936.

NOTE ON UNIFORM CONVERGENCE.

[From the *Proceedings of the Royal Society of Edinburgh*, vol. XIX. (1893), pp. 203—207. Read December 5, 1892.]

It appears to me that the form in which the definition or condition of uniform convergence is usually stated, is (to say the least) not easily intelligible. I call to mind the general notion: We may have a series, to fix the ideas, say of positive terms

$$(0)_x + (1)_x + (2)_x, \dots + (n)_x, \dots$$

the successive terms whereof are continuous functions of x, for all values of x from some value less than a up to and inclusive of a (or from some value greater than a down to and inclusive of a): and the series may be convergent for all such values of x, the sum of the series ϕx is thus a determinate function ϕx of x; but ϕx is not of necessity a continuous function; if it be so, then the series is said to be uniformly convergent; if not, and there is for the value $x = a$ a breach of continuity in the function ϕx, then there is for this value $x = a$ a breach of uniform convergence in the series.

Thus if the limits are say from 0 up to the critical value 1, then in the geometrical series $1 + x + x^2 + \dots$, the successive terms are each of them continuous up to and inclusive of the limit 1, but the series is only convergent up to and exclusive of this limit, viz. for $x = 1$ we have the divergent series $1 + 1 + 1 + \dots$, and this is *not* an instance; but taking, instead, the geometrical series

$$(1 - x) + (1 - x)\,x + (1 - x)\,x^2 + \dots,$$

here the terms are each of them continuous up to and inclusive of the limit 1, and the series is also convergent up to and inclusive of this limit; in fact, at the limit the series is $0+0+0+\ldots$ We have here an instance, and there is in fact a discontinuity in the sum, viz. $x<1$ the sum is

$$(1-x)(1+x+x^2+\ldots),\ =(1-x)\cdot\frac{1}{1-x},\ =1;$$

whereas for the limiting value 1, the sum is $0+0+0+\ldots,\ =0$. The series is thus uniformly convergent up to and exclusive of the value $x=1$, but for this value there is a breach of uniform convergence.

I remark that Du Bois-Reymond in his paper, "Notiz über einen Cauchy'schen Satz, die Stetigkeit von Summen unendlicher Reihen betreffend," *Math. Ann.*, t. IV. (1871), pp. 135—137, shows that, when certain conditions are satisfied, the sum ϕx is a continuous function of x, but he does not use the term "uniform convergence," nor give any actual definition thereof.

M. Jordan, in his "Cours d'Analyse de l'École Polytechnique," t. I. (Paris, 1882), considers p. 116 the series $s=u_1+u_2+u_3+\ldots$, the terms of which are functions of a variable z, and after remarking that such a series is convergent for the values of z included within a certain interval, if for each of these values and for every value of the infinitely small quantity ϵ we can assign a value of n such that for every value of p,

$$\text{Mod}\,(u_{n+1}+u_{n+2}+\ldots+u_{n+p})<\text{Mod}\,\epsilon,$$

ϵ being as small as we please, proceeds:—

"Le nombre des termes qu'il est nécessaire de prendre dans la série pour arriver à ce résultat sera en général une fonction de z et de ϵ. Néanmoins on pourra très habituellement déterminer un nombre n fonction de ϵ seulement telle que la condition soit satisfaite pour toute valeur de z comprise dans l'intervalle considéré. On dira dans ce cas que la série s est *uniformément convergente* dans cet intervalle."

And similarly, Professor Chrystal in his *Algebra*, Part II. (Edinburgh, 1889), after considering, p. 130, the series

$$\frac{x}{x+1\,.\,2x+1}+\frac{x}{2x+1\,.\,3x+1}+\ldots+\frac{x}{(n-1)\,x+1\,.\,nx+1}+\ldots$$

for which the critical value is $x=0$, and in which when $x=0$ the residue R_n of the series or sum of the $(n+1)$th and following terms is $=\frac{1}{nx+1}$ proceeds as follows:— Now when x has any given value, we can by making n large enough make $\frac{1}{nx+1}$ smaller than any given positive quantity α. But on the other hand, the smaller x is the larger must we take n in order that $\frac{1}{nx+1}$ may fall under α; and in general when x is variable there is no finite upper limit for n *independent of* x, say v, such that if $n>v$ then $R_n<\alpha$. When the residue has this peculiarity the series is said to be *non-uniformly convergent;* and if for a particular value of x, such as $x=0$ in the present example, the number of terms required to secure a given degree of approximation to the limit is infinite, the series is said to *converge infinitely slowly.*

And he thereupon gives the formal definition: *If for values of x within a given region in Argand's diagram we can for every value of α, however small Mod. α, assign for n an upper limit v* INDEPENDENT OF x, such that, when $n > v$, Mod. $R_n <$ Mod. α, then the series $\Sigma f(n, x)$ is said to be UNIFORMLY CONVERGENT within the region in question.

The two forms of definition (Jordan and Chrystal) appear to me equivalent, and it seems to me that construing the definition *strictly*, and applying it to the above instance

$$(1-x)+(1-x)x+(1-x)x^2+\ldots,$$

the definition does not in either case indicate a breach of uniform convergency at $x=1$, viz. the definition shows uniform convergency from $x=0$ to $x=1-\epsilon$, ϵ being a positive quantity however small, or (as I have before expressed this) uniform convergency up to and exclusive of the limit 1; and further, it shows uniform convergency at the limit 1. For at this limit, the series of terms is $0+0+0+\ldots$, the residue or sum of the $(n+1)$th and subsequent terms is thus also $0+0+0+\ldots$, and we get the value of this residue, not approximately, but exactly, by taking a single term of the series. Jordan and Chrystal calculate, each of them, the residue from the general expression thereof by writing therein for x or z the critical value; and then, comparing the value thus obtained with the values obtained for the $(n+1)$th and subsequent terms of the series on substituting therein for x or z the critical value, they seem to argue that the discrepancy between these two values indicates the breach of uniform convergency.

It may be said that the objection is a verbal one. But it seems to me that the whole notion of the residue (although very important as regards the general theory of convergence) is irrelevant to the present question of uniform convergency, and that a better method of treating the question is as follows:

Considering as before the series

$$(0)_x+(1)_x+(2)_x+\ldots+(n)_x+\ldots,$$

where the functions $(0)_x$, $(1)_x$, $(2)_x, \ldots$ are each of them continuous up to and inclusive of the limit $x=a$, and the series has thus a definite sum ϕx, this sum is *primâ facie* a continuous function of x, and what we have to explain is the manner in which it may come to be discontinuous. Suppose that it is continuous up to and exclusive of the limit $x=a$, but that there is a breach of discontinuity at this limit: write $x=a-\epsilon$, where ϵ is a positive quantity as small as we please, and consider the two equations

$$\phi x=(0)_x+(1)_x+(2)_x+\ldots,$$

$$\phi a=(0)_a+(1)_a+(2)_a+\ldots,$$

then we have

$$\phi a-\phi x=\epsilon\left\{\frac{(0)_a-(0)_x}{a-x}+\frac{(1)_a-(1)_x}{a-x}+\ldots\right\}.$$

Hence if the sum of the series in { } is a finite magnitude M, not indefinitely large for an indefinitely small value of ϵ, we have $\phi a - \phi x = \epsilon M$, which is indefinitely small for ϵ indefinitely small, and there is no breach of continuity; the only way in which a breach of continuity can arise is by the series in { } having a value indefinitely large for ϵ indefinitely small, viz. if such a value is $\frac{N}{\epsilon}$, then $\phi a - \phi x = \epsilon \,.\, \frac{N}{\epsilon}$, $= N$, and as x changes from $a - \epsilon$ to a, the sum changes abruptly from $\phi(a-\epsilon)$ to $\phi(a-\epsilon) + N$.

The condition for a breach of uniform convergency for the value $x = a$, thus is that, writing $x = a - \epsilon$, ϵ a positive magnitude however small, the series

$$\frac{(0)_a - (0)_x}{a - x} + \frac{(1)_a - (1)_x}{(a - x)} + \ldots,$$

shall have a sum indefinitely large for ϵ indefinitely small, or say as before, a sum $= \frac{N}{\epsilon}$.

For the foregoing example, where the series is

$$(1 - x) + x(1 - x) + x^2(1 - x) + \ldots$$

the critical value is $a = 1$: we have here $(n)_x = x^n(1 - x)$, and consequently

$$\frac{(n)_a - (n)_x}{a - x} = \frac{a^n - x^n}{a - x} - \frac{a^{n+1} - x^{n+1}}{(a - x)}$$

$$= (a^{n-1} + a^{n-2}x + \ldots + x^{n-1}) - (a^n + a^{n-1}x + \ldots + x^n)$$

$$= -x^n \text{ for } a = 1.$$

The series

$$\frac{(0)_a - (0)_x}{a - x} + \frac{(1)_a - (1)_x}{a - x} + \ldots$$

thus is

$$-(1 + x + x^2 + \ldots)$$

$$= -\frac{1}{1 - x}, \; = \frac{-1}{\epsilon}$$

for $x = 1 - \epsilon$, viz. we have

$$\phi 1 - \phi(1 - \epsilon) = \epsilon \,.\, \frac{-1}{\epsilon}, \; = -1;$$

which is right since by what precedes

$$\phi(1 - \epsilon) = 1, \;\; \phi 1 = 0.$$