733.

ON A FORMULA OF ELIMINATION.

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CONSIDER the equations

$$(a, \ldots \between \theta, 1)^n = 0,$$

$$(A, \ldots \between \theta, 1)^m = 0,$$

where $a, \ldots, A, \ldots$ are functions of coordinates. To fix the ideas, suppose that each of these coefficients is a linear function of the four coordinates x, y, z, w. Then, eliminating θ, we obtain $\nabla = 0$, the equation of a surface; and (as is known) this surface has a nodal curve.

It is easy to obtain the equations of the nodal curve in the case where one of the equations, say the second, is a quadric: the process is substantially the same whatever may be the order of the other equation, and I take it to be a cubic; the two equations therefore are

$$(a, b, c, d \between \theta, 1)^3 = 0,$$

$$(A, B, C \between \theta, 1)^2 = 0;$$

giving rise to an equation

$$\nabla, = (a, b, c, d)^2 (A, B, C)^3, = 0.$$

And it is required to perform the elimination so as to put in evidence the nodal line of this surface.

Take θ_1, θ_2 the roots of the second equation, or write

$$(A, B, C \between \theta, 1)^2 = A(\theta - \theta_1)(\theta - \theta_2);$$

that is,

$$\theta_1 + \theta_2 = -\frac{2B}{A}, \quad \theta_1\theta_2 = \frac{C}{A};$$

then, if

$$\Theta_1 = (a,\ b,\ c,\ d \between \theta_1,\ 1)^3,$$

$$\Theta_2 = (a,\ b,\ c,\ d \between \theta_2,\ 1)^3,$$

we have

$$\nabla = A^3 \Theta_1 \Theta_2 ;$$

viz. on the right-hand side, replacing the symmetrical functions of θ_1, θ_2 by their values in terms of A, B, C, we have the expression of ∇ in its known form

$$\nabla = a^2 C^3 + \&c.$$

Form now the expressions

$$\Theta_1 - \Theta_2,\quad \theta_2\Theta_1 - \theta_1\Theta_2,\quad \theta_2^2\Theta_1 - \theta_1^2\Theta_2,\quad \theta_2^3\Theta_1 - \theta_1^3\Theta_2,$$

each divided by $\theta_1 - \theta_2$. These are evidently symmetrical functions of θ_1, θ_2, the values being given by the successive lines of the expression

$$\left(\begin{array}{cccc} 0, & 1, & \theta_1+\theta_2, & \theta_1^2+\theta_1\theta_2+\theta_2^2 \\ -1, & 0, & \theta_1\theta_2, & \theta_1\theta_2(\theta_1+\theta_2) \\ -(\theta_1+\theta_2), & -\theta_1\theta_2, & 0, & \theta_1^2\theta_2^2 \\ -(\theta_1^2+\theta_1\theta_2+\theta_2^2), & -\theta_1\theta_2(\theta_1+\theta_2), & -\theta_1^2\theta_2^2, & 0 \end{array} \between d,\ 3c,\ 3b,\ a\right);$$

and, consequently, these same quantities, each multiplied by A^2, are given by the successive lines of

$$\left(\begin{array}{cccc} 0, & A^2, & -2AB, & -AC+4B^2 \\ -A^2, & 0, & AC, & -2BC \\ 2AB, & -AC, & 0, & C^2 \\ AC-4B^2, & 2BC, & -C^2, & 0 \end{array} \between d,\ 3c,\ 3b,\ a\right).$$

Calling these X, Y, Z, W, that is, writing

$$X = 3A^2c - 6ABb + (-AC + 4B^2)\,a,\ \&c.,$$

then X, Y, Z, W are the values of

$$\Theta_1 - \Theta_2,\quad \theta_2\Theta_1 - \theta_1\Theta_2,\quad \theta_2^2\Theta_1 - \theta_1^2\Theta_2,\quad \theta_2^3\Theta_1 - \theta_1^3\Theta_2,$$

each multiplied by $A^2 \div (\theta_1 - \theta_2)$; and the functions all four of them vanish if only $\Theta_1 = 0$, $\Theta_2 = 0$; or, what is the same thing, the equations $X = 0$, $Y = 0$, $Z = 0$, $W = 0$ constitute only a twofold system.

The functions

$$\left(\begin{array}{ccc} X, & Y, & Z \\ Y, & Z, & W \end{array}\right)$$

contain each of them the factor $\Theta_1\Theta_2$, that is, ∇; they, in fact, each of them vanish if $\Theta_1 = 0$, and they also vanish if $\Theta_2 = 0$; or, by a direct substitution, we have

$$\begin{aligned} XZ - Y^2 &= \frac{A^4}{(\theta_1-\theta_2)^2}\cdot -(\theta_1-\theta_2)^2\,\Theta_1\Theta_2, &&= -A^4\Theta_1\Theta_2,\\ XW - YZ &= \quad ,, \quad -(\theta_1-\theta_2)^2(\theta_1+\theta_2)\,\Theta_1\Theta_2, &&= -A^4\Theta_1\Theta_2(\theta_1+\theta_2),\\ YW - Z^2 &= \quad ,, \quad -(\theta_1-\theta_2)^2\,\theta_1\theta_2\Theta_1\Theta_2, &&= -A^4\Theta_1\Theta_2\theta_1\theta_2. \end{aligned}$$

Or, what is the same thing, these are $= -A\nabla$, $2B\nabla$, $-C\nabla$, respectively; thus the first equation is

$$\{3A^2c - 6ABb + (-AC + 4B^2)\,a\}\,\{2ABd - 3ACc + C^2a\} \\ - (-A^2d + 3ACb - 2BCa)^2 = -A\,(A^3d^2 + \&c.), = -A\nabla;$$

and similarly for the other two equations. The nodal curve is thus given by the twofold system $X = 0$, $Y = 0$, $Z = 0$, $W = 0$.

The method may be extended to the case where, instead of the quadric equation $(A, B, C \between \theta, 1)^2 = 0$, we have an equation of any higher order, but the formulæ are less simple.