939.

ON A CASE OF THE INVOLUTION $AF+BG+CH=0$, WHERE A, B, C, F, G, H ARE TERNARY QUADRICS.

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WE have here the six conics

$$A=0,\ B=0,\ C=0,\ F=0,\ G=0,\ H=0;$$

the curves $AF=0$ and $BG=0$ are quartics intersecting in 16 points, and if 8 of these lie in a conic $H=0$, then the remaining 8 will be in a conic $C=0$. I take the first set of eight points to be 1, 2, 3, 4, 5, 6, 7, 8; the quartics $AF=0$ and $BG=0$ each pass through these eight points; and I assume for the moment

$$A=1234,\ F=5678;\ B=1256,\ G=3478,$$

viz. that $A=0$ is a conic through the points 1, 2, 3, 4, and similarly for F, G, B. Here $H=0$ is a conic through the points 1, 2, 3, 4, 5, 6, 7, 8, or attending only to the last four points it is a conic through 5, 6, 7, 8; we have therefore a linear relation between F, G, H, and supposing the implicit constant factors to be properly determined, this may be taken to be $F+G+H=0$; the identity $AF+BG+CH=0$ thus becomes $F(A-C)+G(B-C)=0$. We have thus F a numerical multiple of $B-C$, and by a proper determination of the implicit factor we may make this relation to be $F=B-C$; the last equation then gives $G=C-A$, and from the equation $F+G+H=0$, we have $H=A-B$; the six functions thus are

$$\begin{array}{ll} A,\ B-C, \text{ or if we please,} & A-D,\ B-C, \\ B,\ C-A & B-D,\ C-A, \\ C,\ A-B & C-D,\ A-B, \end{array}$$

where D is an arbitrary quadric function. The solution

$$(A-D)(B-C)+(B-D)(C-A)+(C-D)(A-B)=0$$

of the involution is an obvious and trivial one.

But the case which I proceed to consider is

$$A=1234,\ F=5678;\ B=1256,\ G=3478;$$

here $AF=0$, and $BG=0$, meet as before in the points 1, 2, 3, 4, 5, 6, 7, 8, and in eight other points, say that

$A=0,\ B=0$	meet in	1, 2	and in two other points	$\alpha,\ \beta$,
$A=0,\ G=0$	„	3, 4	„ „	$\gamma,\ \delta$,
$F=0,\ B=0$	„	5, 6	„ „	$\epsilon,\ \zeta$,
$F=0,\ G=0$	„	7, 8	„ „	$\eta,\ \theta$;

then the 8 points $\alpha, \beta, \gamma, \delta, \epsilon, \zeta, \eta, \theta$ will lie in a conic $C=0$.

I take $y^2-zx=0$ for the conic $H=0$; for any point in this conic we have $x:y:z=1:\theta:\theta^2$, and we may take $\theta_1, \theta_2, \theta_3, \theta_4, \theta_5, \theta_6, \theta_7, \theta_8$ for the parameters of the points 1, 2, 3, 4, 5, 6, 7, 8 respectively.

Write $(a, b, c, f, g, h\between x, y, z)^2=0$ for the conic A, $=1234=0$; therefore we have

$$a+b\theta^2+c\theta^4+f\theta^3+g\theta^2+h\theta=\theta-\theta_1\,.\,\theta-\theta_2\,.\,\theta-\theta_3\,.\,\theta-\theta_4;$$

or, if

$$\begin{aligned}p_{1234}&=\theta_1+\theta_2+\theta_3+\theta_4,\\ q_{1234}&=\theta_1\theta_2+\theta_1\theta_3+\theta_1\theta_4+\theta_2\theta_3+\theta_2\theta_4+\theta_3\theta_4,\\ r_{1234}&=\theta_1\theta_2\theta_3+\theta_1\theta_2\theta_4+\theta_1\theta_3\theta_4+\theta_2\theta_3\theta_4,\\ s_{1234}&=\theta_1\theta_2\theta_3\theta_4,\end{aligned}$$

then

$$c=1,\ f=-p_{1234},\ b+g=q_{1234},\ h=-r_{1234},\ a=s_{1234};$$

or, writing $g=-\lambda$, we have

$$s_{1234}x^2+q_{1234}y^2+z^2-p_{1234}yz-r_{1234}xy+\lambda(y^2-zx)=0$$

for the equation of the conic in question. We may without loss of generality put $\lambda=0$; and then if, in general,

$$\Omega=sx^2+qy^2+z^2-pyz-rxy,$$

we have $A=\Omega_{1234}=0$ for the conic $A=0$. And thus the equations of the four conics are

$$A=\Omega_{1234}=0,\ F=\Omega_{5678}=0;\ B=\Omega_{1256}=0,\ C=\Omega_{3478}=0,$$

or, as for shortness I write them,

$$A=\Omega=0,\ F=\Omega'=0;\ B=\Omega''=0,\ C=\Omega''',$$

viz. in Ω the suffixes are 1, 2, 3, 4, in Ω' they are 5, 6, 7, 8, in Ω'' they are 1, 2, 5, 6, and in Ω''' they are 3, 4, 7, 8.

I find that the implicit constant factors of AF and BG are 1, -1, and consequently that the form of the identity is

$$\Omega\Omega'-\Omega''\Omega'''+(y^2-zx)\,C=0,$$

where C is a quadric function to be determined; or, what is the same thing, we have

$$\begin{aligned}&(sx^2+qy^2+z^2-pyz-rxy)(s'x^2+q'y^2+z^2-p'yz-r'xy),\\ &-(s''x^2+q''y^2+z^2-p''yz-r''xy)(s'''x^2+q'''y^2+z^2-p'''yz-r'''xy),\\ &+(y^2-zx)\,C=0.\end{aligned}$$

Writing for shortness

$$\begin{aligned}\theta_1+\theta_2&=\alpha\ , & \theta_1\theta_2&=\beta\ ,\\ \theta_3+\theta_4&=\alpha'\ , & \theta_3\theta_4&=\beta'\ ,\\ \theta_5+\theta_6&=\alpha''\ , & \theta_5\theta_6&=\beta''\ ,\\ \theta_7+\theta_8&=\alpha''', & \theta_7\theta_8&=\beta''',\end{aligned}$$

we have

$p=\alpha+\alpha'$	$p'=\alpha''+\alpha'''$	$p''=\alpha+\alpha''$	$p'''=\alpha'+\alpha'''$
$q=\alpha\alpha'+\beta+\beta'$	$q'=\alpha''\alpha'''+\beta''+\beta'''$	$q''=\alpha\alpha''+\beta+\beta''$	$q'''=\alpha'\alpha'''+\alpha'\beta'''+\alpha'''\beta'$
$r=\alpha\beta'+\alpha'\beta$	$r'=\alpha''\beta'''+\alpha'''\beta''$	$r''=\alpha\beta''+\alpha''\beta$	$r'''=\alpha'\beta'''+\alpha'''\beta'$
$s=\beta\beta'$	$s'=\beta''\beta'''$	$s''=\beta\beta''$	$s'''=\beta'\beta'''$.

In the last-mentioned equation, the first and second lines together are a quartic function of (x, y, z), say the value is

$$\begin{aligned}=\ &Ax^4+By^4+Cz^4,\\ &+Fy^3z+Gz^3x+Hx^3y,\\ &+Iyz^3+Jzx^3+Kxy^3,\\ &+Lx^2yz+Mxy^2z+Nxyz^2,\\ &+Py^2z^2+Qz^2x^2+Rx^2y^2,\end{aligned}$$

where after all reductions

$$\begin{aligned}
A&=\ ss'-s''s''' &&=0,\\
B&=\ qq'-q''q''' &&=(\alpha\beta'''-\alpha'''\beta)(\alpha'-\alpha'')\\
&&&\qquad+(\alpha'\beta''-\alpha''\beta')(\alpha-\alpha''')-(\beta'-\beta'')(\beta-\beta'''),\\
C&=\ 1-1 &&=0,\\
F&=-pq'-p'q+p''q'''+p'''q'' &&=(\alpha-\alpha''')(\beta'-\beta'')+(\alpha'-\alpha'')(\beta-\beta'''),\\
G&=\ 0-0 &&=0,\\
H&=-rs'-r's+r''s'''+r'''s'' &&=0,\\
I&=-p-p'+p''+p''' &&=0,\\
J&=\ 0-0 &&=0,\\
K&=-qr'-q'r+q''r'''+q'''r'' &&=(\alpha\beta'''-\alpha'''\beta)(\beta''-\beta')+(\alpha'\beta''-\alpha''\beta')(\beta'''-\beta),\\
L&=-ps'-p's+p''s'''+p'''s'' &&=(\alpha\beta'''-\alpha'''\beta)(\beta'-\beta'')+(\alpha'\beta''-\alpha''\beta')(\beta-\beta'''),\\
M&=\ pr'+p'r-p''r'''-p'''r'' &&=(\alpha\beta'''-\alpha'''\beta)(\alpha''-\alpha')+(\alpha'\beta''-\alpha''\beta')(\alpha'''-\alpha),\\
N&=-r-r'+r''+r''' &&=(\alpha-\alpha''')(\beta''-\beta')+(\alpha'-\alpha'')(\beta'''-\beta),\\
P&=\ pp'+q+q'-p''p'''-q''-q''' &&=0,\\
Q&=\ s+s'-s''-s''' &&=(\beta'-\beta'')(\beta-\beta'''),\\
R&=\ rr'+qs'+q's-r''r'''-q''s'''-q'''s''=0:
\end{aligned}$$

values which satisfy

$$\begin{aligned} F + N &= 0, \\ K + L &= 0, \\ B + M + Q &= 0. \end{aligned}$$

The quartic function is thus seen to be

$$= (y^2 - zx)(By^2 + Fyz - Qzx + Kxy) = 0,$$

viz. we have $By^2 + Fyz - Qzx + Kxy = 0$ for the equation of the conic $C = 0$.

Moreover, substituting for p, q, r, s, &c., their values, we have finally for the required involution

$$\begin{gathered}
[\beta\beta' x^2 + (\alpha\alpha' + \beta + \beta')\, y^2 + z^2 - (\alpha + \alpha')\, yz - (\alpha\beta' + \alpha'\beta)\, xy] \\
\times [\beta''\beta''' x^2 + (\alpha''\alpha''' + \beta'' + \beta''')\, y^2 + z^2 - (\alpha'' + \alpha''')\, yz - (\alpha''\beta''' + \alpha'''\beta'')\, xy] \\
- [\beta\beta'' x^2 + (\alpha\alpha'' + \beta + \beta'')\, y^2 + z^2 - (\alpha + \alpha'')\, yz - (\alpha\beta'' + \alpha''\beta)\, xy] \\
\times [\beta'\beta''' x^2 + (\alpha'\alpha''' + \beta' + \beta''')\, y^2 + z^2 - (\alpha' + \alpha''')\, yz - (\alpha'\beta''' + \alpha'''\beta')\, xy],
\end{gathered}$$

$$-(y^2 - zx) \times \left\{ \begin{aligned} & y^2 [(\alpha\beta''' - \alpha'''\beta)(\alpha' - \alpha'') + (\alpha'\beta'' - \alpha''\beta')(\alpha - \alpha''') - (\beta - \beta''')(\beta' - \beta'')] \\ & + yz [(\alpha - \alpha''')(\beta' - \beta'') + (\alpha' - \alpha'')(\beta - \beta''')] \\ & - zx [(\beta - \beta''')(\beta' - \beta'')] \\ & - xy [(\alpha\beta''' - \alpha'''\beta)(\beta' - \beta'') + (\alpha'\beta'' - \alpha''\beta')(\beta - \beta''')] \end{aligned} \right\} = 0.$$

It will be recollected that this is the solution for the case $A = 1234$, $F = 5678$; $B = 1256$, $G = 3478$: being that to which the present paper has reference.