# 937.

## NOTE ON THE ORTHOTOMIC CURVE OF A SYSTEM OF LINES IN A PLANE.

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CONSIDERING *in plano* a singly infinite system of lines, then to each point of the plane there corresponds a line (not in general a unique line), and we can therefore express in terms of the coordinates $(x, y)$ of the point the cosine-inclinations $\alpha$, $\beta$ of the line to the axes. The differential equation of the orthotomic curve is then $\alpha dx + \beta dy = 0$, and it is a well-known and easily demonstrable theorem that $\alpha dx + \beta dy$ is a complete differential, say it is $= dV$; the integral equation of the orthotomic curve is therefore $V = \int (\alpha dx + \beta dy)$, $=$ const., and we see further that $V$ is a solution of the partial differential equation $\left(\frac{dV}{dx}\right)^2 + \left(\frac{dV}{dy}\right)^2 = 1$.

If the lines are the normals of the ellipse $\frac{X^2}{a} + \frac{Y^2}{b} = 1$, then, writing the equation of the normal at the point $X$, $Y$ in the form

$$\frac{a}{X}(x - X) = \frac{b}{Y}(y - Y), \ = \lambda,$$

suppose, we have

$$X = \frac{ax}{a + \lambda}, \quad Y = \frac{by}{b + \lambda};$$

and therefore

$$\frac{ax^2}{(a + \lambda)^2} + \frac{by^2}{(b + \lambda)^2} - 1 = 0,$$

which last equation determines $\lambda$ as a function of $x$, $y$. We have $\alpha$, $\beta$ proportional to $\frac{X}{a}$, $\frac{Y}{b}$; or say we have

$$\alpha = M\frac{x}{a + \lambda}, \quad \beta = M\frac{y}{b + \lambda},$$

whence

$$\frac{1}{M^2} = \frac{x^2}{(a + \lambda)^2} + \frac{y^2}{(b + \lambda)^2};$$

or, writing for convenience

$$\frac{x^2}{(a+\lambda)^2}+\frac{y^2}{(b+\lambda)^2}-\frac{k^2}{\lambda^2}=0,$$

(viz. this equation defines $k$ as a function of $x$, $y$ and $\lambda$, that is, of $x$ and $y$), we have

$$\alpha=\frac{\lambda x}{k(a+\lambda)},\quad \beta=\frac{\lambda y}{k(b+\lambda)};$$

and we ought therefore to have

$$\frac{\lambda}{k}\left(\frac{x\,dx}{a+\lambda}+\frac{y\,dy}{b+\lambda}\right)$$

a complete differential, $=dV$.

This is easily verified, for from the assumed value

$$k=\lambda\left(\frac{x^2}{a+\lambda}+\frac{y^2}{b+\lambda}-1\right)$$

we deduce

$$dk=2\lambda\left(\frac{x\,dx}{a+\lambda}+\frac{y\,dy}{b+\lambda}\right)+d\lambda\left(\frac{ax^2}{(a+\lambda)^2}+\frac{by^2}{(b+\lambda)^2}-1\right),\ =2\lambda\left(\frac{x\,dx}{a+\lambda}+\frac{y\,dy}{b+\lambda}\right);$$

and we have therefore

$$dV=\frac{\lambda}{k}\frac{dk}{2\lambda},\ =\tfrac{1}{2}\frac{dk}{k},$$

where $k$ denotes a function of $(x, y)$ defined as above; hence the equation $V=\text{const.}$ gives $k=\text{const.}$, or the equation of the orthotomic curve is given by the system of equations

$$\frac{ax^2}{(a+\lambda)^2}+\frac{by^2}{(b+\lambda)^2}-1=0,$$

$$\frac{x^2}{(a+\lambda)^2}+\frac{y^2}{(b+\lambda)^2}-\frac{k^2}{\lambda^2}=0,$$

where $k$ is a constant; these equations (eliminating $\lambda$) give, in fact, the equation of the parallel curve of the ellipse, and $k$ denotes the normal distance of a point on the curve from the ellipse. I recall that the first equation may be replaced by

$$\frac{x^2}{a+\lambda}+\frac{y^2}{b+\lambda}-\frac{k}{\lambda}-1=0,$$

and since the derived equation hereof in regard to $\lambda$ is the second equation, we have the equation of the parallel curve in the known form

$$\text{Disct.}\ \{(\lambda+k)(\lambda+a)(\lambda+b)-(b+\lambda)x^2-(a+\lambda)y^2\}=0.$$

I notice further that, considering $k$ a function of $x$, $y$ as above, we have

$$\left(\frac{dV}{dx}\right)^2+\left(\frac{dV}{dy}\right)^2=\frac{1}{4k^2}\left\{\left(\frac{dk}{dx}\right)^2+\left(\frac{dk}{dy}\right)^2\right\},\ =\frac{\lambda^2}{k^2}\left\{\frac{x^2}{(a+\lambda)^2}+\frac{y^2}{(b+\lambda)^2}\right\},$$

that is,

$$\left(\frac{dV}{dx}\right)^2+\left(\frac{dV}{dy}\right)^2=1,$$

as it should be.