# 727.

## EQUATION OF THE WAVE-SURFACE IN ELLIPTIC COORDINATES.

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THE equation of the wave-surface

$$\frac{ax^2}{x^2+y^2+z^2-a}+\frac{by^2}{x^2+y^2+z^2-b}+\frac{cz^2}{x^2+y^2+z^2}=0,$$

when transformed to coordinates $p$, $q$, $r$, such that

$$\frac{x^2}{-a+p}+\frac{y^2}{-b+p}+\frac{z^2}{-c+p}=1,$$

$$\frac{x^2}{-a+q}+\frac{y^2}{-b+q}+\frac{z^2}{-c+q}=1,$$

$$\frac{x^2}{-a+r}+\frac{y^2}{-b+r}+\frac{z^2}{-c+r}=1;$$

(that is, to the elliptic coordinates belonging to the quadric surface $\frac{x^2}{-a}+\frac{y^2}{-b}+\frac{z^2}{-c}=1$), assumes the form

$$(q+r-a-b-c)(r+p-a-b-c)(p+q-a-b-c)=0,$$

(Senate-House Problem, January 14, 1879).

In fact, $p$, $q$, $r$ are the roots of the equation

$$\frac{x^2}{-a+u}+\frac{y^2}{-b+u}+\frac{z^2}{-c+u}=1;$$

we have therefore

$$\begin{aligned}(u-p)(u-q)(u-r)&=(u-a)(u-b)(u-c)\\&\quad-x^2(u-b)(u-c)-y^2(u-c)(u-a)-z^2(u-a)(u-b);\end{aligned}$$

whence, writing for shortness

$$A = a + b + c \quad , \quad P = p + q + r,$$
$$B = bc + ca + ab, \quad Q = qr + rp + pq,$$
$$C = abc \quad , \quad R = pqr,$$

we have

$$x^2 + \quad y^2 + \quad z^2 = P - A,$$
$$(b + c)\, x^2 + (c + a)\, y^2 + (a + b)\, z^2 = Q - B,$$
$$bcx^2 + \quad cay^2 + \quad abz^2 = R - C,$$

and thence also

$$a (b + c)\, x^2 + b (c + a)\, y^2 + c (a + b)\, z^2 = B (P - A) - (R - C),$$
$$ax^2 + \quad by^2 + \quad cz^2 = A (P - A) - (Q - B).$$

The equation of the wave-surface is

$$abc - \{a (b + c)\, x^2 + b (c + a)\, y^2 + c (a + b)\, z^2\} + (x^2 + y^2 + z^2)(ax^2 + by^2 + cz^2) = 0.$$

By the formulæ just obtained, this is

$$C - [B (P - A) - (R - C)] + (P - A) [A (P - A) - (Q - B)] = 0,$$

that is,

$$A^3 - 2A^2P + A (P^2 + Q) - (PQ - R) = 0,$$

that is,

$$\{A - (q + r)\} \{A - (r + p)\} \{A - (p + q) = 0,$$

or, substituting for $A$ its value $a + b + c$, and reversing the sign of each factor, we have the formula in question.

It is easy to see that, taking $a$, $b$, $c$ to be each positive, $(a > b > c)$, and assuming also $p > q > r$, we obtain the different real points of space by giving to these coordinates respectively the different real values from $\infty$ to $a$, $a$ to $b$, and $b$ to $c$ respectively. Hence

| | greatest, | least value, is |
|---|---|---|
| $q + r$, | $a + b$, | $a + c$, |
| $r + p$, | $\infty$ , | $a + c$, |
| $p + q$, | $\infty$ , | $a + b$, |

so that $r + p$, $p + q$, may be either of them $= a + b + c$, but $q + r$ cannot be $= a + b + c$, that is, $q + r = a + b + c$ does not belong to any real point on the wave-surface. We can only have $r + p$ and $p + q$ each $= a + b + c$, if $p = a + c$, $q = r = b$, and these values belong as is easily shown to the nodes on the wave-surface; hence, the equations $r + p = a + b + c$ and $p + q = a + b + c$ being satisfied simultaneously only at the nodes of the surface, must belong to the two sheets respectively. And it can be shown that $p + r = a + b + c$ belongs to the external sheet, and $p + q = a + b + c$ belongs to the internal sheet. In fact, for the point $(0, 0, \sqrt{a})$, which is on the external sheet, we have $p = a + c$, $q = a$, $r = b$, and therefore $p + r = a + b + c$: for the point $(0, 0, \sqrt{b})$, which is on the internal sheet, either

$$(p = b + c,\ q = a,\ r = b) \quad \text{or} \quad (p = a,\ q = b + c,\ r = c),$$

according as $b + c > a$ or $b + c < a$: but in each case

$$p + q = a + b + c.$$