951.

NON-EUCLIDIAN GEOMETRY.

[From the *Transactions of the Cambridge Philosophical Society*, vol. XV. (1894), pp. 37—61. Read January 27, 1890.]

I CONSIDER ordinary three-dimensional space, and use the words point, line, plane, &c., in their ordinary acceptations; only the notion of distance is altered, viz. instead of taking the Absolute to be the circle at infinity, I take it to be a quadric surface: in the analytical developments this is taken to be the imaginary surface $x^2+y^2+z^2+w^2=0$, and the formulæ arrived at are those belonging to the so-called Elliptic Space. The object of the Memoir is to set out, in a somewhat more systematic form than has been hitherto done, the general theory; and in particular, to further develop the analytical formulæ in regard to the perpendiculars of two given lines. It is to be remarked that not only all purely descriptive theorems of Euclidian geometry hold good in the new theory; but that this is the case also (only we in nowise attend to them) with theorems relating to parallelism and perpendicularity, in the Euclidian sense of the words. In Euclidian geometry, infinity is a special plane, the plane of the circle at infinity, and we consider (for instance) parallel lines, that is, lines which meet in a point of this plane: in the new theory, infinity is a plane in nowise distinguishable from any other plane, and there is no occasion to consider (although they exist) lines meeting in a point of this plane, that is, parallel lines in the Euclidian sense. So again, given any two lines, there exists always, in the Euclidian sense, a single line perpendicular to each of the given lines, but this is not in the new sense a perpendicular line; there is nothing to distinguish it from any other line cutting the two given lines, and consequently no occasion to consider it: we do consider the lines—there are, in fact, two such lines—which in the new sense of the word are perpendicular to each of the given lines.

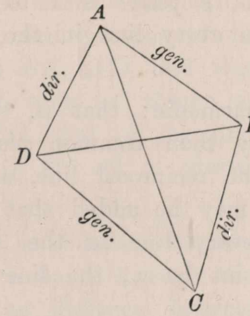
It should be observed that the term distance is used to include inclination: we have, say, a linear distance between two points; an angular distance between two lines which meet; and a dihedral distance between two planes. But all these are distances of the same kind, having a common unit, the quadrant, represented by $\frac{1}{2}\pi$; and in fact, any distance may be considered indifferently as a linear, an angular, or a dihedral distance: the word, perpendicular, usually represented by $\perp$, refers of course to a distance $=\frac{1}{2}\pi$. We have moreover the distance of a point from a plane, that of a point from a line, and that of a plane from a line. Two lines which do not meet may be $\perp$, and in particular they may be reciprocal: in general, they have two distances; and they have also a "moment" and "comoment," the values of which serve to express those of the two distances. Lines may be, in several distinct senses, as will be explained, parallel; and for this reason the word parallel is never used simpliciter; the notion of parallelism does not apply to planes, nor to points.

Elliptic space has been considered and the theory developed in connexion with the imaginaries called by Clifford biquaternions, and as applied to Mechanics: I refer to the names, Ball, Buchheim, Clifford, Cox, Gravelius, Heath, Klein, and Lindemann: in particular, much of the purely geometrical theory is due to Clifford. Memoirs by Buchheim and Heath are referred to further on.

Geometrical Notions. Art. Nos. 1 to 16.

1. The Absolute is a general quadric surface: it has therefore lines of two kinds, which it is convenient to distinguish as directrices and generatrices: through each point of the surface there is a directrix and a generatrix, and the plane through these two lines is the tangent plane at the point. A line meets the surface in two points, say A, C; the generatrix at A meets the directrix at C; and the

Fig. 1.



directrix at A meets the generatrix at C; and we have thus on the surface two new points B, D; joining these we have a line BD, which is the reciprocal of AC; viz. BD is the intersection of the planes BAD, BCD which are the tangent planes at A, C respectively, and similarly AC is the intersection of the planes ABC, ADC which are the tangent planes at B, D respectively.

According to what follows, reciprocal lines are $\perp$, but $\perp$ lines are not in general reciprocal; thus the two epithets are not convertible, and there will be occasion throughout to speak of reciprocal *lines*.

2. Two points may be harmonic; that is, the two points and the intersections of their line of junction with the Absolute may form a harmonic range: the two points are in this case said to be $\perp$.

Two planes may be harmonic: that is, the two planes and the tangent planes of the Absolute through their line of intersection may form a harmonic plane-pencil: the two planes are said to be $\perp$.

Two lines which meet may be harmonic: that is, the two lines and the tangents from their point of intersection to the section of the Absolute by their common plane may form a harmonic pencil: the two lines are said to be $\perp$.

The locus of all the points $\perp$ to a given point is a plane, the reciprocal or polar plane of the given point; and similarly the envelope of all the planes $\perp$ to a given plane is a point, the pole of the given plane: a point and plane reciprocal to each other, or say a pole and polar plane, are said to be $\perp$.

3. If a point is situate anywhere in a given line, the $\perp$ plane passes always through the reciprocal line: each point of the reciprocal line is thus a point of the $\perp$ plane, i.e. it is $\perp$ to the given point: that is, considering two reciprocal lines, any point on the one line and any point on the other line are $\perp$. Similarly any plane through the one line and any plane through the other line are $\perp$.

A line and plane may be harmonic; that is, they may be reciprocal in regard to the cone, vertex their point of intersection, circumscribed to the Absolute; the line and plane are said to be $\perp$. The $\perp$ plane passes through the reciprocal line, and conversely every plane through the reciprocal line is a $\perp$ plane. It may be added that the line passes through the $\perp$ point of the plane; and conversely, that every line through the $\perp$ point of a plane is $\perp$ to the plane. Moreover if a line and plane be $\perp$, the line is $\perp$ to every line in the plane and through the point of intersection.

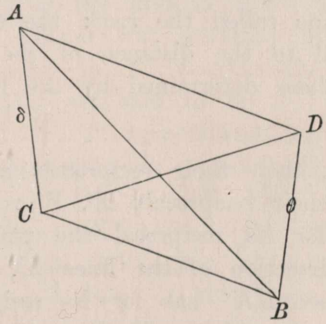
A line and point may be harmonic; that is, they may be reciprocal in regard to the section of the Absolute by their common plane: the line and point are said to be $\perp$. The $\perp$ point lies in the reciprocal line, and conversely every point of the reciprocal line is a $\perp$ point. It may be added that the line lies in the $\perp$ plane of the point: and conversely that every line in the $\perp$ plane of a point is $\perp$ to the point. Moreover if a line and point be $\perp$, the line is $\perp$ to every line through the point and in the plane of junction.

4. We may have a triangle ABC composed of three lines BC, CA, AB in the same plane: the six parts hereof are the linear distances B, C; C, A; A, B of the angular points, and the angular distances of the sides CA, AB; AB, BC; BC, CA. Similarly we may have a trihedral composed of three lines meeting in a point, say the planes through the several pairs of lines are A, B, C respectively: the six parts hereof are the angular distances CA, AB; AB, BC; BC, CA of the three lines, and the dihedral distances B, C; C, A; A, B of the three planes. According to the definitions of distance hereinafter adopted, the relation of the six parts is that of the sides and angles of a spherical triangle: in particular, if two sides are each

$=\frac{1}{2}\pi$, then the opposite angles are each $=\frac{1}{2}\pi$, and the included angle and the opposite side have a common value; and so also if two angles are each $=\frac{1}{2}\pi$, then the opposite sides are each $=\frac{1}{2}\pi$, and the included side and the opposite angle have a common value.

5. Let A, C be points on a line, and B, D points on the reciprocal line: by what precedes, each of the lines AB, AD, CB, CD is $=\frac{1}{2}\pi$: also each of the angles ACD, ACB, CAB, CAD is $=\frac{1}{2}\pi$. The line AC is $\perp$ to the plane BCD and to the lines BC, CD, in that plane; it is also $\perp$ to the plane BAD and to the lines BA, AD in that plane; and similarly for the line BD. From the trihedral of the planes which meet in C, distance of planes ACB, ACD = distance of lines BC, CD, viz. the dihedral distance of two planes through the line AC is equal to the angular distance of their intersections with the $\perp$ plane BCD; and it is therefore equal also to the

Fig. 2.



linear distance of their intersections with the other $\perp$ plane BAD: and so from the triangle BCD, where BC, CD are each $=\frac{1}{2}\pi$, the angular distance BCD is equal to the linear distance BD; that is, the distance of the planes ACB, ACD, that of the lines BC, CD, that of the lines BA, AD, and that of the points B, D, are all of them equal; say the value of each of them is $=\theta$. And in like manner the distance of the planes ABD, CBD, that of the lines AB, BC, that of the lines AD, DC, and that of the points A, C, are all of them equal: say the value of each of them is $=\delta$.

The theorem may be stated as follows: all the planes $\perp$ to a given line intersect in the reciprocal line: and if we have through the given line any two planes, the distance of these two planes, the distance between their lines of intersection with any one of the $\perp$ planes, and the distance between their points of intersection with the reciprocal line, are all of them equal.

And it thus appears also that a distance may be represented indifferently as a linear distance, an angular distance, or a dihedral distance.

6. Consider a point and a plane: we may through the point draw a line $\perp$ to the plane, and intersecting it in a point called the "foot": the distance of the point and plane is then (as a definition) taken to be equal to that of the point and foot. It may be added that the $\perp$ line is, in fact, the line joining the point with the $\perp$

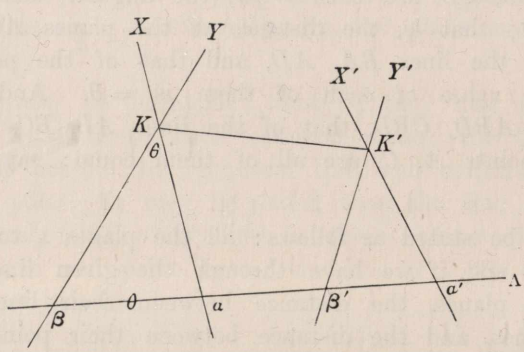
point of the plane; and that the distance of the point and plane is equal to the complement of the distance of the point and the $\perp$ point. Or again, we may in the plane draw a line $\perp$ to the point, and determining with it a plane called the roof: and then (as an equivalent definition) the distance of the plane and point is equal to the distance of the plane and roof. It may be added that the $\perp$ line is, in fact, the intersection of the plane with the $\perp$ plane of the point, and that the distance of the point and plane is also equal to the complement of the distance of the plane and the $\perp$ plane of the point.

7. Consider a point and line: we have through the point a line $\perp$ to the line and cutting it in a point called the foot; the distance of the point and line is then (as a definition) equal to the distance of the point and foot. It may be added that the foot is the intersection with the line of a plane $\perp$ thereto through the point.

Again, consider a plane and line: we have in the plane a line $\perp$ to the line and determining with it a plane called the roof: the distance of the plane and line is then (as a definition) equal to the distance of the plane and roof. It may be added that the roof is the plane determined by the line and a point $\perp$ thereto in the plane.

8. If two lines intersect, then their reciprocals also intersect. Say the intersecting lines are X, Y; and their reciprocals X', Y' respectively; then K, the point of intersection of X, Y, has for its reciprocal the plane of the lines X', Y'; and similarly K', the point of intersection of the lines X', Y', has for its reciprocal the plane of the lines X, Y: hence KK' has for its reciprocal the line of intersection of the planes XY and $X'Y'$; say this is the line Λ, meeting X, Y, X', Y', in the

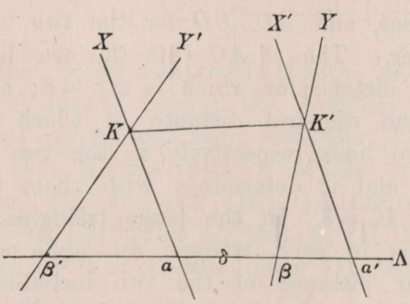
Fig. 3.



points α, β, α', β' respectively. Since K, K' are points in the reciprocal lines X, X' (or in the reciprocal lines Y, Y'), the distance KK' is $=\frac{1}{2}\pi$; and since the plane XY passes through the line Λ which is the reciprocal of KK', the line KK' is $\perp$ to the plane XY and also to each of the lines X, Y: (it is also $\perp$ to the plane $X'Y'$ and to each of the lines X', Y'). Again, since the lines KK' and Λ are reciprocal, each of the distances $K\alpha$, $K\beta$ is $=\frac{1}{2}\pi$; that is, the line Λ is $\perp$ to each of the lines X and Y, (and similarly it is $\perp$ to each of the lines X' and Y'). Moreover the angle at K or distance of the lines X and Y (which is equal to the distance of the planes $K'KX$ and $K'KY$) is equal to the distance $\alpha\beta$ of the intersections of Λ with the lines X and Y respectively. We have thus for the two intersecting lines X and Y, the two lines KK' and Λ each of them $\perp$ to the two lines: where observe that KK' is the line of junction of the point of intersection of the two given lines with the point of intersection of the reciprocal lines; and that Λ is the line of intersection of the plane of the two given lines with the plane of the reciprocal lines. The linear distance along KK' between the two lines is $=0$; the dihedral distance between the planes, which KK' determines with the two lines respectively, is equal to the angular distance between the two lines. The linear distance along Λ is equal to the angular distance between the two lines; the dihedral distance between the two planes, which Λ determines with the two lines respectively, is $=0$.

9. If two lines are such that the first of them intersects the reciprocal of the second of them, then also the second will intersect the reciprocal of the first; the two lines are in this case said to be contrasecting lines; or more simply, to contrasect: and contrasecting lines are said to be $\perp$. Supposing that the two lines are X, Y and their reciprocals X', Y' respectively, we have here X, Y' intersecting in a point K, and X', Y intersecting in a point K': and the planes XY', $X'Y$ intersect in a line Λ which meets the lines X, Y, X', Y' in the points α, β, α', β' respectively. As before, the lines KK' and Λ are reciprocal: the distance KK' is $=\frac{1}{2}\pi$; and KK' is $\perp$ to the plane XY', that is, to each of the lines X, Y'; and also to the plane $X'Y$, that is, to each of the lines X', Y; it is thus $\perp$ to each of the lines X and Y. Again each of the angles at α, β, α', β' is $=\frac{1}{2}\pi$; that is,

Fig. 4.



the line Λ is $\perp$ to each of the lines X, Y', X', Y, or say to each of the lines X and Y. Moreover the angle at K, or say the angular distance of the intersecting lines X and Y', is equal to the distance $\alpha\beta'$; and similarly the angle at K', or say the angular distance of the intersecting lines X' and Y, is equal to the distance $\alpha'\beta$: but the distances $\alpha\alpha'$, $\beta\beta'$ are each equal to $\frac{1}{2}\pi$; and hence the distances $\alpha\beta'$, $\alpha'\beta$ are equal to each other and each of them is equal to the complement of the distance $\alpha\beta$. Thus in the case of two contrasecting lines we have the lines KK' and Λ each of them $\perp$ to the two given lines; where observe that KK' is the line joining the point of intersection of X with the reciprocal of Y and the point of intersection of Y with the reciprocal of X; and that Λ is the line of intersection of the plane through X and the reciprocal of Y with the plane through Y and the reciprocal of X. The linear distance KK' between the two lines along the first of these lines is thus $= \frac{1}{2}\pi$.

10. We have KK' and Λ reciprocal lines; on the first of these, we have the points K, K' which are $\perp$ points: hence also the planes ΛK and $\Lambda K'$ are $\perp$; but the plane ΛK is the plane $\Lambda XY'$ or say the plane ΛX, and the plane $\Lambda K'$ is the plane $\Lambda X'Y$ or say the plane ΛY; hence the planes ΛX and ΛY are $\perp$. Similarly the line Λ cuts the two lines in the points α, β; and the line KK' determines with these two points respectively the plane $KK'\alpha$, that is $KK'X$, and $KK'\beta$, that is $KK'Y$; and thus the linear distance between the two points α, β is equal to the dihedral distance between the two planes $KK'X$ and $KK'Y$. Thus the $\perp$ line Λ cuts the two lines in two points α, β the linear distance of which is, say, δ: and it determines with them two planes the dihedral distance of which is $= \frac{1}{2}\pi$. And the other $\perp$ line KK' cuts the two lines in the points K, K' the linear distance of which is $= \frac{1}{2}\pi$, and it determines with them two planes the dihedral distance of which is $= \delta$.

11. Consider a line X and its reciprocal X': a line intersecting each of these also contrasects each of them and is thus $\perp$ to each of them: and similarly if Y be any other line and Y' its reciprocal, a line intersecting Y and Y' also contrasects each of them and is thus $\perp$ to each of them. Hence a line which meets each of the four lines X, X', Y, Y' is also $\perp$ to each of them, or attending only to the lines X, Y, say it is a $\perp$ of these lines: there are two $\perp$s; and clearly these are reciprocal to each other, for if a line meets X, Y, X', Y', then its reciprocal meets X', Y', X, Y, that is, the same four lines. Looking back to figure 2, we may take AB, CD for the given lines, and AC, BD for the two $\perp$s; as just remarked, these are reciprocal to each other. The $\perp$ AC cuts the two lines respectively in the two points A and C the linear distance of which is say $= \delta$; and it determines with them two planes ACB, ACD, the dihedral distance of which is say $= \theta$. Similarly the other $\perp$ BD meets the two lines respectively in the two points B and D the linear distance of which is $= \theta$, and it determines with them two planes BDA, BDC the dihedral distance of which is $= \delta$. In the plane triangles which are the faces of the tetrahedron $ABCD$, there is in each triangle an angle opposite to AC or BD and which, or say the angular distance of the two including sides, is thus $= \delta$ or θ. Except as aforesaid, the sides, angles, and dihedral angles, or say the linear, angular, and dihedral distances, of the tetrahedron are each of them $= \frac{1}{2}\pi$.

12. Considering the lines X and Y as given, the distances δ and θ depend upon two functions called the Moment and the Comoment: viz. moment $= 0$ is the condition in order that the two lines may intersect (or, what is the same thing, in order that their reciprocals may intersect): comoment $= 0$ is the condition in order that the two lines may contrasect, that is, each line meet the reciprocal of the other one. It may be convenient to mention here that the actual relations are

$$\sin\delta\sin\theta = \text{Moment},\quad \cos\delta\cos\theta = \text{Comoment}.$$

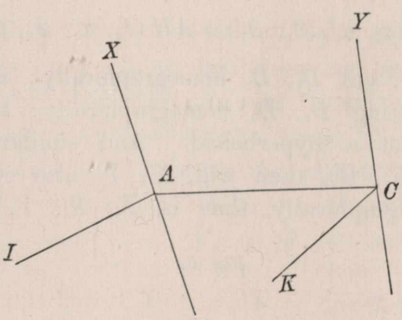
In particular, if moment $= 0$, then the lines intersect; we have, say $\delta = 0$, and therefore $\cos\theta =$ comoment; if comoment $= 0$, then the lines contrasect, that is they are $\perp$: we have, say $\theta = \frac{1}{2}\pi$, that is, $\sin\delta =$ moment. These are the two particular cases which have been considered above.

13. Consider as above the two lines X, Y met by the $\perp$ δ in the two points A and C respectively. Consider at A a line I $\perp$ to the lines X, δ; and take Π the plane of the lines (X, δ) and Ω the plane of the lines (X, I). Similarly consider at C a line K $\perp$ to the lines Y, δ, and take Π_1 the plane of the lines (Y, δ) and Ω_1 the plane of the lines (Y, K): we have thus through A two planes Π, Ω meeting in the line X; and through C two planes Π_1, Ω_1, meeting in the line Y. It requires only a little reflection to see that the distances of these planes are

$$(\Pi, \Pi_1) = \theta, \quad (\Omega, \Omega_1) = \delta;$$

$$(\Pi, \Omega) = \tfrac{1}{2}\pi, \ (\Pi_1, \Omega_1) = \tfrac{1}{2}\pi; \ (\Pi, \Omega_1) = \tfrac{1}{2}\pi, \ (\Pi_1, \Omega) = \tfrac{1}{2}\pi.$$

Fig. 5.



In fact, Π, Π_1 are the before-mentioned planes ACB, ACD the distance of which was $= \theta$: Ω, Ω_1 are planes having the common $\perp$ AC, which is the line through the poles of these planes, and such that the distance AC is equal to the distance of the two poles, that is, the distance of the two planes. Moreover from the definitions, the distances (Π, Ω) and (Π_1, Ω_1) are each $= \frac{1}{2}\pi$: the plane Π passes through the $\perp$ at C to the plane Ω_1 that is, $(\Pi, \Omega_1) = \frac{1}{2}\pi$; and similarly the plane Π_1 passes through the $\perp$ at A to the plane Ω, that is, $(\Pi_1, \Omega) = \frac{1}{2}\pi$; and we have thus the relations in question.

The consideration of these planes leads, (see *post* 31 and 32), to the before-mentioned equation, $\cos\delta\cos\theta =$ comoment; if instead of one of the lines, say Y, we consider the reciprocal line Y', then the angles δ, θ are changed each of them into its complement, and we deduce immediately the other equation, $\sin\delta\sin\theta =$ Moment.

14. It may happen that, instead of the determinate number 2, we have a singly infinite system of $\perp$s: viz. this will be so if the lines X, X', Y, Y' are generating lines (of the same kind) of a hyperboloid. They will be so if the lines X and Y each of them meet the same two lines (of the same kind) of the Absolute, say if X, Y each meet two directrices D_1, D_2, or two generatrices G_1, G_2; but it seems

less easy to prove conversely that the lines X and Y must satisfy one of these two conditions. Suppose first that X, Y each meet the two directrices D_1, D_2; say X meets them in α_1, α_2, and Y in β_1, β_2 respectively. We have at α_1 a generatrix which meets D_2, suppose in α_2', and at α_2 a generatrix which meets D_1, suppose in α_1'; joining α_1', α_2', we have the line X' which is the reciprocal of X; viz. X' meets each of the lines D_1, D_2: similarly the generatrices at β_1, β_2 meet D_2, D_1 in the points β_2', β_1' respectively, and joining these, we have the line Y' which is the reciprocal of Y: thus Y' meets each of the lines D_1 and D_2: the line D_1 meets the four generatrices in the points α_1, α_1', β_1, β_1' respectively, and the line D_2 meets the same four generatrices in the points α_2', α_2, β_2', β_2: thus

$$AH(\alpha_1, \alpha_1', \beta_1, \beta_1') = AH(\alpha_2', \alpha_2, \beta_2', \beta_2),$$

AH denoting anharmonic ratio as usual. But

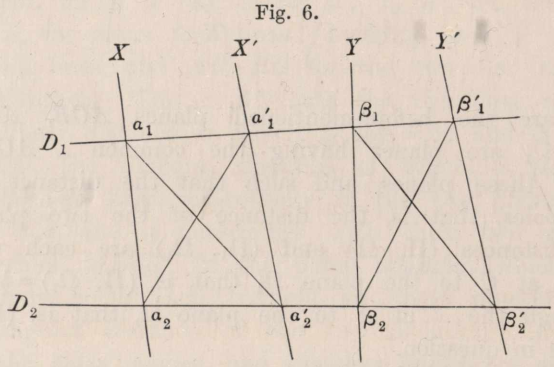
$$AH(\alpha_2', \alpha_2, \beta_2', \beta_2) = AH(\alpha_2, \alpha_2', \beta_2, \beta_2'),$$

and thus the equation may be written

$$AH(\alpha_1, \alpha_1', \beta_1, \beta_1') = AH(\alpha_2, \alpha_2', \beta_2, \beta_2');$$

viz. the lines X, X', Y, Y', cut D_1, D_2 homographically; and there is thus a singly infinite system of lines cutting D_1, D_2 homographically: that is, X, X', Y, Y', are lines (of the same kind) of a hyperboloid. And similarly if X, Y each cut the same two generating lines G_1, G_2, then will X', Y' also cut these lines and X, X', Y, Y' will cut them homographically, that is, X, X', Y, Y' will be lines (of the same kind) of a hyperboloid.

Fig. 6.



The condition may be otherwise stated; if the lines X, Y have for $\perp$s any two directrices D_1, D_2 or any two generatrices G_1, G_2 of the Absolute, then in either case there will be a singly infinite series of $\perp$s: the $\perp$ distances are all of them equal; say we have $\theta = \delta$, and therefore $\sin^2\delta = \text{moment}$, $\cos^2\delta = \text{comoment}$; and therefore moment + comoment = 1; or as the equation is more properly written, $\pm$ moment $\pm$ comoment = 1.

15. Two lines X, Y, each of them meeting the same two directrices D_1, D_2, are said to be "right parallels"; and similarly two lines X, Y each meeting the same two generatrices G_1, G_2, are said to be "left parallels": the selection as to which set

of lines of the Absolute shall be called directrices and which shall be called generatrices will be made further on, (see *post* 35). We have just seen that, if two lines are right parallels, or are left parallels, then in either case there is a singly infinite series of $\perp$ s. It may be remarked that reciprocal lines are at once right parallels and left parallels; and that in this case there is a doubly infinite series of $\perp$ s, viz. every line cutting the two lines is a $\perp$.

Observe that right parallels do not meet, and left parallels do not meet: their doing so would imply in the one case the meeting of two directrices, and in the other case the meeting of two generatrices.

16. If instead of the foregoing definitions by means of two directrices or two generatrices, we consider a directrix and a generatrix of the Absolute, and define parallel lines by reference thereto, then it is at once seen that there are 3 chief forms, and several subforms; the directrix and generatrix meet in a point, or say an ineunt, of the Absolute, and lie in a plane which is a tangent plane of the Absolute: we may have two lines X, Y which

1°. Each pass through the ineunt, neither of them lying in the tangent plane;

2°. Each lie in the tangent plane, neither of them passing through the ineunt;

3°. One passes through the ineunt, but does not lie in the tangent plane: the other lies in the tangent plane, but does not pass through the ineunt.

Observe that in the cases 1° and 2° the lines X and Y intersect, but in the case 3° they do not intersect. The lines in the case 3° are I believe what Buchheim has termed β-parallels, his α-parallels being the foregoing right or left parallels*. The subforms arise by omitting in 1°, 2°, or 3°, as the case may be, the negative condition in regard to the two lines or to one of them; as the question is not here further pursued, I do not attempt to give names to these several kinds of parallel lines.

Point-, line-, and plane-coordinates: General formulæ. Art. Nos. 17 to 20.

17. We consider point-coordinates (x, y, z, w): line-coordinates (a, b, c, f, g, h), where $af+bg+ch=0$: and plane-coordinates $(\xi, \eta, \zeta, \omega)$; if we have a line which is at once through two points and in two planes, then the line-coordinates are given by

$$\begin{array}{ccccccccccc} a & : & b & : & c & : & f & : & g & : & h \\ = y_1z_2 - y_2z_1 & : & z_1x_2 - z_2x_1 & : & x_1y_2 - x_2y_1 & : & x_1w_2 - x_2w_1 & : & y_1w_2 - y_2w_1 & : & z_1w_2 - z_2w_1 \\ = \xi_1\omega_2 - \xi_2\omega_1 & : & \eta_1\omega_2 - \eta_2\omega_1 & : & \zeta_1\omega_2 - \zeta_2\omega_1 & : & \eta_1\zeta_2 - \eta_2\zeta_1 & : & \zeta_1\xi_2 - \zeta_2\xi_1 & : & \xi_1\eta_2 - \xi_2\eta_1. \end{array}$$

Similarly, if a plane be determined by three points thereof, then the coordinates of the plane are given by

$$\xi : \eta : \zeta : \omega = \begin{vmatrix} 1 & & & \\ x_1, & y_1, & z_1, & w_1 \\ x_2, & y_2, & z_2, & w_2 \\ x_3, & y_3, & z_3, & w_3 \end{vmatrix} : \begin{vmatrix} & 1 & & \\ x_1, & y_1, & z_1, & w_1 \\ x_2, & y_2, & z_2, & w_2 \\ x_3, & y_3, & z_3, & w_3 \end{vmatrix} : \begin{vmatrix} & & 1 & \\ x_1, & y_1, & z_1, & w_1 \\ x_2, & y_2, & z_2, & w_2 \\ x_3, & y_3, & z_3, & w_3 \end{vmatrix} : \begin{vmatrix} & & & 1 \\ x_1, & y_1, & z_1, & w_1 \\ x_2, & y_2, & z_2, & w_2 \\ x_3, & y_3, & z_3, & w_3 \end{vmatrix};$$

* See Buchheim, "A Memoir on Biquaternions," *Amer. Math. Jour.* t. VII. (1885), pp. 293—326.

and if a point be given as the intersection of three planes, the coordinates of the point are

$$x : y : z : w = \begin{vmatrix} 1 \\ \xi_1, \eta_1, \zeta_1, \omega_1 \\ \xi_2, \eta_2, \zeta_2, \omega_2 \\ \xi_3, \eta_3, \zeta_3, \omega_3 \end{vmatrix} : \begin{vmatrix} 1 \\ \xi_1, \eta_1, \zeta_1, \omega_1 \\ \xi_2, \eta_2, \zeta_2, \omega_2 \\ \xi_3, \eta_3, \zeta_3, \omega_3 \end{vmatrix} : \begin{vmatrix} 1 \\ \xi_1, \eta_1, \zeta_1, \omega_1 \\ \xi_2, \eta_2, \zeta_2, \omega_2 \\ \xi_3, \eta_3, \zeta_3, \omega_3 \end{vmatrix} : \begin{vmatrix} 1 \\ \xi_1, \eta_1, \zeta_1, \omega_1 \\ \xi_2, \eta_2, \zeta_2, \omega_2 \\ \xi_3, \eta_3, \zeta_3, \omega_3 \end{vmatrix}.$$

18. The conditions in order that a point (x, y, z, w) may be situate on a line (a, b, c, f, g, h) are

$$\begin{aligned} . \quad hy - gz + aw &= 0, \\ -hx \quad . \quad + fz + bw &= 0, \\ gx - fy \quad . \quad + cw &= 0, \\ -ax - by - cz \quad . \quad &= 0, \end{aligned}$$

viz. these constitute a twofold relation.

Similarly, the conditions in order that the plane $(\xi, \eta, \zeta, \omega)$ may contain the line (a, b, c, f, g, h) are

$$\begin{aligned} . \quad c\eta - b\zeta + f\omega &= 0, \\ -c\xi \quad . \quad + a\zeta + g\omega &= 0, \\ b\xi - a\eta \quad . \quad + h\omega &= 0, \\ -f\xi - g\eta - h\zeta \quad . \quad &= 0, \end{aligned}$$

viz. these constitute a twofold relation.

19. The condition in order that two lines (a, b, c, f, g, h), (A, B, C, F, G, H) may meet is

$$Af + Bg + Ch + Fa + Gb + Hc = 0.$$

Supposing that the two lines meet, we have at the point of intersection

$$\begin{aligned} . \quad hy - gz + aw &= 0, & . \quad Hy - Gz + Aw &= 0, \\ -hx \quad . \quad + fz + bw &= 0, & -Hx \quad . \quad + Fz + Bw &= 0, \\ gx - fy \quad . \quad + cw &= 0, & Gx - Fy \quad . \quad + Cw &= 0, \\ -ax - by - cz \quad . \quad &= 0, & -Ax - By - Cz \quad . \quad &= 0; \end{aligned}$$

and from these equations we can find the coordinates x, y, z, w of the point of intersection in a fourfold form, viz. we may write

$$\begin{aligned} x : y : z : w &= fA + bG + cH : gA - aG : hA - aH : hG - gH \\ &= fB - bF : gB + cH + aF : hB - bH : fH - hF \\ &= fC - cF : gC - cG : hC + aF + bG : gF - fG \\ &= bC - cB : cA - aC : aB - bA : fA + gB + hC. \end{aligned}$$

There is no real advantage in any one over any other of these forms, but it is convenient to work with the last of them

$$x : y : z : w = bC - cB : cA - aC : aB - bA : fA + gB + hC.$$

20. In like manner if two lines intersect, the plane which contains each of them is given by

$$\begin{array}{rlllll} \xi : \eta : \zeta : \omega = & aF+gB+hC & : bF-fB & : cF-fC & : cB-bC \\ = & aG-gA & : bG+hC+fA & : cG-gC & : aC-cA \\ = & aH-hA & : bH-hB & : cH+fA+gB & : bA-aB \\ = & gH-hG & : hF-fH & : fG-gF & : aF+bG+cH; \end{array}$$

or say we have

$$\xi : \eta : \zeta : \omega = gH-hG : hF-fH : fG-gF : aF+bG+cH.$$

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21. The equation is

in point coordinates $x^2+y^2+z^2+w^2=0$,

in plane coordinates $\xi^2+\eta^2+\zeta^2+\omega^2=0$,

in line coordinates $a^2+b^2+c^2+f^2+g^2+h^2=0$.

Hence

$\perp$ of plane $(\xi, \eta, \zeta, \omega)$ is point $(\xi, \eta, \zeta, \omega)$,

$\perp$ of point (x, y, z, w) is plane (x, y, z, w).

Reciprocal of line (a, b, c, f, g, h) is line (f, g, h, a, b, c);

Points (x, y, z, w), (x', y', z', w') are $\perp$ if $xx'+yy'+zz'+ww'=0$;

Planes $(\xi, \eta, \zeta, \omega)$, $(\xi', \eta', \zeta', \omega')$ are $\perp$ if $\xi\xi'+\eta\eta'+\zeta\zeta'+\omega\omega'=0$.

22. A line (a, b, c, f, g, h) and plane $(\xi, \eta, \zeta, \omega)$ are $\perp$ when the line passes through the $\perp$ point of the plane, that is, the point $(\xi, \eta, \zeta, \omega)$: the conditions (equivalent to two equations) are

$$\begin{array}{rrrrl} . & h\eta & -g\zeta & +a\omega & =0, \\ -h\xi & . & +f\zeta & +b\omega & =0, \\ g\xi & -f\eta & . & +c\omega & =0, \\ -a\xi & -b\eta & -c\zeta & . & =0. \end{array}$$

A line (a, b, c, f, g, h) and point (x, y, z, w) are $\perp$ when the line lies in the $\perp$ plane of the point, that is, in the plane (x, y, z, w): the conditions (equivalent to two equations) are

$$\begin{array}{rrrrl} . & cy & -bz & +fw & =0, \\ -cx & . & +az & +gw & =0, \\ bx & -ay & . & +hw & =0, \\ -fx & -gy & -hz & . & =0. \end{array}$$

Two lines (a, b, c, f, g, h), (a', b', c', f', g', h') which meet, that is, for which $af'+bg'+ch'+a'f+b'g+c'h=0$, are $\perp$ if

$$aa'+bb'+cc'+ff'+gg'+hh'=0.$$

23. There will be occasion to consider the pair of tangent planes drawn through the line (a, b, c, f, g, h) to the Absolute. Writing for shortness

$$\begin{aligned} P &= \;\;\; . \quad\;\; hy - gz + aw, \\ Q &= - hx \quad . \;\; + fz + bw, \\ R &= \;\; gx - fy \quad . \;\; + cw, \\ S &= - ax - by - cz \quad . \;\; , \end{aligned}$$

it may be shown that the equation of the pair of planes is

$$P^2 + Q^2 + R^2 + S^2 = 0.$$

In fact, writing for a moment $(\xi, \eta, \zeta, \omega)$ and $(\xi', \eta', \zeta', \omega')$ to denote the coefficients of (x, y, z, w) in P and Q respectively, so that

$$(\xi, \eta, \zeta, \omega) = (0, h, -g, a), \quad (\xi', \eta', \zeta', \omega') = (-h, 0, f, b),$$

then the equation of the planes is

$$(\xi' P - \xi Q)^2 + (\eta' P - \eta Q)^2 + (\zeta' P - \zeta Q)^2 + (\omega' P - \omega Q)^2 = 0,$$

that is,

$$(\xi'^2 + \eta'^2 + \zeta'^2 + \omega'^2) P^2 - 2 (\xi\xi' + \eta\eta' + \zeta\zeta' + \omega\omega') PQ + (\xi^2 + \eta^2 + \zeta^2 + \omega^2) Q^2 = 0,$$

viz. this equation is

$$(b^2 + h^2 + f^2) P^2 + 2 (fg - ab) PQ + (a^2 + g^2 + h^2) Q^2 = 0.$$

But P, Q, R, S are connected by the identical equations

$$\begin{aligned} . \quad\;\; cQ - bR + fS &= 0, \\ -cP \quad . \;\; + aR + gS &= 0, \\ bP - aQ \quad . \;\; + hS &= 0, \\ -fP - gQ - hR \quad . \;\; &= 0; \end{aligned}$$

using these equations to express R, S in terms of P, Q, viz. writing

$$R = -\frac{1}{h}(fP + gQ), \quad S = -\frac{1}{h}(bP - aQ),$$

we see that the last preceding equation is equivalent to $P^2 + Q^2 + R^2 + S^2 = 0$.

24. Similarly, if

$$\begin{aligned} P_1 &= \;\;\; . \quad\;\; cy - bz + fw, \\ Q_1 &= - cx \quad . \;\; + az + gw, \\ R_1 &= \;\; bx - ay \quad . \;\; + hw, \\ S_1 &= - fx - gy - hz \quad . \;\; , \end{aligned}$$

functions which are connected by the identical relations

$$\begin{aligned} . \quad\;\; hQ_1 - gR_1 + aS_1 &= 0, \\ -hP_1 \quad . \;\; + fR_1 + bS_1 &= 0, \\ gP_1 - fQ_1 \quad . \;\; + cS_1 &= 0, \\ -aP_1 - bQ_1 - cR_1 \quad . \;\; &= 0; \end{aligned}$$

then in like manner we have

$$P_1^2 + Q_1^2 + R_1^2 + S_1^2 = 0,$$

for the equation of the pair of tangent planes from the reciprocal line (f, g, h, a, b, c) to the Absolute. And we may remark the identity

$$(P^2 + Q^2 + R^2 + S^2) + (P_1^2 + Q_1^2 + R_1^2 + S_1^2) = (a^2 + b^2 + c^2 + f^2 + g^2 + h^2)(x^2 + y^2 + z^2 + w^2).$$

We, in fact, have

$$P^2 + Q^2 + R^2 + S^2 = \begin{array}{c|c|c|c|c|} & x & y & z & w \\ \hline x & a^2 + g^2 + h^2 & ab - fg & ac - hf & cg - bh \\ \hline y & ab - fg & b^2 + h^2 + f^2 & bc - gh & ah - cf \\ \hline z & ac - hf & bc - gh & c^2 + f^2 + g^2 & bf - ag \\ \hline w & cg - bh & ah - cf & bf - ag & a^2 + b^2 + c^2 \\ \hline \end{array};$$

and in like manner

$$P_1^2 + Q_1^2 + R_1^2 + S_1^2 = \begin{array}{c|c|c|c|c|} & x & y & z & w \\ \hline x & b^2 + c^2 + f^2 & -ab + fg & -ac + hf & -cg + bh \\ \hline y & -ab + fg & c^2 + a^2 + g^2 & -bc + gh & -ah + cf \\ \hline z & -ac + hf & -bc + gh & a^2 + b^2 + h^2 & -bf + ag \\ \hline w & -cg + bh & -ah + cf & -bf + ag & f^2 + g^2 + h^2 \\ \hline \end{array}.$$

25. For the distance of two points (x, y, z, w) and (x', y', z', w'), we have

$$\cos\delta = \frac{xx' + yy' + zz' + ww'}{\sqrt{x^2 + y^2 + z^2 + w^2}\sqrt{x'^2 + y'^2 + z'^2 + w'^2}},$$

whence also

$$\sin\delta = \frac{\sqrt{a^2 + b^2 + c^2 + f^2 + g^2 + h^2}}{\sqrt{x^2 + y^2 + z^2 + w^2}\sqrt{x'^2 + y'^2 + z'^2 + w'^2}},$$

where, in the numerator, (a, b, c, f, g, h) stand for the coordinates of the line of junction of the two points, taken to be equal to

$$yz' - y'z,\ zx' - z'x,\ xy' - x'y,\ xw' - x'w,\ yw' - y'w,\ zw' - z'w$$

respectively.

Similarly, for the distance of two planes $(\xi, \eta, \zeta, \omega)$ and $(\xi', \eta', \zeta', \omega')$, we have

$$\cos\delta = \frac{\xi\xi' + \eta\eta' + \zeta\zeta' + \omega\omega'}{\sqrt{\xi^2 + \eta^2 + \zeta^2 + \omega^2}\sqrt{\xi'^2 + \eta'^2 + \zeta'^2 + \omega'^2}},$$

whence also

$$\sin\delta = \frac{\sqrt{a^2 + b^2 + c^2 + f^2 + g^2 + h^2}}{\sqrt{\xi^2 + \eta^2 + \zeta^2 + \omega^2}\sqrt{\xi'^2 + \eta'^2 + \zeta'^2 + \omega'^2}},$$

where, in the numerator, (a, b, c, f, g, h) stand for the coordinates of the line of intersection of the two planes, taken to be equal to

$$\xi\omega' - \xi'\omega,\ \eta\omega' - \eta'\omega,\ \zeta\omega' - \zeta'\omega,\ \eta\zeta' - \eta'\zeta,\ \zeta\xi' - \zeta'\xi,\ \xi\eta' - \xi'\eta$$

respectively.

The distance of a point (x, y, z, w) and plane $(\xi', \eta', \zeta', \omega')$ is the complement of the distance of the point (x, y, z, w) and the point $(\xi', \eta', \zeta', \omega')$ which is the $\perp$ point of the plane; viz. we have

$$\sin\delta = \frac{x\xi' + y\eta' + z\zeta' + w\omega'}{\sqrt{x^2+y^2+z^2+w^2}\sqrt{\xi'^2+\eta'^2+\zeta'^2+\omega'^2}},$$

$$\cos\delta = \frac{\sqrt{a^2+b^2+c^2+f^2+g^2+h^2}}{\sqrt{x^2+y^2+z^2+w^2}\sqrt{\xi'^2+\eta'^2+\zeta'^2+\omega'^2}},$$

where, in the numerator, (a, b, c, f, g, h) stand for the coordinates of the line of junction of the two points. Of course the same result might have been equally well derived from the formula for the distance of two planes.

26. If we now consider a plane triangle ABC, and write

(x_1, y_1, z_1, w_1) for the coordinates of A,

(x_2, y_2, z_2, w_2) „ „ B,

(x_3, y_3, z_3, w_3) „ „ C,

then the coordinates

$$a,\qquad b,\qquad c,\qquad f,\qquad g,\qquad h,$$

of the line BC will be

$$y_2z_3 - y_3z_2,\ z_3x_2 - z_2x_3,\ x_2y_3 - x_3y_2,\ x_2w_3 - x_3w_2,\ y_2w_3 - y_3w_2,\ z_2w_3 - z_3w_2,$$

and similarly for the coordinates of the lines AB, CA; the equations

$$a_1f_2 + b_1g_2 + c_1h_2 + a_2f_1 + b_2g_1 + c_2h_1 = 0, \text{ \&c.},$$

which express that these lines meet in pairs in the points A, B, C respectively, are of course satisfied identically; and we then have for the sides and angles (linear and angular distances) of the triangle

$$\cos a = \frac{x_2x_3 + y_2y_3 + z_2z_3 + w_2w_3}{\sqrt{x_2^2+y_2^2+z_2^2+w_2^2}\sqrt{x_3^2+y_3^2+z_3^2+w_3^2}},$$

$$\sin a = \frac{\sqrt{a_1^2+b_1^2+c_1^2+f_1^2+g_1^2+h_1^2}}{\sqrt{x_2^2+y_2^2+z_2^2+w_2^2}\sqrt{x_3^2+y_3^2+z_3^2+w_3^2}},$$

$$\cos A = \frac{a_2a_3 + b_2b_3 + c_2c_3 + f_2f_3 + g_2g_3 + h_2h_3}{\sqrt{a_2^2+b_2^2+c_2^2+f_2^2+g_2^2+h_2^2}\sqrt{a_3^2+b_3^2+c_3^2+f_3^2+g_3^2+h_3^2}}, \text{ \&c.};$$

and this being so, if with the values of $\cos a$, $\cos b$, $\cos c$, we form the expression for $\cos a - \cos b \cos c$, then reducing to a common denominator, the expression for the numerator is at once found to be

$$= a_2 a_3 + b_2 b_3 + c_2 c_3 + f_2 f_3 + g_2 g_3 + h_2 h_3,$$

and thence easily

$$\cos A = \frac{\cos a - \cos b \cos c}{\sin b \sin c},$$

viz. the expressions for the angles in terms of the sides are those of ordinary spherical trigonometry.

27. Hence also

$$\sin A = \frac{\sqrt{1 - \cos^2 a - \cos^2 b - \cos^2 c + 2 \cos a \cos b \cos c}}{\sin b \sin c};$$

whence

$$\sin A : \sin B : \sin C = \sin a : \sin b : \sin c,$$

and

$$\cos A + \cos B \cos C = \frac{\cos a\,(1 - \cos^2 a - \cos^2 b - \cos^2 c + 2 \cos a \cos b \cos c)}{\sin^2 a \sin b \sin c},$$

and consequently

$$\cos a = \frac{\cos A + \cos B \cos C}{\sin B \sin C},$$

which completes the system of formulæ.

And similarly for a trihedral, that is, if we have three planes A, B, C (meeting of course in a point, O) then the dihedral distances BC, CA, AB and the angular distances CA, AB; AB, BC; BC, CA are related to each other in the same way as the angles and sides of an ordinary spherical triangle.

Distance of a point and line. Art. Nos. 28, 29.

28. The point is taken to be (x_1, y_1, z_1, w_1), the line (a, b, c, f, g, h). Drawing through the point a $\perp$ plane, say $(\xi, \eta, \zeta, \omega)$ meeting the line in the foot, and taking the coordinates hereof to be (x_2, y_2, z_2, w_2), then $\xi x_1 + \eta y_1 + \zeta z_1 + \omega w_1 = 0$ and

$$\begin{aligned} . \quad h\eta - g\zeta + a\omega &= 0, \\ -h\xi \quad . \quad + f\zeta + b\omega &= 0, \\ g\xi - f\eta \quad . \quad + c\omega &= 0, \\ -a\xi - b\eta - c\zeta \quad . \quad &= 0, \end{aligned}$$

giving, say,

$$\begin{aligned} \xi &= \quad . \quad cy_1 - bz_1 + fw_1, \\ \eta &= -cx_1 \quad . \quad + az_1 + gw_1, \\ \zeta &= bx_1 - ay_1 \quad . \quad + hw_1, \\ \omega &= -fx_1 - gy_1 - hz_1 \quad . \end{aligned}$$

We have here

$$\xi^2 + \eta^2 + \zeta^2 + \omega^2 = (b^2 + c^2 + f^2)\, x_1^2 + \&c.,$$

where $(b^2 + c^2 + f^2)\, x_1^2 + \&c.$ denotes the before-mentioned quadric function of (x_1, y_1, z_1, w_1), which, equated to zero and regarding therein (x_1, y_1, z_1, w_1) as current coordinates, represents the pair of tangent-planes from the reciprocal line (f, g, h, a, b, c) to the Absolute.

Resuming the question in hand, we have then

$$\xi x_2 + \eta y_2 + \zeta z_2 + \omega w_2 = 0,$$

which, with

$$\begin{aligned} . \quad\quad hy_2 - gz_2 + aw_2 &= 0, \\ -hx_2 \quad . \quad + fz_2 + bw_2 &= 0, \\ gx_2 - fy_2 \quad . \quad + cw_2 &= 0, \\ -ax_2 - by_2 - cz_2 \quad . \quad &= 0, \end{aligned}$$

gives, say,

$$\begin{aligned} -x_2 &= \quad . \quad c\eta - b\zeta + f\omega, \\ -y_2 &= -c\xi \quad . \quad + a\zeta + g\omega, \\ -z_2 &= -b\xi - a\eta \quad . \quad + h\omega, \\ -w_2 &= -f\xi - g\eta - h\zeta \quad . \quad , \end{aligned}$$

that is,

$$\begin{aligned} x_2 &= (b^2 + c^2 + f^2)\, x_1 + (-ab + fg)\, y_1 + (-ac + hf)\, z_1 + (-cg + bh)\, w_1, \\ y_2 &= (-ab + fg)\, x_1 + (c^2 + a^2 + g^2)\, y_1 + (-bc + gh)\, z_1 + (-ah + cf)\, w_1, \\ z_2 &= (-ca + hf)\, x_1 + (-bc + gh)\, y_1 + (a^2 + b^2 + h^2)\, z_1 + (-bf + ag)\, w_1, \\ w_2 &= (-cg + bh)\, x_1 + (-ah + cf)\, y_1 + (-bf + ag)\, z_1 + (f^2 + g^2 + h^2)\, w_1. \end{aligned}$$

We have therefore

$$x_1x_2 + y_1y_2 + z_1z_2 + w_1w_2 = (b^2 + c^2 + f^2)\, x_1^2 + \&c.,$$

and

$$x_2^2 + y_2^2 + z_2^2 + w_2^2 = (a^2 + b^2 + c^2 + f^2 + g^2 + h^2)\, \{(b^2 + c^2 + f^2)\, x_1^2 + \&c.\},$$

where $(b^2 + c^2 + f^2)\, x_1^2 + \&c.$ denotes in each case the above-mentioned quadric function of (x_1, y_1, z_1, w_1).

In verification of the expression for $x_2^2 + y_2^2 + z_2^2 + w_2^2$, it is to be remarked that we have identically

$$\begin{aligned} &\xi^2 + \eta^2 + \zeta^2 + \omega^2 + (af + bg + ch)^2 (x_1^2 + y_1^2 + z_1^2 + w_1^2) \\ &\qquad = (a^2 + b^2 + c^2 + f^2 + g^2 + h^2)\, \{(b^2 + c^2 + f^2)\, x_1^2 + \&c.\}\,; \end{aligned}$$

here on the left-hand side the whole coefficient of x_1^2 is

$$(b^2 + c^2 + f^2)^2 + (ab - fg)^2 + (ca - hf)^2 + (cg - bh)^2 + (af + bg + ch)^2,$$

where the last four terms are together $= (b^2 + c^2 + f^2)(a^2 + g^2 + h^2)$, and thus the whole coefficient is (as it should be) $= (b^2 + c^2 + f^2)(a^2 + b^2 + c^2 + f^2 + g^2 + h^2)$: and similarly for the coefficients of the remaining terms.

29. Writing then δ for the required distance, we have

$$\cos\delta = \frac{x_1x_2 + y_1y_2 + z_1z_2 + w_1w_2}{\sqrt{x_1^2 + y_1^2 + z_1^2 + w_1^2}\sqrt{x_2^2 + y_2^2 + z_2^2 + w_2^2}},$$

that is,

$$\cos\delta = \frac{\sqrt{(b^2 + c^2 + f^2)\,x_1^2 + \&c.}}{\sqrt{x_1^2 + y_1^2 + z_1^2 + w_1^2}\sqrt{a^2 + b^2 + c^2 + f^2 + g^2 + h^2}},$$

where $(b^2 + c^2 + f^2)\,x_1^2 + \&c.$ is the above-mentioned quadric function

	x_1	y_1	z_1	w_1
x_1	$b^2 + c^2 + f^2$	$-ab + fg$	$-ac + hf$	$-cg + bh$
y_1	$-ab + fg$	$c^2 + a^2 + g^2$	$-bc + gh$	$-ah + cf$
z_1	$-ac + hf$	$-bc + gh$	$a^2 + b^2 + h^2$	$-bf + ag$
w_1	$-cg + bh$	$-ah + cf$	$-bf + ag$	$f^2 + g^2 + h^2$

Distance of a plane and line. Art. No. 30.

30. This may be deduced from the last preceding result: the formula, as written down, gives the distance of the $\perp$ plane (x_1, y_1, z_1, w_1) from the reciprocal line (f, g, h, a, b, c): hence writing $(\xi, \eta, \zeta, \omega)$ for (x_1, y_1, z_1, w_1) and (a, b, c, f, g, h) for (f, g, h, a, b, c), we have for the distance of plane $(\xi, \eta, \zeta, \omega)$ and line (a, b, c, f, g, h) the expression

$$\cos\delta = \frac{\sqrt{(a^2 + g^2 + h^2)\,\xi^2 + \&c.}}{\sqrt{\xi^2 + \eta^2 + \zeta^2 + \omega^2}\sqrt{a^2 + b^2 + c^2 + f^2 + g^2 + h^2}},$$

where $(a^2 + g^2 + h^2)\,\xi^2 + \&c.$ denotes the quadric function

	ξ	η	ζ	ω
ξ	$a^2 + g^2 + h^2$	$ab - fg$	$ac - hf$	$cg - bh$
η	$ab - fg$	$b^2 + h^2 + f^2$	$bc - gh$	$ah - cf$
ζ	$ac - hf$	$bc - gh$	$c^2 + f^2 + g^2$	$bf - ag$
ω	$cg - bh$	$ah - cf$	$bf - ag$	$a^2 + b^2 + c^2$

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31. Considering any two lines X, Y, it has been seen that these have two $\perp$s, viz. each $\perp$ is a line cutting as well the two lines X, Y as the reciprocal lines X', Y', say that one of them cuts the lines X, Y in the points A, C respectively, and the other of them cuts the two lines in the points B, D respectively: and take, as before, the distances AC and BD to be $= \delta$ and θ respectively.

The coordinates of the lines X, Y are

$$(a,\ b,\ c,\ f,\ g,\ h) \text{ and } (a_1,\ b_1,\ c_1,\ f_1,\ g_1,\ h_1)$$

respectively; and if we consider, as before, the planes Π, Ω, Π_1, Ω_1 the coordinates of which are (l, m, n, p), $(\lambda, \mu, \nu, \varpi)$, (l_1, m_1, n_1, p_1), $(\lambda_1, \mu_1, \nu_1, \varpi_1)$ respectively, then X is the intersection of the planes Π, Ω, and we have

$$\begin{aligned} &a : b : c : f : g : h \\ &= l\varpi - \lambda p : m\varpi - \mu p : n\varpi - \nu p : m\nu - n\mu : n\lambda - l\nu : l\mu - m\lambda ; \end{aligned}$$

and similarly Y is the intersection of the planes Π_1, Ω_1, and we have

$$\begin{aligned} &a_1 : b_1 : c_1 : f_1 : g_1 : h_1 \\ &= l_1\varpi_1 - \lambda_1 p_1 : m_1\varpi_1 - \mu_1 p_1 : n_1\varpi_1 - \nu_1 p_1 : m_1\nu_1 - n_1\mu_1 : n_1\lambda_1 - l_1\nu_1 : l_1\mu_1 - m_1\lambda_1 . \end{aligned}$$

Also the planes (Π, Ω), (Π_1, Ω_1), (Π, Ω_1), (Π_1, Ω) being naturally $\perp$, we have

$$\begin{aligned} l\lambda + m\mu + n\nu + p\varpi &= 0, \\ l_1\lambda_1 + m_1\mu_1 + n_1\nu_1 + p_1\varpi_1 &= 0, \\ l\lambda_1 + m\mu_1 + n\nu_1 + p\varpi_1 &= 0, \\ l_1\lambda + m_1\mu + n_1\nu + p_1\varpi &= 0 ; \end{aligned}$$

and for the inclinations to each other of the planes (Π, Π_1) and (Ω, Ω_1), we have

$$\cos\delta = \frac{\lambda\lambda_1 + \mu\mu_1 + \nu\nu_1 + \varpi\varpi_1}{\sqrt{\lambda^2 + \&c.}\ \sqrt{\lambda_1{}^2 + \&c.}},$$

$$\cos\theta = \frac{ll_1 + mm_1 + nn_1 + pp_1}{\sqrt{l^2 + \&c.}\ \sqrt{l_1{}^2 + \&c.}} .$$

32. The expressions for the coordinates of the two lines give

$$\begin{aligned} aa_1 + bb_1 + cc_1 + ff_1 + gg_1 + hh_1 = &\ (ll_1 + mm_1 + nn_1 + pp_1)(\lambda\lambda_1 + \mu\mu_1 + \nu\nu_1 + \varpi\varpi_1) \\ &- (l\lambda_1 + m\mu_1 + n\nu_1 + p\varpi_1)(l_1\lambda + m_1\mu + n_1\nu + p_1\varpi) \\ = &\ (ll_1 + mm_1 + nn_1 + pp_1)(\lambda\lambda_1 + \mu\mu_1 + \nu\nu_1 + \varpi\varpi_1) \\ = &\sqrt{l^2 + \&c.}\ \sqrt{l_1{}^2 + \&c.}\ \sqrt{\lambda^2 + \&c.}\ \sqrt{\lambda_1{}^2 + \&c.}\ \cos\delta\cos\theta . \end{aligned}$$

But we have

$$\begin{aligned} a^2 + b^2 + c^2 + f^2 + g^2 + h^2 &= (l^2 + m^2 + n^2 + p^2)(\lambda^2 + \mu^2 + \nu^2 + \varpi^2) - (l\lambda + m\mu + n\nu + p\varpi)^2 \\ &= (l^2 + \&c.)(\lambda^2 + \&c.) ; \end{aligned}$$

and similarly

$$\begin{aligned} a_1{}^2 + b_1{}^2 + c_1{}^2 + f_1{}^2 + g_1{}^2 + h_1{}^2 &= (l_1{}^2 + m_1{}^2 + n_1{}^2 + p_1{}^2)(\lambda_1{}^2 + \mu_1{}^2 + \nu_1{}^2 + \varpi_1{}^2) - (l_1\lambda_1 + m_1\mu_1 + n_1\nu_1 + p_1\varpi_1)^2 \\ &= (l_1{}^2 + \&c.)(\lambda_1{}^2 + \&c.). \end{aligned}$$

Hence the last result gives

$$\frac{aa_1 + bb_1 + cc_1 + ff_1 + gg_1 + hh_1}{\sqrt{a^2 + \&c.}\ \sqrt{a_1{}^2 + \&c.}} = \cos\delta\cos\theta ;$$

or calling the expression on the left-hand side the comoment of the two lines, and denoting it by M_1, the equation just obtained is

$$\cos\delta\cos\theta = \text{comoment}, = M_1.$$

And if for either of the lines we substitute its reciprocal, then for δ, θ we have $\frac{1}{2}\pi-\delta$, $\frac{1}{2}\pi-\theta$ respectively, and consequently

$$\frac{af_1+bg_1+ch_1+a_1f+b_1g+c_1h}{\sqrt{a^2+\&c.}\,\sqrt{a_1{}^2+\&c.}} = \sin\delta\sin\theta\,;$$

or calling the expression on the left-hand side the moment of the two lines and denoting it by M, the equation is

$$\sin\delta\sin\theta = \text{moment}, = M,$$

where observe that $M=0$ is the condition for the intersection of the two lines, $M_1=0$ the condition for their contrasection*.

33. But to determine the coordinates (A, B, C, F, G, H) of the $\perp$ line AC or BD, and the coordinates of the points A and C or B and D of the points in which it meets the lines X and Y respectively, I employ a different method.

We consider the lines

$$(a, b, c, f, g, h),\quad (a_1, b_1, c_1, f_1, g_1, h_1),$$

and their reciprocals

$$(f, g, h, a, b, c),\quad (f_1, g_1, h_1, a_1, b_1, c_1).$$

A line (A, B, C, F, G, H) meeting each of these four lines is said to be a perpendicular. We have

$$\begin{aligned}(A, B, C, F, G, H)&(a,\ b,\ c,\ f,\ g,\ h)=0,\\ \text{,,}\qquad&(f,\ g,\ h,\ a,\ b,\ c)=0,\\ \text{,,}\qquad&(a_1, b_1, c_1, f_1, g_1, h_1)=0,\\ \text{,,}\qquad&(f_1, g_1, h_1, a_1, b_1, c_1)=0,\end{aligned}$$

equations which determine say A, B, C, F in terms of G, H, and then substituting in $AF+BG+CH=0$ we have two values of $G:H$; i.e. there are two systems of values (A, B, C, F, G, H), that is, two perpendiculars.

The equations may be written

$$\begin{aligned}(A+F)(a+f)+(B+G)(b+g)+(C+H)(c+h)&=0,\\ (A+F)(a_1+f_1)+(B+G)(b_1+g_1)+(C+H)(c_1+h_1)&=0,\\ (A-F)(a-f)+(B-G)(b-g)+(C-H)(c-h)&=0,\\ (A-F)(a_1-f_1)+(B-G)(b_1-g_1)+(C-H)(c_1-h_1)&=0.\end{aligned}$$

* The foregoing demonstration of the fundamental formulæ $\cos\delta\cos\theta=M_1$, $\sin\delta\sin\theta=M$, is, in effect, that given by Heath in his Memoir "On the Dynamics of a Rigid Body in Elliptic Space," *Phil. Trans.* t. 175 (for 1884), pp. 281—324.

Hence we have

$$\begin{array}{ccccc} A+F & : & B+G & : & C+H, = \\ (b+g)(c_1+h_1)-(b_1+g_1)(c+h) & : & (c+h)(a_1+f_1)-(a+f)(c_1+h_1) & : & (a+f)(b_1+g_1)-(a_1+f_1)(b+g), = \\ \mathfrak{A}+\alpha & : & \mathfrak{B}+\beta & : & \mathfrak{C}+\gamma; \\ A-F & : & B-G & : & C-H, = \\ (b-g)(c_1-h_1)-(b_1-g_1)(c-h) & : & (c-h)(a_1-f_1)-(a-f)(c_1-h_1) & : & (a-f)(b_1-g_1)-(a_1-f_1)(b-g), = \\ \mathfrak{A}-\alpha & : & \mathfrak{B}-\beta & : & \mathfrak{C}-\gamma; \end{array}$$

equations which may be written

$$A+F,\ B+G,\ C+H = 2\lambda\,(\mathfrak{A}+\alpha,\ \mathfrak{B}+\beta,\ \mathfrak{C}+\gamma),$$
$$A-F,\ B-G,\ C-H = 2\mu\,(\mathfrak{A}-\alpha,\ \mathfrak{B}-\beta,\ \mathfrak{C}-\gamma),$$

where

$$\mathfrak{A} = bc_1 - b_1c + gh_1 - g_1h, \quad \alpha = bh_1 - b_1h - (cg_1 - c_1g),$$
$$\mathfrak{B} = ca_1 - c_1a + hf_1 - h_1f, \quad \beta = cf_1 - c_1f - (ah_1 - a_1h),$$
$$\mathfrak{C} = ab_1 - a_1b + fg_1 - f_1g, \quad \gamma = ag_1 - a_1g - (bf_1 - b_1f).$$

34. We have

$$(\mathfrak{A}+\alpha)^2+(\mathfrak{B}+\beta)^2+(\mathfrak{C}+\gamma)^2 = \{(a+f)^2+(b+g)^2+(c+h)^2\}\{(a_1+f_1)^2+(b_1+g_1)^2+(c_1+h_1)^2\} - \{(a+f)(a_1+f_1)+(b+g)(b_1+g_1)+(c+h)(c_1+h_1)\}^2,$$
$$(\mathfrak{A}-\alpha)^2+(\mathfrak{B}-\beta)^2+(\mathfrak{C}-\gamma)^2 = \{(a-f)^2+(b-g)^2+(c-h)^2\}\{(a_1-f_1)^2+(b_1-g_1)^2+(c_1-h_1)^2\} - \{(a-f)(a_1-f_1)+(b-g)(b_1-g_1)+(c-h)(c_1-h_1)\}^2;$$

or putting

$$\rho^2 = a^2+b^2+c^2+f^2+g^2+h^2,$$
$$\rho_1{}^2 = a_1{}^2+b_1{}^2+c_1{}^2+f_1{}^2+g_1{}^2+h_1{}^2,$$
$$\sigma_1 = aa_1+bb_1+cc_1+ff_1+gg_1+hh_1,$$
$$\sigma = af_1+bg_1+ch_1+a_1f+b_1g+c_1h,$$

the foregoing values are

$$= \rho^2\rho_1{}^2-(\sigma+\sigma_1)^2, \quad \rho^2\rho_1{}^2-(\sigma-\sigma_1)^2.$$

Hence

$$A^2+B^2+C^2+F^2+G^2+H^2 = 4\lambda^2\{\rho^2\rho_1{}^2-(\sigma+\sigma_1)^2\} = 4\mu^2\{\rho^2\rho_1{}^2-(\sigma-\sigma_1)^2\};$$

or we may write

$$\lambda^2 = \rho^2\rho_1{}^2-(\sigma-\sigma_1)^2, \quad \text{or say} \quad \lambda = \sqrt{\rho^2\rho_1{}^2-(\sigma-\sigma_1)^2},$$
$$\mu^2 = \rho^2\rho_1{}^2-(\sigma+\sigma_1)^2, \qquad \mu = -\sqrt{\rho^2\rho_1{}^2-(\sigma+\sigma_1)^2}.$$

Making a slight change of notation, if we put

$$M = \frac{af_1+bg_1+ch_1+a_1f+b_1g+c_1h}{\sqrt{a^2+\&c.}\,\sqrt{a_1{}^2+\&c.}} = \frac{\sigma}{\rho\rho_1},$$

$$M_1 = \frac{aa_1+bb_1+cc_1+ff_1+gg_1+hh_1}{\sqrt{a^2+\&c.}\,\sqrt{a_1{}^2+\&c.}} = \frac{\sigma_1}{\rho\rho_1},$$

then the values are

$$\lambda = rr_1 \sqrt{1-(M-M_1)^2}, \quad \mu = -rr_1 \sqrt{1-(M+M_1)^2}.$$

And, this being so, the two systems of values of A, B, C, F, G, H, are

$$\begin{array}{l|l} \lambda(\mathfrak{A}+\alpha)+\mu(\mathfrak{A}-\alpha), & \lambda(\mathfrak{A}+\alpha)-\mu(\mathfrak{A}-\alpha), \\ \lambda(\mathfrak{B}+\beta)+\mu(\mathfrak{B}-\beta), & \lambda(\mathfrak{B}+\beta)-\mu(\mathfrak{B}-\beta), \\ \lambda(\mathfrak{C}+\gamma)+\mu(\mathfrak{C}-\gamma), & \lambda(\mathfrak{C}+\gamma)-\mu(\mathfrak{C}-\gamma), \\ \lambda(\mathfrak{A}+\alpha)-\mu(\mathfrak{A}-\alpha), & \lambda(\mathfrak{A}+\alpha)+\mu(\mathfrak{A}-\alpha), \\ \lambda(\mathfrak{B}+\beta)-\mu(\mathfrak{B}-\beta), & \lambda(\mathfrak{B}+\beta)+\mu(\mathfrak{B}-\beta), \\ \lambda(\mathfrak{C}+\gamma)-\mu(\mathfrak{C}-\gamma), & \lambda(\mathfrak{C}+\gamma)+\mu(\mathfrak{C}-\gamma); \end{array}$$

viz. the two perpendiculars are reciprocals each of the other.

35. Before going further I notice that, if

$$\frac{a_1+f_1}{a+f}=\frac{b_1+g_1}{b+g}=\frac{c_1+h_1}{c+h} \quad \text{or} \quad \frac{a_1-f_1}{a-f}=\frac{b_1-g_1}{b-g}=\frac{c_1-h_1}{c-h},$$

then the four equations for (A, B, C, F, G, H) reduce themselves to three equations only: and thus instead of two perpendiculars we have a singly infinite series of perpendiculars (see *ante* 15).

To explain the meaning of the equations, I observe that a line (a, b, c, f, g, h) will be a generating line of the one kind, or say a "generatrix," of the Absolute if

$$a+f=0, \quad b+g=0, \quad c+h=0:$$

and it will be a generating line of the other kind, or say a "directrix," of the Absolute if $a-f=0$, $b-g=0$, $c-h=0$. Or what is the same thing, we have

$$(\mathrm{a},\ \mathrm{b},\ \mathrm{c},\ -\mathrm{a},\ -\mathrm{b},\ -\mathrm{c}), \text{ where } \mathrm{a}^2+\mathrm{b}^2+\mathrm{c}^2=0 \text{ for a generatrix,}$$

and

$$(\mathrm{a},\ \mathrm{b},\ \mathrm{c},\ \ \mathrm{a},\ \ \mathrm{b},\ \ \mathrm{c}), \text{ where } \mathrm{a}^2+\mathrm{b}^2+\mathrm{c}^2=0 \text{ for a directrix, of the Absolute.}$$

Consider now two directrices $(\mathrm{a}, \mathrm{b}, \mathrm{c}, \mathrm{a}, \mathrm{b}, \mathrm{c})$ and $(\mathrm{a}_1, \mathrm{b}_1, \mathrm{c}_1, \mathrm{a}_1, \mathrm{b}_1, \mathrm{c}_1)$: if a line (a, b, c, f, g, h) meets each of these, then

$$(a+f)\,\mathrm{a} + (b+g)\,\mathrm{b} + (c+h)\,\mathrm{c} = 0,$$
$$(a+f)\,\mathrm{a}_1 + (b+g)\,\mathrm{b}_1 + (c+h)\,\mathrm{c}_1 = 0,$$

and consequently

$$a+f : b+g : c+h = \mathrm{bc}_1-\mathrm{b}_1\mathrm{c} : \mathrm{ca}_1-\mathrm{c}_1\mathrm{a} : \mathrm{ab}_1-\mathrm{a}_1\mathrm{b};$$

and similarly if $(a_1, b_1, c_1, f_1, g_1, h_1)$ meets each of the two directrices, then

$$a_1+f_1 : b_1+g_1 : c_1+h_1 = \mathrm{bc}_1-\mathrm{b}_1\mathrm{c} : \mathrm{ca}_1-\mathrm{c}_1\mathrm{a} : \mathrm{ab}_1-\mathrm{a}_1\mathrm{b},$$

that is, if the lines each of them meet the same two directrices of the Absolute, then

$$\frac{a_1+f_1}{a+f}=\frac{b_1+g_1}{b+g}=\frac{c_1+h_1}{c+h}.$$

Conversely, if these relations are satisfied, then the lines each of them meet two directrices of the Absolute.

In like manner, if the lines each meet two generatrices of the Absolute, then

$$\frac{a_1-f_1}{a-f}=\frac{b_1-g_1}{b-g}=\frac{c_1-h_1}{c-h};$$

and conversely, if these relations are satisfied, then the lines each of them meet the same two generatrices of the Absolute. In the former case, the lines are said to be "right parallels": in the latter case, "left parallels."

A line (a, b, c, f, g, h) meets the Absolute in two points, and through each of these we have a directrix and a generatrix: that is, the line meets two directrices and two generatrices.

Through a given point we may draw, meeting the two directrices, or meeting the two generatrices, a line: that is, through a given point we may draw a line

$$(a_1, b_1, c_1, f_1, g_1, h_1)$$

which is a right parallel, and a line which is a left parallel, to a given line. That is, regarding as given the first line, and also a point of the second line, there are two positions of the second line such that for each of them, the $\perp$s of the pair of lines, instead of being two determinate lines, are a singly infinite series of lines.

36. Reverting to the general case, we have found (A, B, C, F, G, H) the coordinates of either of the lines $\perp$ to the given lines (a, b, c, f, g, h) and $(a_1, b_1, c_1, f_1, g_1, h_1)$: supposing that the $\perp$ intersects the first of these lines in the point the coordinates of which are (x, y, z, w), and the second in the point the coordinates of which are

$$(x_1, y_1, z_1, w_1),$$

then we have for each set of coordinates a fourfold expression; the choice of the form is indifferent, and I write

$$x : y : z : w = cB-bC : aC-cA : bA-aB : fA+gB+hC,$$
$$x_1 : y_1 : z_1 : w_1 = c_1B-b_1C : a_1C-c_1A : b_1A-a_1B : f_1A+g_1B+h_1C.$$

We have then, for the distance of these two points,

$$\cos\phi=\frac{xx_1+yy_1+zz_1+ww_1}{\sqrt{x^2+y^2+z^2+w^2}\sqrt{x_1^2+y_1^2+z_1^2+w_1^2}},\quad \sin\phi=\frac{\sqrt{(yz_1-y_1z)^2+\&c.}}{\sqrt{x^2+y^2+z^2+w^2}\sqrt{x_1^2+y_1^2+z_1^2+w_1^2}};$$

where $\phi=\delta$ or θ, according to the sign of the radical $\lambda:\mu$ contained in the expressions for A, B, C, F, G, H.

I have not succeeded in obtaining in this manner the final formulæ for the determination of the distances: these in fact are, by what precedes, given by the equations

$$\sin\delta\sin\theta=M,\quad \cos\delta\cos\theta=M_1.$$

For then, writing ϕ to denote either of the distances δ, θ, at pleasure, we have

$$\frac{M^2}{\sin^2\phi}+\frac{M_1^2}{\cos^2\phi}=1,$$

that is,

$$\cos^4\phi + \cos^2\phi\,(M_1^2 - M^2 + 1) + M_1^2 = 0,$$

or

$$\cos^2\phi = \tfrac{1}{2}\{M_1^2 - M^2 + 1 \pm \sqrt{M_1^4 + M^4 - 2M_1^2M^2 - 2M_1^2 - 2M^2 + 1}\},$$

which is the expression for the cosine of the distance.

In the case where the two lines intersect $M=0$, and if δ be the $\perp$ distance which vanishes, then $\delta=0$, and consequently $\cos\theta = M_1$: the last-mentioned formula, putting therein $M=0$ and taking the radical to be $=M_1^2-1$, gives $\cos^2\phi = M_1^2$, that is, $\phi=\theta$, and $\cos^2\theta = M_1^2$, as it should be.

37. I verify as follows, in the case in question of two *intersecting* lines,

$$(af_1 + bg_1 + ch_1 + a_1f + b_1g + c_1h = 0),$$

the formula

$$\cos\theta = \frac{xx_1 + yy_1 + zz_1 + ww_1}{\sqrt{x^2+y^2+z^2+w^2}\,\sqrt{x_1^2+y_1^2+z_1^2+w_1^2}}.$$

We have here

$$\begin{aligned}
A &= \mathfrak{A} = bc_1 - b_1c + gh_1 - g_1h,\\
B &= \mathfrak{B} = ca_1 - c_1a + hf_1 - h_1f,\\
C &= \mathfrak{C} = ab_1 - a_1b + fg_1 - f_1g,\\
F &= \alpha = bh_1 - b_1h - cg_1 + c_1g,\\
G &= \beta = cf_1 - c_1f - ah_1 + a_1h,\\
H &= \gamma = ag_1 - a_1g - bf_1 + b_1f.
\end{aligned}$$

I stop to notice that these formulæ may be obtained in a different and somewhat more simple manner: the two lines (a, b, c, f, g, h) and $(a_1, b_1, c_1, f_1, g_1, h_1)$ intersect; hence their reciprocals also intersect: the equation of the plane through the two lines and that of the plane through the two reciprocal lines are respectively

$$\begin{aligned}
&(gh_1 - g_1h)\,x + (hf_1 - h_1f)\,y + (fg_1 - f_1g)\,z + (af_1 + bg_1 + ch_1)\,w = 0,\\
&(bc_1 - b_1c)\,x + (ca_1 - c_1a)\,y + (ab_1 - a_1b)\,z + (fa_1 + gb_1 + hc_1)\,w = 0:
\end{aligned}$$

the line (A, B, C, F, G, H) is thus the line of intersection of these two planes, and it is thence easy to obtain the foregoing values.

From the values of A, B, C, F, G, H, we have to find x, y, z, w and x_1, y_1, z_1, w_1 by the formulæ given above. We have

$$\begin{aligned}
x = cB - bC = {} & c^2a_1 - cc_1a + chf_1 - ch_1f\\
& - abb_1 + a_1b^2 - bfg_1 + bgf_1\\
= {} & (bg + ch)f_1 + a_1(b^2+c^2) - b_1ab - c_1ac - bfg_1 - cfh_1\\
= {} & -f(af_1 + bg_1 + ch_1) + a_1(b^2+c^2) - b_1ab - c_1ac;
\end{aligned}$$

or writing here $a_1f + b_1g + c_1h$ in place of $-(af_1 + bg_1 + ch_1)$, this is a linear function of a_1, b_1, c_1; and similarly finding the values of y, z, w, we have

$$\begin{aligned}
x &= a_1(b^2+c^2+f^2) + b_1(fg - ab) + c_1(hf - ca),\\
y &= a_1(fg - ab) + b_1(c^2+a^2+g^2) + c_1(gh - bc),\\
z &= a_1(hf - ca) + b_1(gh - bc) + c_1(a^2+b^2+h^2),\\
w &= a_1(bh - cg) + b_1(cf - ah) + c_1(ag - bf).
\end{aligned}$$

And in like manner, (I introduce for convenience the sign $-$, as is allowable),

$$\begin{aligned}
-x_1 &= a(b_1^2+c_1^2+f_1^2) + b(f_1g_1 - a_1b_1) + c(h_1f_1 - c_1a_1),\\
-y_1 &= a(f_1g_1 - a_1b_1) + b(c_1^2+a_1^2+g_1^2) + c(g_1h_1 - b_1c_1),\\
-z_1 &= a(h_1f_1 - c_1a_1) + b(g_1h_1 - b_1c_1) + c(a_1^2+b_1^2+h_1^2),\\
-w_1 &= a(b_1h_1 - c_1g_1) + b(c_1f_1 - a_1h_1) + c(a_1g_1 - b_1f_1).
\end{aligned}$$

38. Write for shortness

$$p = a^2+b^2+c^2, \qquad p_1 = f^2+g^2+h^2, \qquad \omega = a_1f + b_1g + c_1h,$$

and therefore

$$q = aa_1 + bb_1 + cc_1, \quad q_1 = ff_1 + gg_1 + hh_1, \quad -\omega = af_1 + bg_1 + ch_1,$$

$$r = a_1^2+b_1^2+c_1^2, \qquad r_1 = f_1^2+g_1^2+h_1^2.$$

We have

$$\begin{aligned}
x &= a_1p - aq + f\omega, & x_1 &= -ar + a_1q + f_1\omega,\\
y &= b_1p - bq + g\omega, & y_1 &= -br + b_1q + g_1\omega,\\
z &= c_1p - cq + h\omega, & z_1 &= -cr + c_1q + h_1\omega,
\end{aligned}$$

$$w = -\begin{vmatrix} f, & g, & h \\ a, & b, & c \\ a_1, & b_1, & c_1 \end{vmatrix}, \quad w_1 = -\begin{vmatrix} f_1, & g_1, & h_1 \\ a, & b, & c \\ a_1, & b_1, & c_1 \end{vmatrix};$$

from which we easily obtain

$$x^2+y^2+z^2 = p(pr-q^2) + (p_1+2p)\,\omega^2,$$

and by expressing w^2 in the form of a determinant

$$w^2 = p_1(pr-q^2) - p\omega^2,$$

we obtain

$$x^2+y^2+z^2+w^2 = (p+p_1)(pr-q^2+\omega^2);$$

and in like manner

$$x_1^2+y_1^2+z_1^2+w_1^2 = (r+r_1)(pr-q^2+\omega^2).$$

And again

$$xx_1+yy_1+zz_1 = q(pr-q^2) + (q_1+2q)\,\omega^2,$$

and by expressing ww_1 in the form of a determinant

$$ww_1 = q_1(pr-q^2) - q\omega^2,$$

we find

$$xx_1+yy_1+zz_1+ww_1 = (q+q_1)(pr-q^2+\omega^2).$$

Hence substituting in

$$\cos\theta = \frac{xx_1+yy_1+zz_1+ww_1}{\sqrt{x^2+y^2+z^2+w^2}\,\sqrt{x_1^2+y_1^2+z_1^2+w_1^2}},$$

the factor $pr-q^2+\omega^2$ disappears, and we have

$$\cos\theta = \frac{q+q_1}{\sqrt{p+p_1}\,\sqrt{r+r_1}} = M_1,$$

the required result.