709.

ON THE NUMBER OF CONSTANTS IN THE EQUATION OF A SURFACE $PS-QR=0$.

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THE very important results contained in Mr H. Valentiner's paper "Nogle Sætninger om fuldstændige Skjæringskurver mellem to Flader" may be considered from a somewhat different point of view, and established in a more simple manner, as follows*.

Assuming throughout $n \geqq p+q$, $p \geqq q$, and moreover that P, Q, R, S denote functions of the coordinates (x, y, z, w) of the orders p, q, $n-q$, $n-p$ respectively: then the equation of a surface of the order n containing the curve of intersection of two surfaces of the orders p and q respectively, is

$$\begin{vmatrix} P, & Q \\ R, & S \end{vmatrix} = 0,$$

so that the number of constants in the equation of a surface of the order n satisfying the condition in question is in fact the number of constants contained in an equation of the last-mentioned form. Writing for shortness

$$a_p = \tfrac{1}{6}(p+1)(p+2)(p+3)-1, = \tfrac{1}{6}p(p^2+6p+11),$$

the number of constants contained in a function of the order p is $=a_p+1$; or if we take one of the coefficients (for instance that of x^p) to be unity, then the number

* Idet vi med stor Glæde optage Prof. Cayley's simple Forklaring af den Reduktion af Konstanttallet i Ligningen $PS-QR=0$, som Hr. Valentiner havde paavist (*Tidsskr. f. Math.* 1879, S. 22), skulle vi dog bemærke, at Grunden til, at dennes Bevis er bleven saa vanskeligt, er den, at han tillige har villet bevise, at der ikke finder nogen *yderligere* Reduktion Sted.

of the remaining constants is $=a_p$; viz. a_p is the number of constants in the equation of a surface of the order p. As regards the surface in question

$$\begin{vmatrix} P, & Q \\ R, & S \end{vmatrix} = 0,$$

we may it is clear take P, Q, R each with a coefficient unity as above, but in the remaining function S, the coefficient must remain arbitrary: the apparent number of constants is thus $= a_p + a_q + a_{n-p} + a_{n-q} + 1$; but there is a deduction from this number.

The equation may in fact be written in the form

$$\begin{vmatrix} P + \alpha Q, & Q \\ R + \alpha S + \beta P + \alpha\beta Q, & S + \beta Q \end{vmatrix} = 0,$$

where α represents an arbitrary function of the order $p-q$, and β an arbitrary function of the degree $n-p-q$: we thus introduce $(a_{p-q}+1)+(a_{n-p-q}+1)$, $= a_{p-q} + a_{n-p-q} + 2$, constants, and by means of these we can impose the like number of arbitrary relations upon the constants originally contained in the functions P, Q, R, S respectively (say we can reduce to zero this number $a_{p-q} + a_{n-p-q} + 2$ of the original constants): hence the real number of constants is

$$\begin{aligned} & a_p + a_q + a_{n-p} + a_{n-q} + 1 - (a_{p-q} + a_{n-p-q} + 2), \\ &= a_p + a_q + a_{n-p} + a_{n-q} - a_{p-q} - a_{n-p-q} - 1 \\ &= \omega \text{ suppose}; \end{aligned}$$

viz. this is the required number in the case $n > p+q$, $p > q$.

If however $n = p + q$, or $p = q$, or if these relations are both satisfied, then there is a further deduction of 1, 1, or 2: in fact, calling the last-mentioned determinant $\begin{vmatrix} P', & Q' \\ R', & S' \end{vmatrix}$, then the four cases are

$$n > p+q,\ p > q, \quad \begin{vmatrix} P', & Q' \\ R', & S' \end{vmatrix} = \begin{vmatrix} P', & Q' \\ R', & S' \end{vmatrix}$$

$$n = p+q,\ p > q, \quad \begin{vmatrix} P', & Q' \\ R', & S' \end{vmatrix} = \begin{vmatrix} P' + kR', & Q' + kS' \\ R', & S' \end{vmatrix}$$

$$n > p+q,\ p = q, \quad \begin{vmatrix} P', & Q' \\ R', & S' \end{vmatrix} = \begin{vmatrix} P', & Q' + kP' \\ R', & S' + kR' \end{vmatrix}$$

$$n = p+q,\ p = q, \quad \begin{vmatrix} P', & Q' \\ R', & S' \end{vmatrix} = \begin{vmatrix} P' + kR', & Q' + lP' + kS' + klR' \\ R', & S' + \ lR' \end{vmatrix}$$

where k, l denote arbitrary constants: these, like the constants of α and β, may be used to impose arbitrary relations upon the original constants of P, Q, R, S; and hence the number of constants is $=\omega$, $\omega - 1$, $\omega - 1$, $\omega - 2$ in the four cases respectively; where as above

$$\omega = a_p + a_q + a_{n-p} + a_{n-q} - a_{p-q} - a_{n-p-q} - 1.$$

If $n=4$, there is in each of the four cases one system of values of p, q; viz. the cases are

$p, q =$

$$\begin{array}{llll}
2\ 1 & \text{No.} = a_2 + a_1 + a_2 + a_3 - a_1 - a_1 - 1 = & 9+3+\ 9+19-3-3-1, & =33, \\
3\ 1 & \quad ,, \quad a_3 + a_1 + a_1 + a_3 - a_2 - a_0 - 2 = & 19+3+\ 3+19-9-0-2, & =33, \\
1\ 1 & \quad ,, \quad a_1 + a_1 + a_3 + a_3 - a_0 - a_2 - 2 = & 3+3+19+19-0-9-2, & =33, \\
2\ 2 & \quad ,, \quad a_2 + a_2 + a_2 + a_2 - a_0 - a_0 - 3 = & 9+9+\ 9+\ 9-0-0-3, & =33,
\end{array}$$

and the number of constants is in each case $=33$. This is easily verified: in the first case we have a quartic surface containing a conic, the plane of the conic is therefore a quadruple tangent plane; and the existence of such a plane is 1 condition. In the second case the surface contains a plane cubic; the plane of this cubic is a triple tangent plane, having the points of contact in a line; and this is 1 condition. In the third case the surface contains a line, which is 1 condition: hence in each of these cases the number of constants is $34-1$, $=33$. In the fourth case, where the surface contains a quadriquadric curve, we repeat in some measure the general reasoning: the quadriquadric curve contains 16 constants, and we have thus 16 as the number of constants really contained in the equations $P=0$, $Q=0$ of the quadriquadric curve: the equation $PS-QR=0$, contains in addition $9+10$, $=19$ constants, but writing it in the form $P(S+kQ)-Q(R+kP)=0$, we have a diminution $=1$, or the number apparently is $16+19-1$, $=34$. But the quadriquadric curve is one of a singly infinite series $P+lR=0$, $Q+lS=0$ of such curves, and we have on this account a diminution $=1$; the number of constants is thus $34-1$, $=33$ as above: the reasoning is, in fact, the same as for the case of a plane passing through a line; the line contains 4 constants, hence the plane, quà arbitrary plane through the line, would contain $1+4$, $=5$ constants; but the line being one of a doubly infinite system of lines on the plane the number is really $5-2$, $=3$, as it should be.

Cambridge, 2nd Sept., 1880.