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A PROBLEM IN PARTITIONS.

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TAKE for instance 6 letters; a partition into 3's, such as *abc . def* contains the 6 duads *ab, ac, bc, de, df, ef.* A partition into 2's such as *ab . cd . ef* contains the 3 duads *ab, cd, ef.* Hence if there are α partitions into 3's, and β partitions into 2's, and these contain all the duads each once and only once, $6\alpha+3\beta=15$, or $2\alpha+\beta=5$. The solutions of this last equation are ($\alpha=0$, $\beta=5$), ($\alpha=1$, $\beta=3$), ($\alpha=2$, $\beta=1$), and it is at once seen that the first two sets give solutions of the partition problem, but that the third set gives no solution; thus we have

$\alpha=0,\quad \beta=5$	$\alpha=1,\quad \beta=3$
ab . cd . ef	*abc . def*
ac . be . df	*ad . be . cf*
ad . bf . ce	*ae . bf . cd*
ae . bd . cf	*af . bd . ce.*
af . bc . de	

Similarly for any other number of letters, for instance 15; if we have α partitions into 5's and β partitions into 3's, then, if these contain all the duads, $4\alpha+2\beta=14$, or what is the same $2\alpha+\beta=7$; if $\alpha=0$, $\beta=7$, the partition problem can be solved (this is in fact the problem of the 15 school-girls): but can it be solved for any other values (and if so which values) of α, β? Or again for 30 letters; if we have α partitions into 5's, β partitions into 3's and γ partitions into 2's; then, if these contain all the duads, $4\alpha+2\beta+\gamma=29$; and the question is for what values of α, β, γ, does the partition-problem admit of solution.

The question is important from its connexion with the theory of groups, but it seems to be a very difficult one.

I take the opportunity of mentioning the following theorem: two non-commutative symbols α, β, which are such that $\beta\alpha = \alpha^2\beta^2$ cannot give rise to a group made up of symbols of the form $\alpha^p\beta^q$. In fact, the assumed relation gives $\beta\alpha^2 = \alpha^2\beta\alpha^2\beta^2$; and hence, if $\beta\alpha^2$ be of the form in question, $= \alpha^x\beta^y$ suppose, we have

$$\alpha^x\beta^y = \alpha^2 . \alpha^x\beta^y . \beta^2, \ = \alpha^{x+2}\beta^{y+2};$$

that is, $1 = \alpha^2\beta^2$, and thence $\beta\alpha = 1$, that is, $\beta = \alpha^{-1}$, viz. the symbols are commutative, and the only group is that made up of the powers of α.