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A THEOREM IN ELLIPTIC FUNCTIONS.

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THE theorem is as follows:

If $u+v+r+s=0$, then

$$-k'^2 \operatorname{sn} u \operatorname{sn} v \operatorname{sn} r \operatorname{sn} s + \operatorname{cn} u \operatorname{cn} v \operatorname{cn} r \operatorname{cn} s - \frac{1}{k^2} \operatorname{dn} u \operatorname{dn} v \operatorname{dn} r \operatorname{dn} s = -\frac{k'^2}{k^2}.$$

It is easy to see that, if a linear relation exists between the three products, then it must be this relation: for the relation must be satisfied on writing therein $v=-u$, $s=-r$, and the only linear relation connecting $\operatorname{sn}^2 u \operatorname{sn}^2 r$, $\operatorname{cn}^2 u \operatorname{cn}^2 r$, $\operatorname{dn}^2 u \operatorname{dn}^2 r$ is the relation in question

$$-k'^2 \operatorname{sn}^2 u \operatorname{sn}^2 r + \operatorname{cn}^2 u \operatorname{cn}^2 r - \frac{1}{k^2} \operatorname{dn}^2 u \operatorname{dn}^2 r = -\frac{k'^2}{k^2}.$$

A demonstration of the theorem was recently communicated to me by Mr Glaisher; and this led me to the somewhat more general theorem

$$\begin{aligned} &-k'^2 \operatorname{sn}(\alpha+\beta) \operatorname{sn}(\alpha-\beta) \operatorname{sn}(\gamma+\delta) \operatorname{sn}(\gamma-\delta) \\ &+ \quad \operatorname{cn}(\alpha+\beta) \operatorname{cn}(\alpha-\beta) \operatorname{cn}(\gamma+\delta) \operatorname{cn}(\gamma-\delta) \\ &-\frac{1}{k^2} \operatorname{dn}(\alpha+\beta) \operatorname{dn}(\alpha-\beta) \operatorname{dn}(\gamma+\delta) \operatorname{dn}(\gamma-\delta) \\ &= -\frac{k'^2}{k^2} - \frac{2k'^2 (\operatorname{sn}^2\alpha - \operatorname{sn}^2\gamma)(\operatorname{sn}^2\beta - \operatorname{sn}^2\delta)}{1-k^2\operatorname{sn}^2\alpha \operatorname{sn}^2\beta \, . \, 1-k^2 \operatorname{sn}^2\gamma \operatorname{sn}^2\delta}. \end{aligned}$$

In fact, writing herein $\alpha+\gamma=0$, that is, $\gamma=-\alpha$, the right-hand side becomes $=0$; and the arcs on the left-hand side are $\alpha+\beta$, $\alpha-\beta$, $-\alpha+\delta$, $-\alpha-\delta$, which represent any four arcs the sum of which is $=0$.

Writing in the last-mentioned equation x, y, z, w for the sn's of α, β, γ, δ respectively, also

$$\begin{aligned} P &= x^2-y^2, & P_1 &= z^2-w^2, \\ Q &= 1-x^2-y^2+k^2x^2y^2, & Q_1 &= 1-z^2-w^2+k^2z^2w^2, \\ R &= 1-k^2x^2-k^2y^2+k^2x^2y^2, & R_1 &= 1-k^2z^2-k^2w^2+k^2z^2w^2, \\ D &= 1-k^2x^2y^2, & D_1 &= 1-k^2z^2w^2, \end{aligned}$$

the equation is

$$-k'^2\frac{PP_1}{DD_1}+\frac{QQ_1}{DD_1}-\frac{1}{k^2}\frac{RR_1}{DD_1}=-\frac{k'^2}{k^2}-\frac{2k'^2(x^2-z^2)(y^2-w^2)}{DD_1},$$

that is,

$$-k'^2PP_1+QQ_1-\frac{1}{k^2}RR_1+\frac{k'^2}{k^2}DD_1+2k'^2(x^2-z^2)(y^2-w^2)=0.$$

It is easy to verify that the terms of the orders 0, 1, 2, 3 and 4 in x^2, y^2, z^2, w^2 separately destroy each other; for instance, for the terms of the order 2, we have

$$\begin{aligned} &-k'^2(x^2-y^2)(z^2-w^2)+\{(x^2+y^2)(z^2+w^2)+k^2(x^2y^2+z^2w^2)\} \\ &-\frac{1}{k^2}\{k^4(x^2+y^2)(z^2+w^2)+k^2(x^2y^2+z^2w^2)\} \\ &+\frac{k'^2}{k^2}\{-k^2(x^2y^2+z^2w^2)\}+2k'^2(x^2-z^2)(y^2-w^2)=0, \end{aligned}$$

that is,

$$\begin{aligned} -k'^2(x^2-y^2)(z^2-w^2)&+(1-k^2)(x^2+y^2)(z^2+w^2) \\ &+(k^2-1-k'^2)(x^2y^2+z^2w^2)+2k'^2(x^2-z^2)(y^2-w^2)=0\,; \end{aligned}$$

or, omitting the factor k'^2, this is

$$-(x^2-y^2)(z^2-w^2)+(x^2+y^2)(z^2+w^2)-2(x^2y^2+z^2w^2)+2(x^2-z^2)(y^2-w^2)=0,$$

as it should be.

The theorem in its original form was obtained by me as follows: using the elliptic coordinates p, q, r, such that

$$\frac{x^2}{a+p}+\frac{y^2}{b+p}+\frac{z^2}{c+p}=1,$$

$$\frac{x^2}{a+q}+\frac{y^2}{b+q}+\frac{z^2}{c+q}=1,$$

$$\frac{x^2}{a+r}+\frac{y^2}{b+r}+\frac{z^2}{c+r}=1\,;$$

or, what is the same thing,

$$-\beta\gamma x^2 = a+p\,.\,a+q\,.\,a+r,$$

$$-\gamma\alpha y^2 = b+p\,.\,b+q\,.\,b+r,$$

$$-\alpha\beta z^2 = c+p\,.\,c+q\,.\,c+r,$$

where α, β, γ denote $b-c$, $c-a$, $a-b$ respectively; then, treating r as a constant, the coordinates x, y, z will belong to a point on the ellipsoid

$$\frac{x^2}{a+r}+\frac{y^2}{b+r}+\frac{z^2}{c+r}=1,$$

and the differential equation of the right lines upon this surface is

$$\frac{dp}{\sqrt{a+p\,.\,b+p\,.\,c+p}}=\frac{dq}{\sqrt{a+q\,.\,b+q\,.\,c+q}}.$$

Take x_0, y_0, z_0 the coordinates of a point on the surface, and p_0, q_0 the corresponding values of p, q, so that

$$-\beta\gamma x_0{}^2 = a+p_0\,.\,a+q_0\,.\,a+r,$$

$$-\gamma\alpha y_0{}^2 = b+p_0\,.\,b+q_0\,.\,b+r,$$

$$-\alpha\beta z_0{}^2 = c+p_0\,.\,c+q_0\,.\,c+r,$$

then the equation of the tangent plane at the point (x_0, y_0, z_0) is

$$\frac{xx_0}{a+r}+\frac{yy_0}{b+r}+\frac{zz_0}{c+r}=1,$$

or, substituting for x^2, $x_0{}^2$, &c., their values, we have

$$-\frac{\beta\gamma xx_0}{a+r}=\sqrt{a+p\,.\,a+q\,.\,a+p_0\,.\,a+q_0},\ \text{\&c.},$$

and consequently the equation of the tangent plane is

$$\alpha\sqrt{a+p\,.\,a+q\,.\,a+p_0\,.\,a+q_0}+\beta\sqrt{b+p\,.\,b+q\,.\,b+p_0\,.\,b+q_0}$$
$$+\gamma\sqrt{c+p\,.\,c+q\,.\,c+p_0\,.\,c+q_0}=-\alpha\beta\gamma,$$

the equation of a plane intersecting the ellipsoid in a pair of lines; hence this equation (containing in appearance the two arbitrary constants p_0 and q_0) is the integral of the proposed differential equation.

Writing

$$\text{sn}^2 u = A\,(a+p),\quad \text{cn}^2 u = B\,(b+p),\quad \text{dn}^2 u = C\,(c+p),$$

the values of A, B, C, k are determined; and, assuming for q, p_0, q_0 the like forms with the arguments v, u_0, v_0, the differential equation becomes $du = dv$, having the integral $u-u_0=v-v_0$; while the foregoing integral equation, on reducing the constant coefficients contained therein, takes the form

$$\begin{aligned}&-k'^2 \operatorname{sn} u \operatorname{sn} v \operatorname{sn} u_0 \operatorname{sn} v_0\\ &+\quad \operatorname{cn} u \operatorname{cn} v \operatorname{cn} u_0 \operatorname{cn} v_0\\ &-\frac{1}{k^2} \operatorname{dn} u \operatorname{dn} v \operatorname{dn} u_0 \operatorname{dn} v_0\\ =&-\frac{k'^2}{k^2};\end{aligned}$$

viz. this equation holds good if $u-u_0=v-v_0$. And by a change of signs we have the theorem.

If, as above, $u+v+r+s=0$, the theorem gives a linear relation between the three products $\operatorname{sn} u \operatorname{sn} v \operatorname{sn} r \operatorname{sn} s$, $\operatorname{cn} u \operatorname{cn} v \operatorname{cn} r \operatorname{cn} s$, $\operatorname{dn} u \operatorname{dn} v \operatorname{dn} r \operatorname{dn} s$, and regarding at pleasure the sn's, the cn's, or the dn's as rational, one of these products will be rational while the other two will be each of them a quadric radical; and hence, rationalising, we obtain an equation which contains the product in question linearly, and contains besides only the squares of the sn's, cn's, or dn's; that is, we have three such equations containing the three products respectively. Bringing to one side the terms which contain the product, and again squaring, we obtain an equation involving only the squares of the sn's, cn's, or dn's; but the three equations thus obtained represent, it is clear, one and the same rational equation, which may be expressed as an equation between the squares of the sn's, or of the cn's, or of the dn's, at pleasure. This equation may be obtained, as I will show, from the ordinary addition-equations of the elliptic functions, but it is not obvious how to obtain from them the three equations involving the products respectively, and these last have the advantage of being of a degree which is the half of the equation which involves only the squared functions.

Write x, y, z, w for $\operatorname{sn} u$, $\operatorname{sn} v$, $\operatorname{sn} r$, $\operatorname{sn} s$ respectively; then, writing

$$\begin{aligned}A &= x\sqrt{1-y^2\,.\,1-k^2y^2}, & \alpha &= z\sqrt{1-w^2\,.\,1-k^2w^2},\\ A' &= y\sqrt{1-x^2\,.\,1-k^2x^2}, & \alpha' &= w\sqrt{1-z^2\,.\,1-k^2z^2},\\ P &= x^2-y^2, & \varpi &= z^2-w^2,\\ D &= 1-k^2x^2y^2, & \delta &= 1-k^2z^2w^2,\end{aligned}$$

we have

$$\operatorname{sn}(u+v) = -\operatorname{sn}(r+s),$$

that is,

$$\frac{A+A'}{D} = \frac{P}{A-A'} = -\frac{\alpha+\alpha'}{\delta} = -\frac{\varpi}{\alpha-\alpha'},$$

and consequently

$$\begin{aligned}D\varpi &= -(\alpha-\alpha')(A+A'),\\ P\delta &= -(\alpha+\alpha')(A-A');\end{aligned}$$

whence

$$D\varpi - P\delta = 2(A\alpha' - A'\alpha),$$

that is,
$$(z^2-w^2)(1-k^2x^2y^2)-(x^2-y^2)(1-k^2z^2w^2)$$
$$=2\{xw\sqrt{1-y^2\,.\,1-k^2y^2\,.\,1-z^2\,.\,1-k^2z^2}-yz\sqrt{1-x^2\,.\,1-k^2x^2\,.\,1-w^2\,.\,1-k^2w^2}\}.$$

Rationalising, we obtain, as mentioned above, an equation containing only the squares x^2, y^2, z^2, w^2; it therefore is of a degree twice that of the equation containing the product $xyzw$. I worked out in this way the equation in (x^2, y^2, z^2, w^2), but the calculation was lost, and the easier way of obtaining it is obviously by means of the equation involving $xyzw$.

We have, by the theorem,
$$\begin{aligned}&-k'^2\,xyzw\\ &+\sqrt{1-x^2\,.\,1-y^2\,.\,1-z^2\,.\,1-w^2}\\ &-\frac{1}{k^2}\sqrt{1-k^2x^2\,.\,1-k^2y^2\,.\,1-k^2z^2\,.\,1-k^2w^2}=-\frac{k'^2}{k^2},\end{aligned}$$
that is,
$$\begin{aligned}k'^2(1-k^2xyzw)&=k^2\sqrt{1-x^2\,.\,1-y^2\,.\,1-z^2\,.\,1-w^2}\\ &\quad-\sqrt{1-k^2x^2\,.\,1-k^2y^2\,.\,1-k^2z^2\,.\,1-k^2w^2};\end{aligned}$$
and then, writing
$$\begin{aligned}P&=x^2+y^2+z^2+w^2,\\ Q&=x^2y^2+x^2z^2+x^2w^2+y^2z^2+y^2w^2+z^2w^2,\\ R&=x^2y^2z^2+x^2y^2w^2+x^2z^2w^2+y^2z^2w^2,\\ S&=x^2y^2z^2w^2,\end{aligned}$$
and using $\sqrt{S}$ to denote the rational function $xyzw$, we have
$$\begin{aligned}k'^4&(1-2k^2\sqrt{S}+k^4S)\\ &=k^4(1-P+Q-R+S)\\ &\quad+1-k^2P+k^4Q-k^6R+k^8S\\ &\quad-2k^2\sqrt{(1-P+Q-R+S)(1-k^2P+k^4Q-k^6R+k^8S)};\end{aligned}$$
or, if for a moment the radical is called $\sqrt{\Delta}$, then the factor k^2 divides out, and the equation becomes
$$2\sqrt{\Delta}=2-(1+k^2)P+2k^2Q-(k^2+k^4)R+2k^4S+2k'^4\sqrt{S},$$
whence
$$\begin{aligned}&4(1-P+Q-R+S)(1-k^2P+k^4Q-k^6R+k^8S)\\ &-\{2-(1+k^2)P+2k^2Q-(k^2+k^4)R+2k^4S\}^2-4k'^8S\\ &\qquad=-2k'^4\sqrt{S}\{2-(1+k^2)P+2k^2Q-(k^2+k^4)R+2k^4S\}.\end{aligned}$$
The factor k'^4 divides out; omitting it, we have
$$\begin{aligned}&4Q-P^2-4(1+k^2)R+16k^2S+2k^2PR-4(k^2+k^4)PS-k^4R^2+4k^4QS\\ &\qquad=-2\sqrt{S}\{2-(1+k^2)P+2k^2Q-(k^2+k^4)R+2k^4S\},\end{aligned}$$
or, as this may also be written,
$$\begin{aligned}&\{(-P^2+4Q-4R)+k^2(-4R+2PR+16S-4PS)+k^4(-R^2+4QS-PS)\}\\ &\qquad=-2\sqrt{S}\{2-P+k^2(-P+2Q-R)+k^4(-R+2S)\},\end{aligned}$$
which is the required rational equation involving the product of the sn's.