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ILLUSTRATION OF A THEOREM IN THE THEORY OF EQUATIONS.

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THE knowledge of the value of an unsymmetrical function of the roots of a numerical equation adds something to what is given by the equation itself; but it may or may not add anything to what is given by the equation itself in regard to each root separately. If, for instance, α, β, γ being the roots of a cubic equation, it is known that $\alpha^2\beta+\beta^2\gamma+\gamma^2\alpha =$ a given value k, then α, β, γ must denote the roots, taken not in any order whatever, nor yet in a uniquely determinate order, but with a certain restriction as to order, viz. if the roots in a certain order are a, b, c, these roots being such that $a^2b+b^2c+c^2a=k$, then clearly the relation in question $\alpha^2\beta+\beta^2\gamma+\gamma^2\alpha=k$, will be satisfied if $\alpha, \beta, \gamma = a, b, c$, or $=b, c, a$, or $=c, a, b$ (but not if $\alpha, \beta, \gamma = b, a, c$, or $=$ either of the remaining two arrangements); the relation thus allows α to be $=a$, or $=b$, or $=c$; that is, α is $=$ any one at pleasure of the roots of the cubic equation, and it is thus determined by the cubic equation, and not by any inferior equation; but α being known, the other two roots β and γ will be uniquely, and therefore rationally, determined.

It is worth while to see how the result works out; suppose, for greater simplicity, the cubic equation is $x^3-7x+6=0$ having roots $(1, 2, -3)$, and that the given relation is $\alpha^2\beta+\beta^2\gamma+\gamma^2\alpha=-1$, then the cubic equation gives

$$\alpha+\beta+\gamma=0,\quad \alpha\beta+\alpha\gamma+\beta\gamma=-7,\quad \alpha\beta\gamma=-6,$$

and we have, besides, the relation in question

$$\alpha^2\beta+\beta^2\gamma+\gamma^2\alpha=-1;$$

eliminating γ we have

$$\alpha^2+\alpha\beta+\beta^2=7,\quad \alpha\beta(\alpha+\beta)=6,\quad \alpha^3+3\alpha^2\beta-\beta^3+1=0;$$

or, as it is convenient to write these equations,

$$\begin{aligned}\beta^2+\quad \alpha\beta+\alpha^2-7&=0,\\ \beta^2+\quad \alpha\beta-\frac{6}{\alpha}\qquad&=0,\\ \beta^3-3\alpha^2\beta-\alpha^3-1&=0.\end{aligned}$$

If from these equations we eliminate β, we obtain two equations in α, which it might be supposed would determine α uniquely; but, by what precedes, α is any root at pleasure of the cubic equation and can thus be determined only by the cubic equation itself, and it follows that any equation obtained by the elimination of β must contain as a factor the cubic function $\alpha^3 - 7\alpha + 6$, and be thus of the form $M(\alpha^3 - 7\alpha + 6) = 0$, where M is a function of α; one result of the elimination is $\alpha^3 - 7\alpha + 6 = 0$, and every other result is of the form just referred to, $M(\alpha^3 - 7\alpha + 6) = 0$; hence we have definitely $\alpha^3 - 7\alpha + 6 = 0$, viz. the roots of the equation $M = 0$ do not apply to the question.

In verification, observe that the first and second equations give $\alpha^2 - 7 = \frac{6}{\alpha}$, that is, $\alpha^2 - 6\alpha + 7 = 0$. To eliminate β from the first and third equations we first find

$$\alpha\beta^2 + (4\alpha^2 - 7)\,\beta + \alpha^3 + 1 = 0,$$

or say

$$\beta^2 + \left(4\alpha - \frac{7}{\alpha}\right)\beta + \alpha^2 + \frac{1}{\alpha} = 0,$$

and combining herewith the first equation

$$\beta^2 + \alpha\beta + \alpha^2 - 7 = 0,$$

we obtain

$$\beta\left(3\alpha - \frac{7}{\alpha}\right) + 7 + \frac{1}{\alpha} = 0,$$

that is,

$$\beta = \frac{7\alpha + 1}{-3\alpha^2 + 7};$$

substituting in the first equation,

$$\begin{aligned}&(7\alpha + 1)^3\\ &+ \alpha(7\alpha + 1)(-3\alpha^2 + 7)\\ &+ \quad (\alpha^2 - 7)(-3\alpha^2 + 7)^2 = 0,\end{aligned}$$

that is,

$$\begin{array}{ccccccc} & & & & 49 & 14 & 1\\ & & -21 & -3 & +49 & +7 & \\ 9 & 0 & -105 & & +343 & & -343\\ \hline 9 & 0 & -126 & -3 & +441 & +21 & -342,\end{array}$$

or, dividing by 3,

$$3\alpha^6 - 42\alpha^4 - \alpha^3 + 147\alpha^2 + 7\alpha - 114 = 0,$$

which, in fact, is

$$(\alpha^3 - 7\alpha + 6)(3\alpha^3 - 21\alpha - 19) = 0,$$

of the form in question $M(\alpha^3 - 7\alpha + 6) = 0$. Thus α has any one at pleasure of the three values 1, 2, -3, but α being known we have $\beta = \frac{7\alpha + 1}{-3\alpha^2 + 7}$, and thence

$$\gamma = -\alpha + \frac{-7\alpha - 1}{-3\alpha^2 + 7}, \quad = \frac{3\alpha^3 - 14\alpha - 1}{-3\alpha^2 + 7};$$

in particular, as $\alpha = 1$, then $\beta = 2$ and $\gamma = -3$.