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ADDITION TO MR ROWE'S MEMOIR ON ABEL'S THEOREM.

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In Abel's general theorem y is an irrational function of x determined by an equation $\chi(y)=0$, or say $\chi(x, y)=0$, of the order n as regards y: and it was shown by him that the sum of any number of the integrals considered may be reduced to a sum of γ integrals; where γ is a determinate number depending only on the form of the equation $\chi(x, y)=0$, and given in his equation (62), [*Œuvres Complètes*, (1881), t. I. p. 168]: viz. if, solving the equation so as to obtain from it developments of y in descending series of powers of x, we have*

$$\begin{array}{lll} n_1\mu_1 \text{ series each of the form} & & y = x^{\frac{m_1}{\mu_1}} + \ldots, \\ n_2\mu_2 \quad " & " & y = x^{\frac{m_2}{\mu_2}} + \ldots, \\ \vdots & & \vdots \\ n_k\mu_k \quad " & " & y = x^{\frac{m_k}{\mu_k}} + \ldots, \end{array}$$

* The several powers of x have coefficients: the form really is $y=A_1x^{\frac{m_1}{\mu_1}}+\ldots$, which is regarded as representing the μ_1 different values of y obtained by giving to the radical $x^{\frac{1}{\mu_1}}$ each of its μ_1 values, and the corresponding values to the radicals which enter into the coefficients of the series: and (so understanding it) the meaning is that there are n_1 such series each representing μ_1 values of y. It is assumed that the series contains *only* the radical $x^{\frac{1}{\mu_1}}$, that is, the indices after the leading index $\frac{m_1}{\mu_1}$ are $\frac{m_1-1}{\mu_1}$, $\frac{m_1-2}{\mu_1}$, ...; a series such as $y=A_1x^{\frac{4}{3}}+B_1x^{\frac{3}{5}}+\ldots$, depending on the two radicals $x^{\frac{1}{3}}$, $x^{\frac{1}{5}}$ represents 15 different values, and would be written $y=A_1x^{\frac{20}{15}}+\ldots$, or the values of m_1 and μ_1 would be 20 and 15 respectively: in a case like this where $\frac{m_1}{\mu_1}$ is not in its least terms, the number of values of the leading coefficient A_1 is equal, not to μ_1, but to a submultiple of μ_1. But the case is excluded by Abel's assumption that $\frac{m_1}{\mu_1}$, $\frac{m_2}{\mu_2}$,..., are fractions each of them in its least terms.

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(so that $n = n_1\mu_1 + n_2\mu_2 + \ldots + n_k\mu_k$), then γ is a determinate function of n_1, m_1, μ_1; n_2, m_2, μ_2; ...; n_k, m_k, μ_k.

Mr Rowe has expressed Abel's γ in the following form, viz. assuming

$$\frac{m_1}{\mu_1} > \frac{m_2}{\mu_2} > \ldots > \frac{m_k}{\mu_k},$$

then this expression is

$$\gamma = \sum_{s>r} n_r m_r n_s \mu_s + \tfrac{1}{2}\Sigma n^2 m\mu - \tfrac{1}{2}\Sigma nm - \tfrac{1}{2}\Sigma n - \tfrac{1}{2}n + 1,$$

or, what is the same thing, for n writing its value $\Sigma n\mu$,

$$\gamma = \sum_{s>r} n_r m_r n_s \mu_s + \tfrac{1}{2}\Sigma n^2 m\mu - \tfrac{1}{2}\Sigma nm - \tfrac{1}{2}\Sigma n\mu - \tfrac{1}{2}\Sigma n + 1,$$

where in the first sum r, s have each of them the values 1, 2, ..., k, subject to the condition $s > r$; in each of the other sums n, m, and μ are considered as having the suffix r, which has the values 1, 2, ..., k.

It is a leading result in Riemann's theory of the Abelian integrals that γ is the deficiency (Geschlecht) of the curve represented by the equation $\chi(x, y) = 0$: and it must consequently be demonstrable *à posteriori* that the foregoing expression for γ is in fact = deficiency of curve $\chi(x, y) = 0$. I propose to verify this by means of the formulæ given in my paper "On the Higher Singularities of a Plane Curve," *Quart. Math. Jour.*, vol. VII., (1866), pp. 212—223, [374].

It is necessary to distinguish between the values of $\frac{m}{\mu}$ which are >, =, and < 1; and to fix the ideas I assume $k = 7$, and

$$\frac{m_1}{\mu_1},\ \frac{m_2}{\mu_2},\ \frac{m_3}{\mu_3}, \text{ each } > 1,$$

$$\frac{m_4}{\mu_4} = 1; \text{ say } m_4 = \mu_4 = \lambda, \text{ and } n_4 = \theta;$$

$$\frac{m_5}{\mu_5},\ \frac{m_6}{\mu_6},\ \frac{m_7}{\mu_7}, \text{ each } < 1,$$

but it will be easily seen that the reasoning is quite general. I use Σ' to denote a sum in regard to the first set of suffixes 1, 2, 3, and Σ'' to denote a sum in regard to the second set of suffixes 5, 6, 7. The foregoing value of n is thus

$$n = \Sigma' n\mu + \lambda\theta + \Sigma'' n\mu.$$

Introducing a third coordinate z for homogeneity, the equation $\chi(x, y) = 0$ of the curve will be

$$0 = \left(yz^{\frac{m_1}{\mu_1}-1} - x^{\frac{m_1}{\mu_1}}\right)^{n_1\mu_1} \ldots \left(y - x^{\frac{\lambda}{\lambda}}\right)^{\lambda\theta} \left(y - x^{\frac{m_5}{\mu_5}} z^{1-\frac{m_5}{\mu_5}}\right) \ldots,$$

where it is to be observed that $(\quad)^{n_1\mu_1}$ is written to denote the product of $n_1\mu_1$ different series each of the form $yz^{\frac{m_1}{\mu_1}-1} - A_1 x^{\frac{m_1}{\mu_1}} - \ldots$; these divide themselves into n_1

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groups, each a product of μ_1 series; and in each such product the μ_1 coefficients A_1 are in general the μ_1 values of a function containing a radical $a^{\frac{1}{\mu_1}}$ and are thus different from each other: it is in what follows in effect assumed not only that this is so, but that all the $n_1\mu_1$ coefficients A_1 are different from each other*: the like remarks apply to the other factors. It applies in particular to the term $\left(y - x^{\frac{\lambda}{\lambda}}\right)^{\lambda\theta}$, viz. it is assumed that the coefficients A in the $\lambda\theta$ series $y = Ax^{\frac{\lambda}{\lambda}} + \ldots$ are all of them different from each other. These assumptions as to the leading coefficients really imply Abel's assumption that $\frac{m_1}{\mu_1}, \ldots, \frac{m_k}{\mu_k}$ are all of them fractions in their least terms, and in particular that $\frac{\lambda}{\lambda}$ is a fraction in its least terms, viz. that $\lambda = 1$: I retain however for convenience the general value λ, putting it ultimately $= 1$.

In the product of the several infinite series, the terms containing negative powers all disappear of themselves; and the product is a rational and integral function $F(x, y, z)$ of the coordinates, which on putting therein $z = 1$ becomes $= \chi(x, y)$. The equation of the curve thus is $F(x, y, z) = 0$; and the order is

$$= \frac{m_1}{\mu_1} n_1\mu_1 + \ldots + \lambda\theta + n_5\mu_5 + \ldots, = m_1 n_1 + \ldots + \lambda\theta + n_5\mu_5 + \ldots;$$

viz. if K is the order of the curve $\chi(x, y) = 0$, then $K = \Sigma' nm + \lambda\theta + \Sigma'' n\mu$.

The curve has singularities (singular points) at infinity, that is, on the line $z = 0$: viz.—

First, a singularity at $(z = 0,\ x = 0)$, where the tangent is $x = 0$, and which, writing for convenience $y = 1$, is denoted by the function

$$\left(z - x^{\frac{m_1}{m_1 - \mu_1}}\right)^{n_1 (m_1 - \mu_1)} \ldots;$$

where observe that the expressed factor indicates n_1 branches $\left(z - x^{\frac{m_1}{m_1 - \mu_1}}\right)^{m_1 - \mu_1}$, or say $n_1 (m_1 - \mu_1)$ partial branches $z - x^{\frac{m_1}{m_1 - \mu_1}}$, that is, $n_1 (m_1 - \mu_1)$ partial branches $z = A_1 x^{\frac{m_1}{m_1 - \mu_1}} + \ldots$, with in all $n_1 (m_1 - \mu_1)$ distinct values of A_1: and the like as regards the unexpressed factors with the suffixes 2 and 3.

Secondly, a singularity at $(z = 0,\ y = 0)$, where the tangent is $y = 0$, and which, writing for convenience $x = 1$, is denoted by the function

$$\left(z - y^{\frac{\mu_5}{\mu_5 - m_5}}\right)^{n_5 (\mu_5 - m_5)} \ldots;$$

* This assumption is virtually made by Abel, (*l. c.*) p. 162, in the expression "alors on aura en général, excepté quelques cas particuliers que je me dispense de considérer: $h(y' - y'') = hy'$, &c.": viz. the meaning is that the degree of y' being greater than or equal to that of y'', then the degree of $y' - y''$ is equal to that of y'': of course when the degrees are equal, this implies that the coefficients of the two leading terms must be unequal.

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where observe that the expressed factor indicates n_5 branches $\left(z - y^{\frac{\mu_5}{\mu_5 - m_5}}\right)^{\mu_5 - m_5}$, or say $n_5(\mu_5 - m_5)$ partial branches $z - y^{\frac{\mu_5}{\mu_5 - m_5}}$, that is, $n_5(\mu_5 - m_5)$ partial branches $z = A_5 y^{\frac{\mu_5}{\mu_5 - m_5}} + \dots$, with in all $n_5(\mu_5 - m_5)$ distinct values of A_5: and the like as regards the unexpressed factors with the suffixes 6 and 7.

Thirdly, singularities at the θ points $(z = 0,\ y - Ax = 0)$, A having here θ distinct values, at any one of which the tangent is $y - Ax = 0$, and which are denoted by the function

$$\left(y - x^{\frac{\lambda}{\lambda}}\right)^{\lambda\theta}:$$

but in the case ultimately considered λ is $= 1$; and these are then the θ ordinary points at infinity, $(z = 0,\ y - Ax = 0)$.

According to the theory explained in my paper above referred to, these several singularities are together equivalent to a certain number $\delta' + \kappa'$ of nodes and cusps; viz. we have

$$\delta' = \tfrac{1}{2}M - \tfrac{3}{2}\Sigma(\alpha - 1),$$
$$\kappa' = \qquad \Sigma(\alpha - 1),$$

hence

$$\delta' + \kappa' = \tfrac{1}{2}M - \tfrac{1}{2}\Sigma(\alpha - 1).$$

Assuming that there are no other singularities, the deficiency

$$\tfrac{1}{2}(K-1)(K-2) - \delta' - \kappa'$$

is

$$= \tfrac{1}{2}(K-1)(K-2) - \tfrac{1}{2}M + \tfrac{1}{2}\Sigma(\alpha - 1).$$

This should be equal to the before-mentioned value of γ; viz. we ought to have

$$(K-1)(K-2) - M + \Sigma(\alpha - 1) = 2\sum_{s>r} n_r m_r n_s \mu_s - \Sigma n^2 m\mu - \Sigma nm - \Sigma n\mu - \Sigma n + 2,$$

or, as it will be convenient to write it,

$$M = K^2 - 3K + \Sigma(\alpha - 1) - 2\sum_{s>r} n_r m_r n_s \mu_s - \Sigma n^2 m\mu + \Sigma nm + \Sigma n\mu + \Sigma n,$$

which is the equation which ought to be satisfied by the values of M and $\Sigma(\alpha - 1)$ calculated, according to the method of my paper, for the foregoing singularities of the curve.

We have as before

$$K = \Sigma' nm + \Sigma'' n\mu + \theta\lambda.$$

The term $\sum_{s>r} n_r m_r n_s \mu_s$, written at length, is

$$\begin{aligned}
= \quad & n_1 m_1 (n_2\mu_2 + n_3\mu_3 + \theta\lambda + n_5\mu_5 + n_6\mu_6 + n_7\mu_7) \\
+ \; & n_2 m_2 (\qquad\quad n_3\mu_3 + \theta\lambda + n_5\mu_5 + n_6\mu_6 + n_7\mu_7) \\
+ \; & n_3 m_3 (\qquad\qquad\qquad \theta\lambda + n_5\mu_5 + n_6\mu_6 + n_7\mu_7) \\
+ \; & \theta\lambda \; (\qquad\qquad\qquad\qquad\quad n_5\mu_5 + n_6\mu_6 + n_7\mu_7) \\
+ \; & n_5 m_5 (\qquad\qquad\qquad\qquad\qquad\qquad n_6\mu_6 + n_7\mu_7) \\
+ \; & n_6 m_6 (\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad n_7\mu_7),
\end{aligned}$$

which is

$$=\sum_{s>r}{}'n_r m_r n_s \mu_s + \theta\lambda\,(\Sigma' nm + \Sigma'' n\mu) + \Sigma' nm\,.\,\Sigma'' n\mu + \sum_{s>r}{}'' n_r m_r n_s \mu_s.$$

We have moreover

$$\begin{aligned}\Sigma n^2 m\mu &= \Sigma' n^2 m\mu + \theta^2\lambda^2 + \Sigma'' n^2 m\mu,\\ \Sigma nm &= \Sigma' nm + \theta\lambda + \Sigma'' nm,\\ \Sigma n\mu &= \Sigma' n\mu + \theta\lambda + \Sigma'' n\mu,\\ \Sigma n &= \Sigma' n + \theta + \Sigma'' n.\end{aligned}$$

We next calculate $\Sigma(\alpha-1)$.

For the singularity

$$\left(z - x^{\frac{m_1}{m_1-\mu_1}}\right)^{n_1(m_1-\mu_1)}\;\ldots,$$

each branch $\left(z - x^{\frac{m_1}{m_1-\mu_1}}\right)^{m_1-\mu_1}$ gives $\alpha = m_1 - \mu_1$, and the value of $\Sigma(\alpha-1)$ for this singularity is

$$n_1(m_1-\mu_1-1) + n_2(m_2-\mu_2-1) + n_3(m_3-\mu_3-1),$$

which is

$$= \Sigma' nm - \Sigma' n\mu - \Sigma' n.$$

For the singularity

$$\left(z - y^{\frac{\mu_5}{\mu_5-m_5}}\right)^{n_5(\mu_5-m_5)}\;\ldots,$$

each branch $\left(z - y^{\frac{\mu_5}{\mu_5-m_5}}\right)^{\mu_5-m_5}$ gives $\alpha = \mu_5 - m_5$, and the value of $\Sigma(\alpha-1)$ for this singularity is

$$n_5(\mu_5-m_5-1) + n_6(\mu_6-m_6-1) + n_7(\mu_7-m_7-1),$$

which is

$$= \Sigma'' n\mu - \Sigma'' nm - \Sigma'' n.$$

For each of the θ singularities

$$\left(y - x^{\frac{\lambda}{\lambda}}\right)^{\lambda\theta},$$

we have $\alpha = \lambda$ and the value of $\Sigma(\alpha-1)$ is $=\theta(\lambda-1)$: this is $=0$ for the value $\lambda=1$, which is ultimately attributed to λ.

The complete value of $\Sigma(\alpha-1)$ is thus

$$= \Sigma' nm - \Sigma'' nm - \Sigma' n\mu + \Sigma'' n\mu - \Sigma' n - \Sigma'' n + \theta\lambda - \theta.$$

Substituting all these values, we have

$$\begin{aligned} M = {} & (\Sigma' nm + \Sigma'' n\mu)^2 + 2\theta\lambda\,(\Sigma' nm + \Sigma'' n\mu) + (\theta\lambda)^2\\ & - 3\,(\Sigma' nm + \Sigma'' n\mu) - 3\theta\lambda\\ & + \Sigma' nm - \Sigma'' nm - \Sigma' n\mu + \Sigma'' n\mu - \Sigma' n - \Sigma'' n + \theta\lambda - \theta\\ & - 2\sum_{s>r}{}' n_r m_r n_s \mu_s - 2\theta\lambda\,(\Sigma' nm + \Sigma'' n\mu) - 2\Sigma' nm\,.\,\Sigma'' n\mu - 2\sum_{s>r}{}'' n_r m_r n_s \mu_s\\ & - \Sigma' n^2 m\mu - (\theta\lambda)^2 - \Sigma'' n^2 m\mu\\ & + \Sigma' nm + \theta\lambda + \Sigma'' nm\\ & + \Sigma' n\mu + \theta\lambda + \Sigma'' n\mu\\ & + \Sigma' n + \theta + \Sigma'' n,\end{aligned}$$

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or, reducing,

$$M = (\Sigma' nm)^2 - \Sigma' nm - \Sigma' n^2 m\mu - 2\sum_{s>r}' n_r m_r n_s \mu_s$$
$$+ (\Sigma'' n\mu)^2 - \Sigma'' n\mu - \Sigma'' n^2 m\mu - 2\sum_{s>r}'' n_r m_r n_s \mu_s;$$

and it is to be shown that the two lines of this expression are in fact the values of M belonging to the singularities

$$\left(z - x^{\frac{m_1}{m_1-\mu_1}}\right)^{n_1(m_1-\mu_1)} \ldots, \text{ and } \left(z - y^{\frac{\mu_5}{\mu_5-m_5}}\right)^{n_5(\mu_5-m_5)} \ldots,$$

respectively. We assume $\lambda = 1$, and there is thus no singularity $\left(y - x^{\frac{\lambda}{\bar{\lambda}}}\right)^{\lambda\theta}$.

I recall that, considering the several partial branches which meet at a singular point, M denotes the sum of the number of the intersections of each partial branch by every other partial branch: so that for each pair of partial branches the intersections are to be counted *twice*. Supposing that the tangent is $x=0$, and that for any two branches we have $z_1 = A_1 x^{p_1}$, $z_2 = A_2 x^{p_2}$ (where p_1, p_2 are each equal to or greater than 1), then if $p_2 = p_1$, and $z_1 - z_2 = (A_1 - A_2) x^{p_1}$ where $A_1 - A_2$ not $=0$ (an assumption which has been already made as regards the cases about to be considered), then the number of intersections is taken to be $=p_1$; and if p_1 and p_2 are unequal, then *taking p_2 to be the greater of them*, the leading term of $z_1 - z_2$ is $= A_1 x^{p_1}$, and the number of intersections is taken to be $=p_1$; viz. in the case of unequal exponents, it is equal to the smaller exponent.

Consider now the singularity $\left(z - x^{\frac{m_1}{m_1-\mu_1}}\right)^{n_1(m_1-\mu_1)} \ldots$; and first the intersections of a partial branch $z - x^{\frac{m_1}{m_1-\mu_1}}$ by each of the remaining $n_1(m_1-\mu_1)-1$ partial branches of the same set: the number of intersections with any one of these is $=\frac{m_1}{m_1-\mu_1}$; and consequently the number with all of them is $=\frac{m_1}{m_1-\mu_1}[n_1(m_1-\mu_1)-1]$. But we obtain this same number from each of the $n_1(m_1-\mu_1)$ partial branches, and thus the whole number is

$$n_1(m_1-\mu_1)\frac{m_1}{m_1-\mu_1}[n_1(m_1-\mu_1)-1], = n_1 m_1 [n_1(m_1-\mu_1)-1].$$

Taking account of the other sets, each with itself, the whole number of such intersections is

$$n_1 m_1 [n_1(m_1-\mu_1)-1] + n_2 m_2 [n_2(m_2-\mu_2)-1] + n_3 m_3 [n_3(m_3-\mu_3)-1],$$

which is

$$= \Sigma' n^2 m^2 - \Sigma' n^2 m\mu - \Sigma' nm.$$

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Observe now that $\frac{m_1}{\mu_1} > \frac{m_2}{\mu_2}$, that is, $\frac{\mu_1}{m_1} < \frac{\mu_2}{m_2}$, and that, these being each < 1, we thence have $1 - \frac{\mu_1}{m_1} > 1 - \frac{\mu_2}{m_2}$, that is, $\frac{m_1 - \mu_1}{m_1} > \frac{m_2 - \mu_2}{m_2}$: and we thus have

$$\frac{m_1}{m_1 - \mu_1} < \frac{m_2}{m_2 - \mu_2} < \frac{m_3}{m_3 - \mu_3}.$$

Considering now the intersections of partial branches of the two sets

$$\left(z - x^{\frac{m_1}{m_1 - \mu_1}}\right)^{n_1 (m_1 - \mu_1)} \text{ and } \left(z - x^{\frac{m_2}{m_2 - \mu_2}}\right)^{n_2 (m_2 - \mu_2)}$$

respectively, a partial branch $z - x^{\frac{m_1}{m_1 - \mu_1}}$ gives with each partial branch of the other set a number $= \frac{m_1}{m_1 - \mu_1}$; and in this way taking each partial branch of each set, the number is

$$n_1 (m_1 - \mu_1) \,.\, n_2 (m_2 - \mu_2) \,.\, \frac{m_1}{m_1 - \mu_1}, \; = n_1 m_1 n_2 (m_2 - \mu_2);$$

and thus for all the sets the number is

$$= n_1 m_1 n_2 (m_2 - \mu_2) + n_1 m_1 n_3 (m_3 - \mu_3) + n_2 m_2 n_3 (m_3 - \mu_3),$$

which is

$$= \Sigma' n_r m_r n_s m_s - \underset{s>r}{\Sigma'} n_r m_r n_s \mu_s,$$

where in the first sum the Σ' refers to each pair of values of the suffixes. But the intersections are to be taken twice; the number thus is

$$= 2\Sigma' n_r m_r n_s m_s - 2\underset{s>r}{\Sigma'} n_r m_r n_s \mu_s.$$

Adding the foregoing number

$$\Sigma' n^2 m^2 - \Sigma' n^2 m\mu - \Sigma' nm,$$

the whole number for the singularity in question is

$$= (\Sigma' nm)^2 - \Sigma' nm - \Sigma' n^2 m\mu - 2\underset{s>r}{\Sigma'} n_r m_r n_s \mu_s.$$

Similarly for the singularity $\left(z - y^{\frac{\mu_5}{\mu_5 - m_5}}\right)^{n_5 (\mu_5 - m_5)} \ldots$; taking each set with itself, the number of intersections is

$$n_5 \mu_5 [n_5 (\mu_5 - m_5) - 1] + n_6 \mu_6 [n_6 (\mu_6 - m_6) - 1] + n_7 \mu_7 [n_7 (\mu_7 - m_7) - 1],$$

which is

$$= \Sigma'' n^2 \mu^2 - \Sigma'' n^2 m\mu - \Sigma'' n\mu.$$

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We have here $\frac{m_5}{\mu_5} > \frac{m_6}{\mu_6}$; each of these being less than 1, we have $1 - \frac{m_5}{\mu_5} < 1 - \frac{m_6}{\mu_6}$, that is, $\frac{\mu_5 - m_5}{\mu_5} < \frac{\mu_6 - m_6}{\mu_6}$, or $\frac{\mu_5}{\mu_5 - m_5} > \frac{\mu_6}{\mu_6 - m_6}$; and so

$$\frac{\mu_7}{\mu_7 - m_7} < \frac{\mu_6}{\mu_6 - m_6} < \frac{\mu_5}{\mu_5 - m_5}.$$

Hence considering the two sets

$$\left(z - y^{\frac{\mu_5}{\mu_5 - m_5}}\right)^{n_5(\mu_5 - m_5)} \text{ and } \left(z - y^{\frac{\mu_6}{\mu_6 - m_6}}\right)^{n_6(\mu_6 - m_6)},$$

a partial branch of the first set gives with a partial branch of the second set $\frac{\mu_6}{\mu_6 - m_6}$ intersections: and the number thus obtained is

$$n_5(\mu_5 - m_5) \,.\, n_6(\mu_6 - m_6) \,.\, \frac{\mu_6}{\mu_6 - m_6}, \; = n_5 n_6 \mu_6 (\mu_5 - m_5).$$

For all the sets the number is

$$n_5 n_6 \mu_6 (\mu_5 - m_5) + n_5 n_7 \mu_7 (\mu_5 - m_5) + n_6 n_7 \mu_7 (\mu_6 - m_6)$$

or taking this twice, the number is

$$= 2\Sigma'' n_r \mu_r n_s \mu_s - 2\underset{s>r}{\Sigma''} n_r m_r n_s \mu_s$$

where in the first sum the Σ'' refers to each pair of suffixes. Adding the foregoing value

$$\Sigma'' n^2 \mu^2 - \Sigma'' n^2 m\mu - \Sigma'' n\mu,$$

the whole number for the singularity in question is

$$= (\Sigma'' n\mu)^2 - \Sigma'' n\mu - \Sigma'' n^2 m\mu - 2\underset{s>r}{\Sigma''} n_r m_r n_s \mu_s;$$

and the proof is thus completed.

Referring to the foot-note (ante, p. 31), I remark that the theorem $\gamma =$ deficiency, is absolute, and applies to a curve with any singularities whatever: in a curve which has singularities not taken account of in Abel's theory, the "quelques cas particuliers que je me dispense de considérer," the singularities not taken account of give rise to a diminution in the deficiency of the curve, and also to an equal diminution of the value of γ as determined by Abel's formula; and the actual deficiency will be $=$ Abel's $\gamma -$ such diminution, that is, it will be $=$ true value of γ.

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