742.

ON THE TRANSFORMATION OF COORDINATES.

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THE formulæ for the transformation between two sets of oblique coordinates assume a very elegant form when presented in the notation of matrices. I call to mind that a matrix denotes a system of quantities arranged in a square form

$$\begin{array}{|ccc|} \multicolumn{3}{c}{(\ \alpha,\ \ \beta,\ \ \gamma\ \ ),} \\ \alpha', & \beta', & \gamma' \\ \alpha'', & \beta'', & \gamma'' \end{array}$$

see my "Memoir on the Theory of Matrices," *Phil. Trans.* t. CXLVIII. (1858), pp. 17—37, [152]; moreover $(\alpha, \beta, \gamma \big)\!\big(x, y, z)$ denotes $\alpha x + \beta y + \gamma z$, and so

$$\begin{array}{|ccc|l} \multicolumn{3}{c}{(\ \alpha,\ \ \beta,\ \ \gamma\ } & \big)\!\big(x, y, z) \\ \alpha', & \beta', & \gamma' & \\ \alpha'', & \beta'', & \gamma'' & \end{array}$$

denotes

$$(\alpha x + \beta y + \gamma z,\ \alpha' x + \beta' y + \gamma' z,\ \alpha'' x + \beta'' y + \gamma'' z),$$

and again

$$\begin{array}{|ccc|l} \multicolumn{3}{c}{(\ \alpha,\ \ \beta,\ \ \gamma\ } & \big)\!\big(x, y, z \big)\!\big(\xi, \eta, \zeta) \text{ denotes } & \xi\,(\alpha\, x + \beta\, y + \gamma\, z) \\ \alpha', & \beta', & \gamma' & & +\,\eta\,(\alpha' x + \beta' y + \gamma' z) \\ \alpha'', & \beta'', & \gamma'' & & +\,\zeta\,(\alpha'' x + \beta'' y + \gamma'' z). \end{array}$$

Consequently

$$\begin{array}{|ccc|l|ccc|l} \multicolumn{3}{c}{(\ \alpha,\ \ \beta,\ \ \gamma\ } & \big)\!\big(x, y, z \big)\!\big(\xi, \eta, \zeta) = & \multicolumn{3}{c}{(\ \alpha,\ \ \alpha',\ \ \alpha''\ } & \big)\!\big(\xi, \eta, \zeta \big)\!\big(x, y, z). \\ \alpha', & \beta', & \gamma' & & \beta, & \beta', & \beta'' & \\ \alpha'', & \beta'', & \gamma'' & & \gamma, & \gamma', & \gamma'' & \end{array}$$

In the case of a symmetrical matrix

$$\begin{array}{|ccc|}\hline a, & h, & g \\ h, & b, & f \\ g, & f, & c \end{array},$$

the equal expressions

$$\begin{array}{|ccc|}\hline a, & h, & g \\ h, & b, & f \\ g, & f, & c \end{array})\!(x,\ y,\ z)\!(\xi,\ \eta,\ \zeta), \ = \begin{array}{|ccc|}\hline a, & h, & g \\ h, & b, & f \\ g, & f, & c \end{array})\!(\xi,\ \eta,\ \zeta)\!(x,\ y,\ z),$$

are also written

$$(a,\ b,\ c,\ f,\ g,\ h)\!(x,\ y,\ z)\!(\xi,\ \eta,\ \zeta), \text{ or } (a, \ldots)\!(\xi,\ \eta,\ \zeta)\!(x,\ y,\ z).$$

In particular, if

$$(\xi,\ \eta,\ \zeta) = (x,\ y,\ z),$$

then

$$\begin{array}{|ccc|}\hline a, & h, & g \\ h, & b, & f \\ g, & f, & c \end{array})\!(x,\ y,\ z)^2 \text{ is written } (a,\ b,\ c,\ f,\ g,\ h)\!(x,\ y,\ z)^2.$$

Two matrices are compounded together according to the law

$$\begin{array}{|ccc|}\hline a, & b, & c \\ a', & b', & c' \\ a'', & b'', & c'' \end{array})\!(\begin{array}{|ccc|}\hline \alpha, & \beta, & \gamma \\ \alpha', & \beta', & \gamma' \\ \alpha'', & \beta'', & \gamma'' \end{array} = \begin{array}{c|ccc} & (\alpha,\ \alpha',\ \alpha''), & (\beta,\ \beta',\ \beta''), & (\gamma,\ \gamma',\ \gamma'') \\ \hline (a,\ b,\ c) & \text{,,} & \text{,,} & \text{,,} \\ (a',\ b',\ c') & \text{,,} & \text{,,} & \text{,,} \\ (a'',\ b'',\ c'') & \text{,,} & \text{,,} & \text{,,} \end{array};$$

viz. in the compound matrix, the top-line is

$$(a,\ b,\ c)\!(\alpha,\ \alpha',\ \alpha''), \quad (a,\ b,\ c)\!(\beta,\ \beta',\ \beta''), \quad (a,\ b,\ c)\!(\gamma,\ \gamma',\ \gamma''),$$

and the other two lines are the like functions with $(a',\ b',\ c')$, and $(a'',\ b'',\ c'')$, respectively, in the place of $(a,\ b,\ c)$.

The inverse matrix is the matrix the terms of which are the minors of the determinant formed out of the original matrix, each minor being divided by this determinant, viz.

$$\begin{array}{|ccc|}\hline \alpha, & \beta, & \gamma \\ \alpha', & \beta', & \gamma' \\ \alpha'', & \beta'', & \gamma'' \end{array}^{-1} = \frac{1}{\nabla}\begin{array}{|ccc|}\hline \beta'\gamma''-\beta''\gamma', & \beta''\gamma-\beta\gamma'', & \beta\gamma'-\beta'\gamma \\ \gamma'\alpha''-\gamma''\alpha', & \gamma''\alpha-\gamma\alpha'', & \gamma\alpha'-\gamma'\alpha \\ \alpha'\beta''-\alpha''\beta', & \alpha''\beta-\alpha\beta'', & \alpha\beta'-\alpha'\beta \end{array},$$

where ∇ is the determinant

$$\begin{vmatrix} \alpha, & \beta, & \gamma \\ \alpha', & \beta', & \gamma' \\ \alpha'', & \beta'', & \gamma'' \end{vmatrix}.$$

Coming now to the question of transformation, write

$$\begin{array}{c|ccc:ccc|c:c}
 & x & y & z & x_1 & y_1 & z_1 & x \;\; y \;\; z & x_1 \;\; y_1 \;\; z_1 \\ \hline
x & 1 & \nu & \mu & \alpha & \alpha' & \alpha'' \; = x & & \\
y & \nu & 1 & \lambda & \beta & \beta' & \beta'' \;\;\;\; y & \Omega & W \\
z & \mu & \lambda & 1 & \gamma & \gamma' & \gamma'' \;\;\;\; z & & \\ \hdashline
x_1 & \alpha & \beta & \gamma & 1 & \nu_1 & \mu_1 \;\;\;\; x_1 & & \\
y_1 & \alpha' & \beta' & \gamma' & \nu_1 & 1 & \lambda_1 \;\;\;\; y_1 & V & \Omega_1 \\
z_1 & \alpha'' & \beta'' & \gamma'' & \mu_1 & \lambda_1 & 1 \;\;\;\; z_1 & &
\end{array}$$

viz. the axes of x, y, z are inclined to each other at angles the cosines whereof are λ, μ, ν: those of x_1, y_1, z_1 are inclined to each other at angles the cosines whereof are λ_1, μ_1, ν_1: and the cosines of the inclinations of the two sets of axes to each other are α, β, γ; α', β', γ'; $\alpha'', \beta'', \gamma''$: as is more clearly indicated in the diagram, the top-line showing that cosine-inclinations of x to

$$x, y, z, x_1, y_1, z_1,$$

are

$$1, \nu, \mu, \alpha, \alpha', \alpha'',$$

respectively, and the like for the other lines of the diagram. The letters Ω, Ω_1, V, W are used to denote matrices, viz. as appearing by the diagram, these are

$$\left(\begin{array}{ccc} 1, & \nu, & \mu \\ \nu, & 1, & \lambda \\ \mu, & \lambda, & 1 \end{array}\right), \quad \left(\begin{array}{ccc} 1, & \nu_1, & \mu_1 \\ \nu_1, & 1, & \lambda_1 \\ \mu_1, & \lambda_1, & 1 \end{array}\right), \quad \left(\begin{array}{ccc} \alpha, & \beta, & \gamma \\ \alpha', & \beta', & \gamma' \\ \alpha'', & \beta'', & \gamma'' \end{array}\right), \quad \left(\begin{array}{ccc} \alpha, & \alpha', & \alpha'' \\ \beta, & \beta', & \beta'' \\ \gamma, & \gamma', & \gamma'' \end{array}\right),$$

respectively.

The coordinates (x, y, z) and (x_1, y_1, z_1) form each set a broken line extending from the origin to the point; hence projecting on the axes of x, y, z and on those of x_1, y_1, z_1 respectively, we have two sets, each of three equations, which may be written

$$(\Omega \between x, y, z) = (W \between x_1, y_1, z_1),$$
$$(V \between x, y, z) = (\Omega_1 \between x_1, y_1, z_1);$$

where of course each set implies the other set.

We have

$$(x\,,\; y\,,\; z\,) = (\Omega^{-1} W \between x_1, y_1, z_1), \;\; = (V^{-1} \Omega_1 \between x_1, y_1, z_1),$$
$$(x_1, y_1, z_1) = (W^{-1} \Omega \between x\,,\; y\,,\; z\,), \;\; = (\Omega_1{}^{-1} V \between x\,,\; y\,,\; z\,),$$

the first giving in two forms (x, y, z) as linear functions of (x_1, y_1, z_1), and the second giving in two forms (x_1, y_1, z_1) as linear functions of (x, y, z); comparing the two forms for each set, we have

$$\Omega^{-1} W = V^{-1} \Omega_1, \quad W^{-1} \Omega = \Omega_1{}^{-1} V,$$

or, what is the same thing,

$$V\Omega^{-1}W = \Omega_1, \quad W\Omega_1^{-1}V = \Omega,$$

where in each equation the two sides are matrices which must be equal term by term to each other; but, the matrices being symmetrical, the equation thus gives (not nine but only) six equations. Writing

$$(a, b, c, f, g, h) = (1-\lambda^2, 1-\mu^2, 1-\nu^2, \mu\nu-\lambda, \nu\lambda-\mu, \lambda\mu-\nu),$$

and

$$K = 1-\lambda^2-\mu^2-\nu^2+2\lambda\mu\nu,$$

we have

$$\Omega^{-1} = \frac{1}{K}\left(\begin{array}{ccc} a, & h, & g \\ h, & b, & f \\ g, & f, & c \end{array}\right).$$

The first equation, written in the form

$$V\left(\begin{array}{ccc} a, & h, & g \\ h, & b, & f \\ g, & f, & c \end{array}\right)W = K\Omega_1,$$

denotes the six equations

$$\begin{array}{llll} (a, b, c, f, g, h)(\alpha, & \beta, & \gamma)^2 & = K, \\ \quad" & (\alpha', & \beta', & \gamma')^2 = K, \\ \quad" & (\alpha'', & \beta'', & \gamma'')^2 = K, \\ \quad" & (\alpha', & \beta', & \gamma')(\alpha'', \beta'', \gamma'') = K\lambda_1, \\ \quad" & (\alpha'', & \beta'', & \gamma'')(\alpha, \beta, \gamma) = K\mu_1, \\ \quad" & (\alpha, & \beta, & \gamma)(\alpha', \beta', \gamma') = K\nu_1. \end{array}$$

And, similarly, writing

$$(a_1, b_1, c_1, f_1, g_1, h_1) = (1-\lambda_1^2, 1-\mu_1^2, 1-\nu_1^2, \mu_1\nu_1-\lambda_1, \nu_1\lambda_1-\mu_1, \lambda_1\mu_1-\nu_1),$$

and

$$K_1 = 1-\lambda_1^2-\mu_1^2-\nu_1^2+2\lambda_1\mu_1\nu_1,$$

then

$$\Omega_1^{-1} = \frac{1}{K_1}\left(\begin{array}{ccc} a_1, & h_1, & g_1 \\ h_1, & b_1, & f_1 \\ g_1, & f_1, & c_1 \end{array}\right);$$

and the second equation, written in the form

$$W\left(\begin{array}{ccc} a_1, & h_1, & g_1 \\ h_1, & b_1, & f_1 \\ g_1, & f_1, & c_1 \end{array}\right)V = K_1\Omega,$$

denotes the six equations

$$\begin{array}{lll}
(\mathrm{a}_1,\ \mathrm{b}_1,\ \mathrm{c}_1,\ \mathrm{f}_1,\ \mathrm{g}_1,\ \mathrm{h}_1 \between \alpha,\ \alpha',\ \alpha'')^2 & & = K_1\ , \\
\quad ,, & (\beta,\ \beta',\ \beta'')^2 & = K_1\ , \\
\quad ,, & (\gamma,\ \gamma',\ \gamma'')^2 & = K_1\ , \\
\quad ,, & (\beta,\ \beta',\ \beta'' \between \gamma,\ \gamma',\ \gamma'') & = K_1\lambda_1, \\
\quad ,, & (\gamma,\ \gamma',\ \gamma'' \between \alpha,\ \alpha',\ \alpha'') & = K_1\mu_1, \\
\quad ,, & (\alpha,\ \alpha',\ \alpha'' \between \beta,\ \beta',\ \beta'') & = K_1\nu_1.
\end{array}$$

The two sets each of six equations are, in fact, equivalent to a single set of six equations, and serve to express the relations between the nine cosines

$$(\alpha,\ \beta,\ \gamma,\ \alpha',\ \beta',\ \gamma',\ \alpha'',\ \beta'',\ \gamma''),$$

and the cosines (λ, μ, ν) and $(\lambda_1, \mu_1, \nu_1)$. Observe that the nine cosines are *not* (as in the rectangular transformation) the coefficients of transformation between the two sets of coordinates.

From the original linear relations between the coordinates, multiplying the equations of the first set by x, y, z and adding, and again multiplying the equations of the second set by (x_1, y_1, z_1) and adding, we have

$$(\Omega \between x\ ,\ y\ ,\ z\ )^2 = (W \between x_1,\ y_1,\ z_1 \between x\ ,\ y\ ,\ z\ ),$$
$$(\Omega_1 \between x_1,\ y_1,\ z_1)^2 = (V \between x\ ,\ y\ ,\ z \between x_1,\ y_1,\ z_1).$$

But

$$(W \between x_1,\ y_1,\ z_1 \between x\ ,\ y\ ,\ z\ )$$

and

$$(V \between x\ ,\ y\ ,\ z \between x_1,\ y_1,\ z_1)$$

denote one and the same function; hence

$$(\Omega \between x,\ y,\ z)^2 = (\Omega_1 \between x_1,\ y_1,\ z_1)^2,$$

that is,

$$(1,\ 1,\ 1,\ \lambda,\ \mu,\ \nu \between x,\ y,\ z)^2 = (1,\ 1,\ 1,\ \lambda_1,\ \mu_1,\ \nu_1 \between x_1,\ y_1,\ z_1)^2,$$

or the linear relations between (x, y, z) and (x_1, y_1, z_1) are such as to transform one of these quadric functions into the other: the two quadrics, in fact, denote the squared distance from the origin expressed in terms of the coordinates (x, y, z) and (x_1, y_1, z_1) respectively.

Since the nine cosines are connected by six equations, there should exist values containing three arbitrary constants, and satisfying these equations identically: but, by what just precedes, it appears that the problem of determining these values is, in fact, that of finding the linear transformation between two given quadric functions: the problem of the linear transformation of a quadric function into itself has an elegant solution; but it would seem that this is not the case for the transformation between two different functions.

The foregoing equation

$$K = (\text{a, b, c, f, g, h} \between \alpha, \beta, \gamma)^2,$$

is a relation between λ, μ, ν, the cosines of the sides of a spherical triangle, and (α, β, γ) the cosines of the distances of a point P from the three vertices: it can be at once verified by means of the relation $A + B + C = 2\pi$, and thence

$$1 - \cos^2 A - \cos^2 B - \cos^2 C + 2 \cos A \cos B \cos C = 0,$$

which connects the angles A, B, C which the sides subtend at P. Writing a, b, c for λ, μ, ν, and f, g, h for α, β, γ, the relation is

$$\begin{aligned} 1 - a^2 - b^2 - c^2 + 2abc = (1 - a^2) f^2 + (1 - b^2) g^2 + (1 - c^2) h^2 \\ + 2(bc - a) gh + 2(ca - b) hf + 2(ab - c) fg, \end{aligned}$$

viz. this is

$$\begin{aligned} 1 - a^2 - b^2 - c^2 - f^2 - g^2 - h^2 + 2abc + 2agh + 2bhf + 2cfg \\ - a^2f^2 - b^2g^2 - c^2h^2 + 2bcgh + 2cahf + 2abfg = 0; \end{aligned}$$

where (a, b, c, f, g, h) are the cosines of the sides of a spherical quadrangle; (a, b, c), (a, h, g), (h, b, f), (g, f, c) belong respectively to sides forming a triangle, and the remaining sides (f, g, h), (b, c, f), (c, a, g), (a, b, h) are sides meeting in a vertex.

The equation

$$K\nu_1 = (\text{a, b, c, f, g, h} \between \alpha, \beta, \gamma)(\alpha', \beta', \gamma')$$

is a relation between λ, μ, ν, the cosines of the sides of a spherical triangle; α, β, γ, the cosines of the distances of a point P from the three vertices; α', β', γ', the cosines of the distances of a point Q from the three vertices; and ν_1, the cosine of the distance PQ.

Drawing a figure, it is at once seen that

$$\nu_1 = \alpha\alpha' + \sqrt{1 - \alpha^2}\sqrt{1 - \alpha'^2}\cos(\theta - \theta'),$$

where

$$\cos\theta = \frac{\beta - \alpha\nu}{\sqrt{1 - \alpha^2}\sqrt{1 - \nu^2}},$$

and therefore

$$\sin\theta = \frac{\sqrt{\nabla}}{\sqrt{1 - \alpha^2}\sqrt{1 - \nu^2}};$$

also

$$\cos\theta' = \frac{\beta' - \alpha'\nu}{\sqrt{1 - \alpha'^2}\sqrt{1 - \nu^2}},$$

and therefore

$$\sin\theta' = \frac{\sqrt{\nabla'}}{\sqrt{1 - \alpha'^2}\sqrt{1 - \nu^2}},$$

the values of ∇, ∇' being

$$\nabla = 1 - \alpha^2 - \beta^2 - \nu^2 + 2\alpha\beta\nu,$$

$$\nabla' = 1 - \alpha'^2 - \beta'^2 - \nu^2 + 2\alpha'\beta'\nu;$$

the resulting value of ν_1 is therefore

$$\nu_1 = \alpha\alpha' + \frac{1}{1-\nu^2}\{(\beta - \alpha\nu)(\beta' - \alpha'\nu) + \sqrt{\nabla\nabla'}\}.$$

The equations

$$K = (\mathrm{a, b, c, f, g, h})(\alpha, \beta, \gamma)^2, \quad K = (\mathrm{a}, \ldots)(\alpha', \beta', \gamma')^2,$$

give

$$(\mathrm{g}\alpha + \mathrm{f}\beta + \mathrm{c}\gamma)^2 = K\nabla,$$

$$(\mathrm{g}\alpha' + \mathrm{f}\beta' + \mathrm{c}\gamma')^2 = K\nabla':$$

and we therefore have

$$(\mathrm{g}\alpha + \mathrm{f}\beta + \mathrm{c}\gamma)(\mathrm{g}\alpha' + \mathrm{f}\beta' + \mathrm{c}\gamma') = K\sqrt{\nabla\nabla'};$$

recollecting that $1 - \nu^2 = \mathrm{c}$, the formula thus is

$$\nu_1 = \alpha\alpha' + \frac{1}{\mathrm{c}}\left\{(\beta - \alpha\nu)(\beta' - \alpha'\nu) + \frac{1}{K}(\mathrm{g}\alpha + \mathrm{f}\beta + \mathrm{c}\gamma)(\mathrm{g}\alpha' + \mathrm{f}\beta' + \mathrm{c}\gamma')\right\},$$

or say,

$$K\nu_1 = K\alpha\alpha' + \frac{1}{\mathrm{c}}\{K(\beta - \alpha\nu)(\beta' - \alpha'\nu) + (\mathrm{g}\alpha + \mathrm{f}\beta)(\mathrm{g}\alpha' + \mathrm{f}\beta')\} + \mathrm{g}(\alpha\gamma' + \alpha'\gamma) + \mathrm{f}(\beta\gamma' + \beta'\gamma) + \mathrm{c}\gamma\gamma'.$$

The sum of the first and second terms is readily found to be

$$= \mathrm{a}\alpha\alpha' + \mathrm{b}\beta\beta' + \mathrm{h}(\alpha\beta' + \alpha'\beta);$$

and the equation thus becomes

$$K\nu_1 = (\mathrm{a, b, c, f, g, h})(\alpha, \beta, \gamma)(\alpha', \beta', \gamma'),$$

as it should do.