954.

ON THE NINE-POINTS CIRCLE OF A PLANE TRIANGLE.

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I CONSIDER the circle which meets the sides of a triangle ABC in the points F, L; G, M; H, N respectively, where ultimately F, G, H are the feet of the perpendiculars let fall from the angles on the opposite sides, and L, M, N are the mid-points of the sides: but in the first instance, they are taken to be arbitrary points. Taking the radius of the circle to be unity, the coordinates of the point F may be taken to be $\cos F$, $\sin F$, and these may be expressed rationally in terms of the tangent of the half-angle, $f = \tan \frac{1}{2}F$; and similarly for the other points, viz. we may determine the six points by the parameters f, g, h, l, m, n respectively. The sides of the triangle are the lines joining the points L, F; M, G; N, H respectively: thus the equations of the sides are

$$\begin{aligned} &\text{for } BC: \ x(1-lf\ ) + y(l\ +f) - (1+lf\ ) = 0, \ \text{ say } U = 0, \\ &\ ,,\ \ CA: \ x(1-mg) + y(m+g) - (1+mg) = 0, \ ,,\ \ V = 0, \\ &\ ,,\ \ AB: \ x(1-nh\ ) + y(n\ +h) - (1+nh\ ) = 0, \ ,,\ \ W = 0. \end{aligned}$$

We have AF, a line through the intersections of BC and CA; its equation is therefore of the form $BV - CW = 0$, and to determine B, C we have $BV_0 - CW_0 = 0$, if V_0, W_0 are the values of V, W belonging to the point F, the coordinates of which are

$$\frac{1-f^2}{1+f^2}, \ \frac{2f}{1+f^2};$$

we find

$$\begin{aligned} V_0 &= -2(f-g)(f-m) \div (1+f^2); \\ W_0 &= \ \ 2(h-f)(f-n) \div (1+f^2), \end{aligned}$$

and then $B \div C = W_0 \div V_0$: we thus find the following equations:

$$\begin{aligned} &\text{that of } AF \text{ is } BV\ - CW = 0, \\ &\ ,,\ \ \ BG\ ,,\ C'W - A'U = 0, \\ &\ ,,\ \ \ CH\ ,,\ A''U - B''V = 0, \end{aligned}$$

where

$$B\ :\ C\ = -(h-f)(f-n)\ :\ (f-g)(f-m),$$
$$C'\ :\ A' = -(f-g)(g-l)\ :\ (g-h)(g-n),$$
$$A''\ :\ B'' = -(g-h)(h-m)\ :\ (h-f)(h-l).$$

The condition in order that the three lines may meet in a point is $BC'A'' = CA'B''$, viz. this is

$$(f-n)(g-l)(h-m)+(f-m)(g-n)(h-l)=0,$$

or, as this may also be written,

$$2fgh-gh(m+n)-hf(n+l)-fg(l+m)+mn(g+h)+nl(h+f)+lm(f+g)-2lmn=0.$$

Similarly, the equation of

$$AL \text{ is } \mathfrak{B}V - \mathfrak{C}W = 0,$$
$$BM \text{ „ } \mathfrak{C}'W - \mathfrak{A}'U = 0,$$
$$CN \text{ „ } \mathfrak{A}''U - \mathfrak{B}''V = 0,$$

where

$$\mathfrak{B}\ :\ \mathfrak{C}\ = -(n-l)(h-l)\ :\ (l-m)(g-l),$$
$$\mathfrak{C}'\ :\ \mathfrak{A}' = -(l-m)(f-m)\ :\ (m-n)(h-m),$$
$$\mathfrak{A}''\ :\ \mathfrak{B}'' = -(m-n)(g-n)\ :\ (n-l)(f-n).$$

The condition in order that the three lines may meet in a point is $\mathfrak{B}\mathfrak{C}'\mathfrak{A}'' = \mathfrak{C}\mathfrak{A}'\mathfrak{B}''$, viz. this is the same condition as before; that is, if the lines AF, BG, CH meet in a point, then also the lines AL, BM, CN will meet in a point.

In the case of the nine-points circle, we have MN, NL, LM parallel to LF, MG, NH, respectively: the equation of MN is

$$x(l-mn)+y(m+n)-(l+mn)=0,$$

and this is parallel to LF, if

$$\frac{m+n}{1-mn}=\frac{l+f}{1-lf}, \text{ that is, } L+F=M+N.$$

Hence, for the nine-points circle, we have

$$L+F=M+N, \quad M+G=N+L, \quad N+H=L+M,$$

or, as these equations may be written,

$$2L=G+H, \quad 2M=H+F, \quad 2N=F+G,$$

viz. it thus appears that the radii to the points L, M, N respectively, or say the radii L, M, N, bisect the angles made by the radii G and H, H and F, F and G respectively.

It may be added that we have

$$m+n-l+lmn=f\{1-mn+l(m+n)\},$$
$$n+l-m+lmn=g\{1-nl+m(n+l)\},$$
$$l+m-n+lmn=h\{1-lm+n(l+m)\},$$

viz. f, g, h are expressible each of them as a rational function of l, m, n.