738.

NOTE ON A HYPERGEOMETRIC SERIES.

[From the *Quarterly Journal of Pure and Applied Mathematics*, vol. XVI. (1879), pp. 268—270.]

IN the memoir on hypergeometric series, Schwarz, "Ueber diejenigen Fälle, &c.," *Crelle*, t. LXXV. (1873), pp. 292—335, the author shows, as part of his general theory, that the equation

$$\frac{d^2y}{dx^2} - \frac{\frac{2}{3} - \frac{7}{6}x}{x\,.\,1-x}\frac{dy}{dx} + \frac{\frac{1}{48}}{x\,.\,1-x}y = 0,$$

which belongs to the hypergeometric series $F(\frac{1}{4}, -\frac{1}{12}, \frac{2}{3}, x)$, is algebraically integrable, having in fact the two particular integrals

$$y^2 = \sqrt{(\alpha - \alpha^5 x^{\frac{1}{3}})} \pm \sqrt{(-\alpha^5 + \alpha x^{\frac{1}{3}})},$$

where α is a prime sixth root of -1, $\alpha^6 + 1 = 0$, or say $\alpha^4 - \alpha^2 + 1 = 0$ (see p. 326, α being for greater simplicity written instead of δ^2, and the form being somewhat simplified).

It is interesting to verify this directly; writing first $y = \sqrt{(Y)}$ and then $x = X^3$, the equation between Y, X is easily found to be

$$Y\frac{d^2Y}{dX^2} - \frac{\frac{3}{2}X^2}{1-X^3}Y\frac{dY}{dX} - \frac{1}{2}\left(\frac{dY}{dX}\right)^2 + \frac{\frac{3}{8}X}{1-X^3}Y^2 = 0,$$

and the theorem in effect is that that equation has the two particular integrals

$$Y = \sqrt{(P)} \pm \sqrt{(Q)},$$

P and Q being linear functions of X: in fact,

$$P = \alpha - \alpha^5 X,$$
$$Q = -\alpha^5 + \alpha X.$$

Starting say from the equation

$$Y = \sqrt{(P)} + \sqrt{(Q)},$$

or, as it is convenient to write it,

$$Y = P^{\frac{1}{2}} + Q^{\frac{1}{2}},$$

where P and Q are assumed to be linear functions of X, we have

$$\frac{dY}{dX} = \quad \tfrac{1}{2}P^{-\frac{1}{2}}P' + \tfrac{1}{2}Q^{-\frac{1}{2}}Q',$$

$$\frac{d^2Y}{dX^2} = -\tfrac{1}{4}P^{-\frac{3}{2}}P'^2 - \tfrac{1}{4}Q^{-\frac{3}{2}}Q'^2,$$

and thence

$$Y\frac{d^2Y}{dX^2} = -\tfrac{1}{4}P^{-1}P'^2 - \tfrac{1}{4}Q^{-1}Q'^2 - \tfrac{1}{4}Q^{\frac{1}{2}}P^{-\frac{3}{2}}P'^2 - \tfrac{1}{4}P^{\frac{1}{2}}Q^{-\frac{3}{2}}Q'^2$$

$$Y\frac{dY}{dX} = \quad \tfrac{1}{2}(P' + Q') \qquad + \tfrac{1}{2}P^{-\frac{1}{2}}Q^{\frac{1}{2}}P' + \tfrac{1}{2}P^{\frac{1}{2}}Q^{-\frac{1}{2}}Q',$$

$$\left(\frac{dY}{dX}\right)^2 = \quad \tfrac{1}{4}P^{-1}P'^2 + \tfrac{1}{4}Q^{-1}Q'^2 + \tfrac{1}{2}P^{-\frac{1}{2}}Q^{-\frac{1}{2}}P'Q',$$

where P', Q' are written to denote the derived functions of P, Q respectively.

Substituting these values, the resulting equation contains on the left-hand side a rational part, and a part with the factor $P^{-\frac{3}{2}}Q^{-\frac{3}{2}}$, and it is clear the equation can only be true if these two parts are separately $= 0$. We have thus two equations which ought to be verified; viz. after a slight reduction these are found to be

$$\frac{1}{PQ}(QP'^2 + PQ'^2) + \frac{2X^2}{1 - X^3}(P' + Q') - \frac{X}{1 - X^3}(P + Q) = 0,$$

$$P^2Q'^2 + Q^2P'^2 + PQP'Q' + \frac{3X^2}{1 - X^3}PQ(PQ' + P'Q) - \frac{3X}{1 - X^3}P^2Q^2 = 0,$$

and it is very interesting to observe the manner in which these equations are, in fact, verified by the foregoing values of P, Q.

We have

$$P + Q = (\alpha - \alpha^5)(1 + X), \quad P' + Q' = \alpha - \alpha^5,$$

and hence

$$2X(P' + Q') - X(P + Q) = -(\alpha - \alpha^5)(1 - X),$$

or, in the first equation, the second part

$$\frac{2X^2}{1 - X^3}(P' + Q') - \frac{X}{1 - X^3}(P + Q)$$

is

$$= -(\alpha - \alpha^5)\frac{X(1 - X)}{1 - X^3};$$

viz. this is

$$=\frac{-(\alpha-\alpha^5)X}{1+X+X^2}.$$

We have

$$QP'^2+PQ'^2=\alpha^{10}(-\alpha^5+\alpha X)+\alpha^2(\alpha-\alpha^5X),$$
$$=\alpha^3-\alpha^{15}-(\alpha^7-\alpha^{11})X,\quad =(\alpha-\alpha^5)X;$$

and

$$PQ=-\alpha^6+(\alpha^2+\alpha^{10})X-\alpha^6X^2,\quad =1+X+X^2;$$

hence

$$\frac{1}{PQ}(QP'^2+PQ'^2)=\frac{(\alpha-\alpha^5)X}{1+X+X^2},$$

and the sum of the two parts is $=0$.

Similarly as regards the second equation, the second part

$$\frac{3X^2}{1-X^3}PQ(PQ'+P'Q)-\frac{3X}{1-X^3}P^2Q^2$$

is

$$=\frac{3PQX}{1-X^3}\{(PQ'+P'Q)X-PQ\}.$$

Here $PQ'+P'Q$ is $\alpha(\alpha-\alpha^5X)-\alpha^5(-\alpha^5+\alpha X)$, which is $=1+2X$; and PQ being $=1+X+X^2$, the term in $\{\ \}$ is

$$(1+2X)X-(1+X+X^2),\quad =-(1-X)(1+X);$$

hence, outside the $\{\ \}$ writing for PQ its value $=1+X+X^2$, the term is

$$=\frac{-3X(1+X+X^2)(1-X)(1+X)}{1-X^3},\quad =-3X(1+X),$$

which is the value of the second part in question; the first part is

$$(PQ'+QP')^2-PQP'Q',\quad =(1+2X)^2-(1+X+X^2),\quad =3X(1+X);$$

and the sum of the two terms is thus $=0$.