725.

NEW FORMULÆ FOR THE INTEGRATION OF $\frac{dx}{\sqrt{X}}+\frac{dy}{\sqrt{Y}}=0$.

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I HAVE found in regard to the differential equation

$$\frac{dx}{\sqrt{(a-x\,.\,b-x\,.\,c-x\,.\,d-x)}}+\frac{dy}{\sqrt{(a-y\,.\,b-y\,.\,c-y\,.\,d-y)}}=0,$$

a system of formulæ analogous to those given, p. 63, of my *Treatise on Elliptic Functions,* for the values of $\operatorname{sn}(u+v)$, $\operatorname{cn}(u+v)$, $\operatorname{dn}(u+v)$. Writing for shortness

$$\begin{aligned}\mathrm{a}\ ,\ \mathrm{b}\ ,\ \mathrm{c}\ ,\ \mathrm{d}&=a-x,\ b-x,\ c-x,\ d-x,\\ \mathrm{a}_1,\ \mathrm{b}_1,\ \mathrm{c}_1,\ \mathrm{d}_1&=a-y,\ b-y,\ c-y,\ d-y,\end{aligned}$$

and $(bc,\ ad)$ to denote the determinant

$$\begin{vmatrix} 1, & x+y, & xy \\ 1, & b+c, & bc \\ 1, & a+d, & ad \end{vmatrix},$$

and $(cd,\ ab)$, $(bd,\ ac)$ to denote the like determinants; then the formulæ are

$$\begin{aligned}\sqrt{\left(\frac{a-z}{d-z}\right)}&=\frac{\sqrt{(a-b\,.\,a-c)}\,\{\sqrt{(\mathrm{adb_1c_1})}+\sqrt{(\mathrm{a_1d_1bc})}\}}{(bc,\ ad)},\\ &=\frac{\sqrt{(a-b\,.\,a-c)}\,(x-y)}{\sqrt{(\mathrm{adb_1c_1})}-\sqrt{(\mathrm{a_1d_1bc})}},\\ &=\frac{\sqrt{(a-b\,.\,a-c)}\,\{\sqrt{(\mathrm{abc_1d_1})}+\sqrt{(\mathrm{a_1b_1cd})}\}}{(a-c)\sqrt{(\mathrm{bdb_1d_1})}-(b-d)\sqrt{(\mathrm{aca_1c_1})}},\\ &=\frac{\sqrt{(a-b\,.\,a-c)}\,\{\sqrt{(\mathrm{acb_1d_1})}+\sqrt{(\mathrm{a_1c_1bd})}\}}{(a-b)\sqrt{(\mathrm{cdc_1d_1})}-(c-d)\sqrt{(\mathrm{aba_1b_1})}},\end{aligned}$$

$$\sqrt{\left(\frac{b-z}{d-z}\right)}=\frac{\sqrt{\left(\frac{a-b}{a-d}\right)}\{(a-c)\surd(\mathrm{bdb_1d_1})+(b-d)\surd(\mathrm{aca_1c_1})\}}{(bc,\ ad)},$$

$$=\frac{\sqrt{\left(\frac{a-b}{a-d}\right)}\{\surd(\mathrm{abc_1d_1})-\surd(\mathrm{a_1b_1cd})\}}{\surd(\mathrm{adb_1c_1})-\surd(\mathrm{a_1d_1bc})},$$

$$=\frac{\sqrt{\left(\frac{a-b}{a-d}\right)}(cd,\ ab)}{(a-c)\surd(\mathrm{bdb_1d_1})-(b-d)\surd(\mathrm{aca_1c_1})},$$

$$=\frac{\sqrt{\left(\frac{a-b}{a-d}\right)}\{(a-d)\surd(\mathrm{bcb_1c_1})+(b-c)\surd(\mathrm{ada_1d_1})\}}{(a-b)\surd(\mathrm{cdc_1d_1})-(c-d)\surd(\mathrm{aba_1b_1})},$$

$$\sqrt{\left(\frac{c-z}{d-z}\right)}=\frac{\sqrt{\left(\frac{a-c}{a-d}\right)}\{(a-b)\surd(\mathrm{cdc_1d_1})+(c-d)\surd(\mathrm{aba_1b_1})\}}{(bc,\ ad)},$$

$$=\frac{\sqrt{\left(\frac{a-c}{a-d}\right)}\{\surd(\mathrm{acb_1d_1})-\surd(\mathrm{a_1c_1bd})\}}{\surd(\mathrm{adb_1c_1})-\surd(\mathrm{a_1d_1bc})},$$

$$=\frac{\sqrt{\left(\frac{a-c}{a-d}\right)}\{(a-d)\surd(\mathrm{bcb_1c_1})-(b-c)\surd(\mathrm{ada_1d_1})\}}{(a-c)\surd(\mathrm{bdb_1d_1})-(b-d)\surd(\mathrm{aca_1c_1})},$$

$$=\frac{\sqrt{\left(\frac{a-c}{a-d}\right)}(bd,\ ac)}{(a-b)\surd(\mathrm{cdc_1d_1})-(c-d)\surd(\mathrm{aba_1b_1})}.$$

The twelve equations are equivalent to each other, each giving z as one and the same function of x, y; and regarding z as a constant of integration, any one of the equations is a form of the integral of the proposed differential equation.

Writing in the formulæ $x=a$, b, c, d successively, the formulæ become

$$\begin{array}{llll} x=a, & x=b, & x=c, & x=d, \\ \frac{a-z}{d-z}=\frac{\mathrm{a_1}}{\mathrm{d_1}}, & -\frac{c-a}{d-b}\frac{\mathrm{b_1}}{\mathrm{c_1}}, & -\frac{b-a}{d-c}\frac{\mathrm{c_1}}{\mathrm{b_1}}, & \frac{a-b\,.\,a-c}{d-b\,.\,d-c}\frac{\mathrm{d_1}}{\mathrm{a_1}}, \\ \frac{b-z}{d-z}=\frac{\mathrm{b_1}}{\mathrm{d_1}}, & -\frac{c-b}{d-a}\frac{\mathrm{a_1}}{\mathrm{c_1}}, & \frac{b-a\,.\,b-c}{d-a\,.\,d-c}\frac{\mathrm{d_1}}{\mathrm{b_1}}, & -\frac{a-b}{d-c}\frac{\mathrm{c_1}}{\mathrm{a_1}}, \\ \frac{c-z}{d-z}=\frac{\mathrm{c_1}}{\mathrm{d_1}}, & \frac{c-a\,.\,c-b}{d-a\,.\,d-b}\frac{\mathrm{d_1}}{\mathrm{c_1}}, & -\frac{b-c}{d-a}\frac{\mathrm{a_1}}{\mathrm{b_1}}, & -\frac{a-c}{d-b}\frac{\mathrm{b_1}}{\mathrm{a_1}}, \end{array}$$

viz. in the first case we have $z=y$, and in each of the other cases z equal to a linear function $\frac{\alpha y+\beta}{\gamma y+\delta}$ of y.

Cambridge, July 3, 1878.