsilon_4\epsilon_6$	$-$
$-\epsilon_5\epsilon_6$	$+$
$-\epsilon_6$	$-\epsilon_4$
ϵ_7	$-\epsilon_5$
$-\epsilon_5\epsilon_7$	$+$
$+$	$-\epsilon_6\epsilon_7$
$-$	$-\epsilon_4\epsilon_5$
$-\epsilon_4\epsilon_5$	$-$
$-\epsilon_6\epsilon_7$	$+$

We hence see at once that the pairs of signs in the two columns respectively cannot be made identical: to make them so, we should have $\epsilon_6 = \epsilon_4$, $\epsilon_7 = -\epsilon_5$, $\epsilon_7 = \epsilon_4$, that is, $\epsilon_4 = \epsilon_6 = \epsilon_7 = -\epsilon_5$, which is inconsistent with the last equation of the system $-\epsilon_6\epsilon_7 = +$. Hence the imaginaries **1**, **2**, **3**, **4**, **5**, **6**, **7**, as defined by the original conditions, are not in any case associative.

If we have $\epsilon_1 = \epsilon_2 = \epsilon_3 = +$ and also $-\epsilon_4 = \epsilon_5 = \epsilon_6 = \epsilon_7 = \theta$, that is, if the imaginaries belong to the 8-square formula, then it is at once seen that each pair consists of two opposite signs; that is, for the several triads **123**, **145**, **167**, **246**, **257**, **347**, **356** used for the definition of the imaginaries, the associative property holds good, **12 . 3 = 1 . 23**, etc.; but for each of the remaining twenty-eight triads, *the two terms are equal but of opposite signs*, viz. **12 . 4 = − 1 . 24**, etc.; so that the product **124** of any such three symbols has no determinate meaning.

Baltimore, March 5th, 1882.