710.

ON A DIFFERENTIAL EQUATION.

[From *Collectanea Mathematica: in memoriam Dominici Chelini*, (Milan, Hoepli, 1881), pp. 17—26.]

In the Memoir on hypergeometric series, *Crelle*, t. xv. (1836), Kummer in effect considers a differential equation

$$\frac{(a'z^2 + 2b'z + c')\,dz^2}{z^2(z-1)^2} = \frac{(ax^2 + 2bx + c)\,dx^2}{x^2(x-1)^2},$$

viz. he seeks for solutions of an equation of this form which also satisfy a certain differential equation of the third order. The coefficients a, b, c are either all arbitrary, or they are two or one of them, arbitrary; but this last case (or say the case where the function of x is the completely determinate function $x^2 + 2bx + c$) is scarcely considered: a', b', c' are regarded as determinable in terms of a, b, c; and z is to be found as a function of x independent of a, b, c: so that when these coefficients are arbitrary, the equation breaks up into three equations, and when two of the coefficients are arbitrary, it breaks up into two equations, satisfied in each case by the same value of z; and the value of z is thus determined without any integration: these cases will be considered in the sequel, but they are of course included in the general case where the coefficients a, b, c are regarded as having any given values whatever.

Writing for shortness $X = ax^2 + 2bx + c$, in general the integral

$$\int \frac{N dx}{D\sqrt{X}},$$

where D is the product of any number n of distinct linear factors $x - p$, and N is a rational and integral function of x of the order n at most, and therefore also the integral

$$\int \frac{N\sqrt{X}dx}{D} = \int \frac{NXdx}{D\sqrt{X}},$$

where N is now of the order $n-2$ at most, is expressible as the logarithm of a quasi-algebraical function, that is, a function containing powers the exponents of which are incommensurable (for instance, $x^{\sqrt{2}}$ is a quasi-algebraical function): in fact, the integral is of the form

$$\int\left(M+\frac{A}{x-p}+\frac{B}{x-q}+\ldots\right)\frac{dx}{\sqrt{X}},$$

where each term is separately integrable,

$$\int\frac{dx}{\sqrt{X}}=\quad\frac{1}{\sqrt{a}}\log\{ax+b+\sqrt{a}\,.\,\sqrt{X}\},$$

$$\int\frac{dx}{(x-p)\sqrt{X}}=-\frac{1}{\sqrt{P}}\log\left\{\frac{(ap+b)\,x+(bp+c)+\sqrt{P}\,.\,\sqrt{X}}{x-p}\right\},$$

where P is written to denote $ap^2+2bp+c$: the integral is thus $=\log\Omega$, where Ω is a product of factors

$$ax+b+\sqrt{a}\,.\,\sqrt{X},\quad \frac{(ap+b)\,x+(bp+c)+\sqrt{P}\,.\,\sqrt{X}}{x-p},\text{ etc.,}$$

raised to powers $\dfrac{M}{\sqrt{a}}$, $\dfrac{-A}{\sqrt{P}}$, etc.: hence, if we have a differential equation

$$\frac{N'dz}{D'\sqrt{Z}}=\frac{Ndx}{D\sqrt{X}},\quad\text{or}\quad\frac{N'\sqrt{Z}dz}{D'}=\frac{N\sqrt{X}dx}{D},$$

where $Z\,(=a'z^2+2b'z+c')$, and N', D' are functions of z such as X, N, D are of x; then, taking $\log C$ for the constant of integration, the general integral is

$$\log\Omega'=\log C+\log\Omega:$$

viz. we have the quasi-algebraical integral $\Omega'-C\Omega=0$.

The constants a, b, c, p, q, ... etc. may be such that the exponents are rational, and the integral is then algebraical: in particular, for the differential equation

$$\frac{\sqrt{z^2+14z+1}\,dz}{z\,(z-1)}=\frac{\sqrt{x^2+14x+1}\,dx}{x\,(x-1)},$$

the general integral is in the first instance obtained in the form

$$\frac{(z+1+\sqrt{Z})\,(z-1)^2}{\sqrt{z}\,(2z+2+\sqrt{Z})^2}=C\,\frac{(x+1+\sqrt{X})\,(x-1)^2}{\sqrt{x}\,(2x+2+\sqrt{X})^2}$$

which, observing that $(2x+2)^2-X=3\,(x-1)^2$, may also be written

$$\frac{(z+1)\,(z^2-34z+1)+Z\sqrt{Z}}{\sqrt{z}\,(z-1)^2}=C\,\frac{(x+1)\,(x^2-34x+1)+X\sqrt{X}}{\sqrt{x}\,(x-1)^2}.$$

I had previously obtained the solution

$$z=\left(\frac{1-\sqrt[4]{x}}{1+\sqrt[4]{x}}\right)^4,$$

and I wish to show that this is, in fact, the particular integral belonging to the value $C=1$ of the constant of integration: for this purpose I proceed to rationalise the general integral as regards z.

Writing for a moment

$$P=(z+1)(z^2-34z+1),$$
$$Q=(z^2+14z+1)\sqrt{z^2+14z+1},$$
$$R=M\sqrt{z}\,(z-1)^2,$$

where

$$M=C\,\frac{(x+1)(x^2-34x+1)+(x^2+14x+1)\sqrt{x^2+14x+1}}{\sqrt{x}\,(x-1)^2},$$

the integral is $P+Q+R=0$; or rationalising, it is

$$(P^2-Q^2)^2-2R^2(P^2+Q^2)+R^4=0;$$

we have

$$P^2=(1,\ -66,\ 1023,\ 2180,\ 1023,\ -66,\ 1 \between z,\ 1)^6,$$
$$Q^2=(1,\ \ 42,\ \ 591,\ 2828,\ \ 591,\ \ 42,\ 1 \between z,\ 1)^6,$$

and thence

$$P^2-Q^2=(0,\ -108,\ 432,\ -648,\ 432,\ -108,\ 0 \between z,\ 1)^6,$$
$$=-108z(z-1)^4;$$
$$P^2+Q^2=2\,(1,\ -12,\ 807,\ 2504,\ 807,\ -12,\ 1 \between z,\ 1)^6.$$

Writing the equation in the form

$$\tfrac{1}{2}(P^2+Q^2)-\tfrac{1}{4}\left\{R^2+\frac{(P^2-Q^2)^2}{R^2}\right\}=0,$$

it thus becomes

$$(1,\ -12,\ 807,\ 2504,\ 807,\ -12,\ 1 \between z,\ 1)^6-z(z-1)^4\left\{M^2+\frac{(108)^2}{M^2}\right\}=0,$$

where M has its above-mentioned value; and if we now assume $C=1$, then

$$M=\frac{(x+1)(x^2-34x+1)+(x^2+14x+1)\sqrt{x^2+14x+1}}{\sqrt{x}\,(x-1)^2},$$

$$\frac{108}{M}=\frac{(x+1)(x^2-34x+1)-(x^2+14x+1)\sqrt{x^2+14x+1}}{\sqrt{x}\,(x-1)^2},$$

and thence

$$M^2+\frac{(108)^2}{M^2},\ =\left(M-\frac{108}{M}\right)^2+216,\ =4\,\frac{(x+1)^2(x^2-34x+1)^2}{x(x-1)^4}+216,$$

$$=\frac{4}{x(x-1)^4}\cdot(1,\ -12,\ 807,\ 2504,\ 807,\ -12,\ 1 \between x,\ 1)^6:$$

and the rationalised equation is

$$(1, -12, 807, 2504, 807, -12, 1 \between z, 1)^6$$
$$-\frac{z(z-1)^4}{x(x-1)^4}(1, -12, 807, 2504, 807, -12, 1 \between x, 1)^6 = 0.$$

This is a sextic equation in z, of the form

$$z^3 + \frac{1}{z^3} + \lambda\left(z^2 + \frac{1}{z^2}\right) + \mu\left(z + \frac{1}{z}\right) + \nu = 0,$$

where

$$\lambda, \mu, \nu = -12 - \Omega, \quad 807 + 4\Omega, \quad 2504 - 6\Omega,$$

if Ω denote the function of x which enters into the equation; and writing $z + \frac{1}{z} = \theta$, this becomes

$$\theta^3 - 3\theta + \lambda(\theta^2 - 2) + \mu\theta + \nu = 0.$$

But the equation in z is satisfied by the value $z = x$, and therefore the equation in θ by the value $\theta = x + \frac{1}{x} = \alpha$ suppose, we have therefore

$$\alpha^3 - 3\alpha + \lambda(\alpha^2 - 2) + \mu\alpha + \nu = 0,$$

and thence subtracting, and throwing out the factor $\theta - \alpha$,

$$\theta^2 + \theta\alpha + \alpha^2 - 3 + \lambda(\theta + \alpha) + \mu = 0;$$

viz. writing for λ, μ, α their values, this is

$$\theta^2 + \theta\left(x + \frac{1}{x} - 12 - \Omega\right) + x^2 - 1 + \frac{1}{x^2} - \left(x + \frac{1}{x}\right)(12 + \Omega) + 807 + 4\Omega = 0,$$

or, what is the same thing,

$$\theta^2 + \theta\left(x - 12 + \frac{1}{x} - \Omega\right) + x^2 - 12x + 806 - \frac{12}{x} + \frac{1}{x^2} - \left(x - 4 - \frac{1}{x}\right)\Omega = 0,$$

where

$$\Omega = \frac{1}{x(x-1)^4}(1, -12, 807, 2504, 807, -12, 1 \between x, 1)^6.$$

Hence in the quadric equation, the coefficients, each multiplied by $(x-1)^4$, are

$$(x-1)^4\left(x - 12 + \frac{1}{x}\right) - \frac{1}{x}(1, -12, 807, 2504, 807, -12, 1 \between x, 1)^6,$$

and

$$(x-1)^4\left(x^2 - 12x + 806 - \frac{12}{x} + \frac{1}{x^2}\right)$$
$$-\frac{1}{x}\left(x - 4 + \frac{1}{x}\right)(1, -12, 807, 2504, 807, -12, 1 \between x, 1)^6,$$

which are respectively rational and integral quartic functions of x; and, writing for θ its value, the equation finally is

$$\left(z + \frac{1}{z}\right)^2 - 4\left(z + \frac{1}{z}\right)\frac{(1, 188, 646, 188, 1 \between x, 1)^4}{(x-1)^4} + 4\,\frac{(1, -644, 3334, -644, 1 \between x, 1)^4}{(x-1)^4} = 0$$

Writing

$$\xi = \sqrt[4]{x},\quad A = \frac{1-\xi}{1+\xi},\quad B = \frac{1+\xi}{1-\xi},\quad C = \frac{1-i\xi}{1+i\xi},\quad D = \frac{1+i\xi}{1-i\xi},\qquad (i = \sqrt{-1}\text{ as usual}),$$

this is

$$(z-A^4)(z-B^4)(z-C^4)(z-D^4) = 0,$$

or, what is the same thing,

$$\left\{z + \frac{1}{z} - (A^4+B^4)\right\}\left\{z + \frac{1}{z} - (C^4+D^4)\right\} = 0,$$

that is,

$$\left(z+\frac{1}{z}\right)^2 - \left(z+\frac{1}{z}\right)(A^4+B^4+C^4+D^4) + (A^4+B^4)(C^4+D^4) = 0;$$

for we have

$$\tfrac{1}{2}(A^4+B^4) = \frac{(1,\ 28,\ 70,\ 28,\ 1)(\xi^2,\ 1)^4}{(\xi^2-1)^4},$$

$$\tfrac{1}{2}(C^4+D^4) = \frac{(1,\ -28,\ 70,\ -28,\ 1)(\xi^2,\ 1)^4}{(\xi^2+1)^4}.$$

And substituting these values, the coefficients will be rational functions of ξ^4, that is, of x, and it is easy to verify that they have in fact their foregoing values.

It thus appears that for $C=1$, besides the values x and $\frac{1}{x}$, we have for z only the values

$$\left(\frac{1-\xi}{1+\xi}\right)^4,\quad \left(\frac{1+\xi}{1-\xi}\right)^4,\quad \left(\frac{1-i\xi}{1+i\xi}\right)^4,\quad \left(\frac{1+i\xi}{1-i\xi}\right)^4;$$

viz. that the only solution is

$$z = \left(\frac{1-\sqrt[4]{x}}{1+\sqrt[4]{x}}\right)^4.$$

The example shows that although the differential equation

$$\frac{\sqrt{a'z^2+2b'z+c'}\,dz}{z(z-1)} = \frac{\sqrt{ax^2+2bx+c}\,dx}{x(x-1)}$$

can be integrated generally in a quasi-algebraical or algebraical form as above, yet we cannot from the general solution deduce, at once or easily, the various particular integrals comprised therein: nor can we find for what values of the constants a, b, c and a', b', c' the differential equation admits of a simple solution, or say of a solution where z is expressed as an explicit (irrational) function of x.

In the cases considered by Kummer there is a second (or it may be also a third) differential equation of the like form, the equations being each of them satisfied by the same value of z: hence eliminating the differentials dx, dz, the relation between x and z is of the form

$$\frac{P'}{Q'} = \frac{P}{Q},$$

where P, Q are quadric functions of x; P', Q' quadric functions of z. But P and Q may contain a common factor, and the integral is then expressible in the form $x=\frac{P'}{Q'}$, the quotient of two quadric functions of z; or P' and Q' may have a common factor, and the integral is then expressible in the form $z=\frac{P}{Q}$, the quotient of two quadric functions of x; or there may be a common factor of P, Q, and also a common factor of P' and Q', and the integral is then of the form $z=\frac{L}{M}$, the quotient of two linear functions of x.

In the general case the differential equation is

$$\frac{\lambda\,(aP'+bQ')\,dz^2}{z^2(z-1)^2}=\frac{(aP+bQ)\,dx^2}{x^2(x-1)^2},$$

where a, b are arbitrary constants, λ is a constant the value of which can in each particular case be at once determined; so when the integral is $z=\frac{P}{Q}$, the differential equation is

$$\frac{\lambda\,(az+b)\,dz^2}{z^2(z-1)^2}=\frac{(aP+bQ)\,dx^2}{x^2(x-1)^2},$$

where a, b are arbitrary constants, but λ is now a linear function of z the value of which can in each particular case be at once determined. When the integral is $z=\frac{L}{M}$, the differential equation is

$$\frac{\lambda\,(az^2+2bz+c)\,dz^2}{z^2(z-1)^2}=\frac{(aL^2+2bLM+cM^2)\,dx^2}{x^2(x-1)^2}$$

containing the three arbitrary constants a, b, c; λ is a constant the value of which can be at once determined.

There are in all 6 integrals of the form $z=\frac{L}{M}$, for which the differential equation contains three arbitrary constants: 18 integrals of the form $z=\frac{P}{Q}$ $\left(\text{and of course the same number of integrals of the form } x=\frac{P'}{Q'}\right)$, and 9 integrals of the form $\frac{P}{Q}=\frac{P'}{Q'}$, for all of which the differential equation contains two arbitrary constants. It is to be remarked that Kummer, considering the values of z as a function of x, obtains the 72 rational and irrational values mentioned in his equations (31), (35), (36), (37), (38), and (39): but the 72 values are made up as follows, viz. the 18 values of z as a rational function of x, the 36 irrational values obtained from the 18 expressions of x as a rational function of z, and the 18 irrational values of z obtained from the 9 integrals in which neither of the variables is a rational function of the other: $18+36+18=72$.

The several integrals together with the expressions of the functions

$$a'z^2 + 2b'z + c' \quad \text{and} \quad ax^2 + 2bx + c$$

which enter into the differential equation are as follows:

	$z =$	$a'z^2 + 2b'z + c' =$	$ax^2 + 2bx + c =$
1.	x	$az^2 + 2bz + c$	$ax^2 + 2bx + c$
	$1 - x$	,,	$a(x-1)^2 - 2b(x-1) + c$
	$\dfrac{1}{x}$	,,	$a + 2bx + cx^2$
	$\dfrac{1}{1-x}$	,,	$a - 2b(x-1) + c(x-1)^2$
	$\dfrac{x}{x-1}$	,,	$ax^2 + 2bx(x-1) + c(x-1)^2$
	$\dfrac{x-1}{x}$	,,	$a(x-1)^2 + 2bx(x-1) + cx^2$
2.	$\left(\dfrac{x+1}{x-1}\right)^2$	$az^2 + bz$	$a(x+1)^2 + b(x-1)^2$
	$(2x-1)^2$	,,	$a(2x-1)^2 + b$
	$\left(\dfrac{x-2}{x}\right)^2$	,,	$a(x-2)^2 + bx^2$
	$\dfrac{(x+1)^2}{4x}$	,,	$a(x+1)^2 + 4bx$
	$\dfrac{(2x-1)^2}{4x(x-1)}$	,,	$a(2x-1)^2 + 4bx(x-1)$
	$-\dfrac{(x-2)^2}{4(x-1)}$	,,	$a(x-2)^2 - 4b(x-1)$
3.	$\left(\dfrac{x-1}{x+1}\right)^2$	$bz + c$	$b(x-1)^2 + c(x+1)^2$
	$\left(\dfrac{1}{2x-1}\right)^2$	,,	$b + c(2x-1)^2$
	$\left(\dfrac{x}{x-2}\right)^2$	,,	$bx^2 + c(x-2)^2$
	$\dfrac{4x}{(x+1)^2}$	,,	$4bx + c(x+1)^2$
	$\dfrac{4x(x-1)}{(2x-1)^2}$	,,	$4bx(x-1) + c(2x-1)^2$
	$-\dfrac{4(x-1)}{(x-2)^2}$	,,	$-4b(x-1) + c(x-2)^2$

	$z=$	$a'z^2+2b'z+c'=$	$ax^2+2bx+c=$
4.	$-\dfrac{(x-1)^2}{4x}$	$az^2-(a+c)z+c$	$a(x-1)^2+4cx$
	$-4x(x-1)$	,,	$4ax(x-1)+c$
	$\dfrac{4(x-1)}{x^2}$	,,	$-4a(x-1)+cx^2$
	$-\dfrac{4x}{(x-1)^2}$	,,	$4ax+c(x-1)^2$
	$\dfrac{-1}{4x(x-1)}$	,,	$a+4cx(x-1)$
	$\dfrac{x^2}{4(x-1)}$	,,	$ax^2-4c(x-1)$

5., 6., 7. } same as 2, 3, 4 interchanging x and z.

	$z=$	$a'z^2+2b'z+c'=$	$ax^2+2bx+c=$
8.	$\dfrac{(z-1)^2}{4z}=\dfrac{4x}{(x-1)^2}$	$a(z-1)^2+4bz$	$4ax+b(x-1)^2$
	$\dfrac{z^2}{4(z-1)}=\dfrac{4(x-1)}{x^2}$	$az^2+4b(z-1)$	$-4a(x-1)-bx^2$
	$4z(z-1)=\dfrac{1}{4x(x-1)}$	$4az(z-1)+b$	$a+4bx(x-1)$
9.	$\dfrac{(z-1)^2}{4z}=4x(x-1)$	$a(z-1)^2+4bz$	$4ax(x-1)+b$
	$\dfrac{z^2}{4(z-1)}=-4x(x-1)$	$az^2-4b(z-1)$	$4ax(x-1)+b$
	$\dfrac{(z-1)^2}{4z}=-\dfrac{4(x-1)}{x^2}$	$a(z-1)^2+4bz$	$-4a(x-1)+bx^2$
10.	$4z(z-1)=\dfrac{(x-1)^2}{4x}$	$4az(z-1)+b$	$a(x-1)^2+4bx$
	$4z(z-1)=-\dfrac{x^2}{4(x-1)}$	$4az(z-1)+b$	$ax^2-4b(x-1)$
	$\dfrac{4(z-1)}{z^2}=-\dfrac{(x-1)^2}{4x}$	$-4a(z-1)+bz^2$	$a(x-1)^2+4bx$

The six functions of the set (1), that is,

$$x, \quad 1-x, \quad \frac{1}{x}, \quad \frac{1}{1-x}, \quad \frac{x}{x-1}, \quad \frac{x-1}{x},$$

form a group; and by operating with the substitutions of this group, and of the like group

$$z, \quad 1-z, \quad \frac{1}{z}, \quad \frac{1}{1-z}, \quad \frac{z}{z-1}, \quad \frac{z-1}{z},$$

upon any value of z in the sets (2), (3), (4), for instance upon $z=\left(\frac{x+1}{x-1}\right)^2$, we form all the 18 functions of these sets.

In any one of these sets (2), (3), and (4), comparing two forms (the same or different), for instance in the set (2), writing y for z and then in one form z for x,

$$y=\left(\frac{x+1}{x-1}\right)^2 \text{ and } y=\left(\frac{z+1}{z-1}\right)^2, \text{ whence } \left(\frac{x+1}{x-1}\right)^2=\left(\frac{z+1}{z-1}\right)^2,$$

or

$$y=\left(\frac{x+1}{x-1}\right)^2 \text{ and } y=\frac{(z+1)^2}{4z}, \text{ whence } \left(\frac{x+1}{x-1}\right)^2=\frac{(z+1)^2}{4z},$$

we obtain either the equations of the set (1) or those of the sets (8), (9) and (10); and whether we use the set (2), (3) or (4), the only new equations obtained are thus the 9 equations of the sets (8), (9) and (10). These several equations present themselves however in different forms: for instance, instead of the equation

$$\frac{(z-1)^2}{4z}=\frac{4x}{(x-1)^2},$$

we may obtain

$$\frac{(z+1)^2}{4z}=\left(\frac{x+1}{x-1}\right)^2.$$

If, to get rid of this variety of form, we multiply out the denominators, the 9 equations are

$$\begin{aligned}
0 &= x^2z^2 - 2x^2z - 2xz^2 + x^2 - 12xz + z^2 - 2x - 2z + 1, \\
0 &= x^2z^2 - 16xz + 16x + 16z - 16, \\
0 &= 16x^2z^2 - 16x^2z - 16xz^2 + 16xz - 1, \\
0 &= x^2z^2 - 2x^2z + x^2 + 16xz - 16z, \\
0 &= 16x^2z - 16xz - z^2 + 2z - 1, \\
0 &= 16x^2z - 16x^2 - 16xz + z^2 + 16x, \\
0 &= x^2z^2 - 2xz^2 + 16xz + z^2 - 16x, \\
0 &= 16xz^2 - x^2 - 16xz + 2x + 1, \\
0 &= 16xz^2 + x^2 - 16xz - 16z^2 + 16z.
\end{aligned}$$

These 9 equations are derivable all from any one of them by the changes of the set (1) upon x and z.

Cambridge, 3rd June, 1879.