932.

ON SYMMETRIC FUNCTIONS AND SEMINVARIANTS.

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THE principal object of the present memoir is to develop further the theory of seminvariants, but in connexion therewith I was led to some investigations on symmetric functions, and I have consequently included this subject in the title. The two theories, if we adopt the MacMahon form of equation,

$$0 = 1 + bx + \frac{c}{2}x^2 + \frac{d}{6}x^3 + \dots,$$

may be regarded as identical; but there are still two branches of the theory, viz. we may seek to obtain for the symmetric functions of the roots expressions in terms of the coefficients (which expressions, in the case of non-unitary symmetric functions, are in fact seminvariants), or we may attend to the properties of the functions of the coefficients thus obtained and which we call seminvariants. But I do not in the first instance use the MacMahon form, but retain the ordinary form of equation $0 = 1 + bx + cx^2 + dx^3 + \dots$, and we have thus only a parallelism of the two theories, and in place of seminvariants we have functions which I call non-unitariants. In regard as well to these as to unitariant functions, I consider certain operators Θ_σ, Δ, $P - \delta b$, and $Q - 2\omega b$, which under altered forms present themselves also in the theory of seminvariants.

As regards seminvariants, I consider what I call the blunt and sharp forms respectively: the great problem is, it appears to me, that of sharp seminvariants, otherwise the *I*-and-*F* problem—viz. for any given weight we have to determine the correspondence between the initial and final terms in such wise as to obtain a system of sharp seminvariants. I obtain a "square diagram" solution, which is so far theoretically complete that for any given weight I can, without any tentative operation, determine by a laborious process the correspondence in question: but I am not thereby enabled to establish or enunciate for successive weights any general rule of correspondence; and my process is in fact, as regards practicability, far inferior to that which I call the MacMahon linkage, but of the validity of this I have not succeeded in obtaining any satisfactory proof.

I establish an umbral theory of seminvariants which will be presently again referred to, and I consider the question of the reduction of seminvariants. The final term of a seminvariant may be composite (that is, the product of two or more final terms), and that in one way only or in two or more ways, or it may be non-composite. In the case of a composite final term the seminvariant is reducible, but the converse theorem that a seminvariant with a non-composite final term is irreducible is in nowise true; the reason of this is explained. An irreducible seminvariant is a perpetuant. In regard to perpetuants, I reproduce and simplify a demonstration recently obtained by Dr Stroh as to the perpetuants for any given degree whatever: viz. the generating function for perpetuants of degree n is

$$= x^{2^{n-1}-1} \div 1 - x^2 \,.\, 1 - x^3 \ldots 1 - x^n;$$

the theorem was previously known, and more or less completely proved, for the values $n = 4$, 5, 6, and 7. Dr Stroh's investigation is conducted by an umbral representation,

$$(\alpha x + \beta y + \gamma z + \ldots)^n, \quad x + y + z + \ldots = 0,$$

of the blunt seminvariants of a given weight.

I consider in regard to seminvariants the theory of the symbols $P - \delta b$ and $Q - 2\omega b$, and the derived symbols Y and Z, each of which operating on a seminvariant gives a seminvariant. These are, in fact, connected with the derivatives (f, F) of a quantic f and any covariant thereof F; but except to point out this connexion, I do not in the present memoir consider the theory of covariants.

The Coefficients $(a, b, c, d, e, \ldots)$ *or* $(1, b, c, d, e, \ldots)$. Art. Nos. 1 to 9.

1. I consider the series $(a, b, c, d, e, \ldots)$, or putting as we most frequently do $a = 1$, say the series $(1, b, c, d, e, \ldots)$ of coefficients, the several terms whereof are taken to be of the weights 0, 1, 2, 3, 4, ... respectively. We form with these sets of isobaric terms, or say columns of the weights 0, 1, 2, 3, 4, ... respectively, for instance,

0	1	2	3	4	5	6
1	b	c	d	e	f	g
		b^2	bc	bd	be	bf
			b^3	c^2	cd	ce
				b^2c	b^2d	d^2
				b^4	bc^2	b^2e
					b^3c	bcd
					b^5	c^3
						b^3d
						b^2c^2
						b^4c
						b^6

and generally a set or column of any given weight. In each term, the letters are written in alphabetical order.

Taking the whole or any part of a column, for instance the whole column (d, bc, b^3), or the part (e, bd, c^2) of the next column, we may by supplying powers of a in such wise as to leave unaltered the terms of the highest degree, that is, by reading these as (a^2d, abc, b^3) and (ae, bd, c^2) respectively, regard them as homogeneous sets of a given degree in $(a, b, c, d, e, \ldots)$; and thus generally we may speak of the degree of a set of terms.

The terms of the several columns as above written down are in alphabetical order, AO; viz. we supply as above the proper powers of a, reading for instance col. 4 as a^3e, a^2bd, a^2c^2, ab^2c, b^4, where the terms are in alphabetical or dictionary order.

Each column is derived from the preceding one by Arbogast's rule, it being understood, for instance, that b^4, that is, ab^4, gives the two terms ab^3c and b^5, that is, b^3c and b^5; and so in other cases.

2. We attend in particular to the non-unitary terms, or non-unitaries, e.g. in col. 5, f, cd, which contain no b; and to the power-ending terms or power-enders, bc^2, b^5, which end in a power. It will be observed that, whenever by Arbogast's rule a term in one column gives two terms in the next column, the second of these is a power-ender; and thus in any column the excess of the number of terms above that in the preceding column is equal to the number of power-enders.

3. I consider the notion of conjugate terms: representing, for instance, the terms

$$f \qquad be \qquad cd$$

by dots in the form

$$\cdot\cdot\cdot\cdot\cdot \qquad :\cdot\cdot\cdot \qquad ::\cdot$$

and reading the number of dots in columns instead of in lines we derive the conjugate terms

$$b^5 \qquad b^3c \qquad bc^2,$$

and so in other cases. It is clear that the relation is a reciprocal one (thus the conjugates of b^5, b^3c, bc^2 are f, be, cd respectively). Moreover, a term may be its own conjugate; thus cd^2, arranging the dots in lines and reading them in columns, $\vdots\ \vdots\ :$ is again cd^2.

It is at once seen that non-unitaries and power-enders are conjugate to each other; hence in any column, the non-unitaries and the power-enders are equal in number, and a preceding result may be stated in the more complete form: in any column the excess of the number of terms above that in the preceding column is equal to the number of non-unitaries or to the number of power-enders.

4. The terms of the several columns may be arranged in counter-order CO, thus:

0	1	2	3	4	5	6
1	b	c	d	e	f	g
		b^2	bc	bd	be	bf
			b^3	c^2	cd	ce
				b^2c	b^2d	b^2e
				b^4	bc^2	d^2
					b^3c	bcd
					b^5	b^3d
						c^3
						b^2c^2
						b^4c
						b^6

viz. we arrange here according to the highest letters. The counter-order is, in fact, the alphabetical order with the reversed arrangement (..., g, f, e, d, c, b, a) of the alphabet, but in the separate terms we retain the alphabetical order, thus writing as before bf and not fb. Observe that the difference between the two arrangements, AO and CO, first presents itself in the col. 6.

In this CO arrangement, each column is derived from the next preceding one by a rule as follows: We operate on the lowest letter of each term, being a simple letter, not a power, by changing it into the next highest letter, and we further operate upon each term by multiplying it by b, the operation or (as the case may be) two operations upon any term being performed before operating upon the next term.

5. If we compare a column in AO with the same column in CO, for instance

AO	CO	AO	CO rev.
g	g	g	b^6
bf	bf	bf	b^4c
ce	ce	ce	b^2c^2
d^2	b^2e	d^2	c^3
b^2e	d^2	b^2e	b^3d
bcd	bcd	bcd	bcd
c^3	b^3d	c^3	d^2
b^3d	c^3	b^3d	b^2e
b^2c^2	b^2c^2	b^2c^2	ce
b^4c	b^4c	b^4c	bf
b^6	b^6	b^6	g

it will be seen that the terms are conjugates of each other, the first and last, the second and last but one terms, and so on; or, what is the same thing, if we reverse the order of either column, then the pairs of conjugate terms will appear each in the same line; of course, here a self-conjugate term such as *bcd* is put in evidence.

6. By writing $a, b, c, d, \ldots = a_0, a_1, a_2, a_3, \ldots$, or more simply 0, 1, 2, 3, ...; we connect the theory with that of the partition of numbers: in particular, the terms of a given weight correspond to the partitions of that weight, or number of ways in which that weight can be made up with the parts 1, 2, 3, ... It may be remarked that, in a partition, the parts are usually written in decreasing order, whereas (as remarked above) in a literal term the letters are written in alphabetical order. Thus we have 321 and *bcd*; it would be more correct to write the partition as 123.

It is frequently convenient, retaining the letters $b, c, d, \ldots$, to write for instance $q = a_\sigma$ (σ a numerical suffix), meaning thereby that q is the letter corresponding to the place σ in the series 1, 2, 3, ... If instead of the indefinite series $(1, b, c, d, \ldots)$ we consider, as is sometimes convenient, a definite series of terms $(1, b, c, \ldots, q = a_\sigma)$, then σ is said to be the "extent" of the system. The next preceding letter p will naturally be $= a_{\sigma-1}$; and if, increasing the extent by unity, we introduce a new letter r, this will be $a_{\sigma+1}$, and so in other cases, the notation being for the most part used merely as a convenient way of showing the place of a letter in the series.

7. Considering the terms of a given weight, or say a column, in AO or CO, we may represent any portion of the column by means of its initial and final terms, say I and F, by the notations $I\text{ao}F$ and $I\text{co}F$ respectively. But a much more important notation is $I\text{ca}F$; viz. this represents the series of terms of given weight which are in CO not superior to I, and in AO not inferior to F (a like notation, which however I do not employ, would be $I\text{ac}F$; viz. this would denote the series of terms which are in AO not superior to I and in CO not inferior to F). The definition of $I\text{ca}F$ has been given in the above general form, but we are in fact exclusively or chiefly concerned with the case where I is a non-unitary and F a power-ender. It is to be observed that, considering the AO column as given, then to form from it the set or interval $I\text{ca}F$ we may disregard altogether the terms which are in the AO column inferior (posterior) to F, for by the definition none of these enter into $I\text{ca}F$, but it may very well be that there are in $I\text{ca}F$ terms which are in the AO column superior (anterior) to I. An instance of this first presents itself for the weight 11; viz. here a portion of the AO column is $(fg, b^2j, bci, bdh, beg, bf^2, c^2h, cdg, \ldots)$: hence in $I\text{ca}F$, if the initial term be c^2h, for instance in $c^2h\text{ca}b^3e^2$, we have terms fg, beg, bf^2 which are in AO anterior to the initial term c^2h. In order therefore to form $I\text{ca}F$ from the AO column, we must first take the terms (if any) which being in CO posterior to I are in the AO column anterior to I, and then from the portion $I\text{ao}F$ of the AO column reject the terms (if any) which are in CO anterior to I. In particular, starting from the AO column, and

arranging the non-unitaries thereof in CO and the power-enders in AO, for instance, weight 12, these are

$$\begin{array}{ll} m & g^2 \\ ck & cf^2 \\ dj & e^3 \\ ei & b^2f^2 \\ c^2i & bde^2 \\ fh & c^2e^2 \\ \vdots & \vdots \end{array}$$

There is no difficulty in writing down the terms of the several sets or intervals

$$mcag^2,\ mcacf^2,\ mcae^3,\ \ldots,\ ckcag^2,\ ckcacf^2,\ \ldots.$$

Instead of ca we may, if we please, use, and in fact I generally use the conventional symbol ∞, or write $m\infty g^2$, $m\infty cf^2$, &c. In any such set, the terms need not be arranged in AO; if for any purpose it is more convenient, they may be arranged in CO; but of course the definition of the meaning must not be departed from. The expressed initial is the highest term in CO, and the expressed final the lowest term in AO.

8. I diminish a term by replacing successively each letter thereof by the next inferior letter; for instance, if the term is cdf, then the diminished terms are $Dcdf$, $=bdf$, c^2f, cde, and so Db^2df, $=bdf$, b^2cf, b^2de (where the diminished b is a, that is, 1). Conversely, we may augment a term by replacing successively each letter thereof by the next superior letter; for instance, $Abdf$, $=b^2df$, cdf, bef, bdg, where the first augmentation b^2df is obtained from the a (which may be regarded as latent in the term operated upon). Operating upon the letters in order beginning with the lowest, the several diminutions may be called D_1, D_2, D_3, ..., and the several augmentations A_0, A_1, A_2, ... (where A_0 is in fact multiplication by b). We diminish a set by diminishing successively the several terms thereof (the diminished terms being taken without repetition; that is, each such term once only). Similarly, we may augment a set by augmenting successively the several terms thereof (the augmented terms being taken without repetition). It is to be noticed that the two operations are not reciprocal to each other; if we diminish a set, and then augment the diminished set, we obtain indeed all the terms of the original set, but in general we obtain also terms which are not included in the original set.

9. It requires some consideration to see that we have $D(I\infty F)=(D_1I\infty D_\theta F)$, where $D_\theta F$ is the diminution performed upon the highest letter of F. Take any term M of $D(I\infty F)$, the several diminutions D_1M, D_2M, ..., $D_\phi M$ are arranged in descending order: D_1M the highest and $D_\phi M$ the lowest, as well in CO as in AO. If then D_1M is in CO not superior to D_1I, then all the DM's will be in CO not superior to D_1I; and similarly, if $D_\phi M$ is in AO not inferior to $D_\theta F$, then all the DM's will be in AO not inferior to $D_\theta F$. And this being seen, then if we take

N a term of $(D_1 I \propto D_\theta F)$, and consider the successive augmentations $A_0 N, A_1 N, \ldots, A_\phi N$ of N, then these will be in ascending order $A_0 N$ the lowest and $A_\phi N$ the highest in CO as well as in AO. It may happen that $A_\phi N$ or this and neighbouring terms are in CO higher than I, and that $A_0 N$ or this and neighbouring terms are in AO lower than F, but there will always be a term or terms which is or are in CO lower than I and in AO higher than F; and thus not only every term of $D(I \propto F)$ will be a term of $(D_1 I \propto D_\theta F)$, but conversely every term of $(D_1 I \propto D_\theta F)$ will be a term of $D(I \propto F)$, and we thus have the required relation $D(I \propto F) = (D_1 I \propto D_\theta F)$.

Symmetric Functions of the Roots. Art. Nos. 10 to 31.

10. We consider a set of roots $\alpha, \beta, \gamma, \delta, \epsilon, \ldots$ either indefinite in number, or else definite, for instance $\alpha, \beta, \gamma, \delta$. The symmetric functions (rational and integral functions) are in the first instance denoted in the usual manner

$$S\alpha = \alpha + \beta + \gamma + \delta + \ldots, \quad S\alpha\beta = \alpha\beta + \alpha\gamma + \beta\gamma + \ldots,$$
$$S\alpha^2\beta = \alpha^2\beta + \alpha\beta^2 + \alpha^2\gamma + \alpha\gamma^2 + \beta^2\gamma + \beta\gamma^2 + \ldots,$$

viz. the S refers to all the distinct combinations of like form with the combination (α, $\alpha\beta$, or $\alpha^2\beta$, as the case may be) to which it is prefixed. By omitting the S and instead of the roots considering merely their indices, these same symmetric functions would be 1, 11 (or 1^2), 21, &c., and then if instead of the numbers 1, 2, 3, &c., we introduce the symbolic capital letters $B, C, D, \ldots$, the same symmetric functions will be represented as B, B^2, BC, &c. (21, that is, 12 is written as BC, and so in other cases, the letters in alphabetical order). The letters $B, C, D, \ldots$ are considered as being of the weights 1, 2, 3, ... respectively, and thus the symmetric functions of a given degree in the roots are represented by the terms of that weight in the symbolic letters $B, C, D, \ldots$, thus the symmetric functions of the degree 4 are E, BD, C^2, B^2C, B^4; of course these terms may be arranged in AO or in CO as may be most convenient for the purpose in hand. The capital letters $B, C, D, \ldots$ are in fact umbræ, but to avoid confusion with subsequent notations I do not in general thus speak of them. A form such as $S\alpha^2$ or $S\alpha^4\beta^2$, in which there is no index 1, is said to be non-unitary; but a form $S\alpha$ or $S\alpha^2\beta$, in which there is an index $=1$ or two or more indices each $=1$, is said to be unitary: or, what is the same thing, in the symbolic representation by capital letters, the form is non-unitary or unitary according as it does not or does contain the letter B.

11. In the ordinary theory of symmetric functions, we connect the coefficients $(1, b, c, d, \ldots)$ with the roots $(\alpha, \beta, \gamma, \ldots)$ by the equation

$$1 + bx + cx^2 + dx^3 + \ldots = 1 - \alpha x \,.\, 1 - \beta x \,.\, 1 - \gamma x \ldots,$$

and we thus have

$$\begin{aligned}
-b &= \alpha + \beta + \gamma + \ldots, &&= S\alpha, &&= 1, &&= B,\\
+c &= \alpha\beta + \alpha\gamma + \beta\gamma + \ldots, &&= S\alpha\beta, &&= 1^2, &&= B^2,\\
-d &= \alpha\beta\gamma + \ldots &&= S\alpha\beta\gamma, &&= 1^3, &&= B^3,
\end{aligned}$$

&c., &c.;

and it is to be remarked that, for any given number of roots, there will be this same number of coefficients: we may for instance have

$$1+bx+cx^2+dx^3=1-\alpha x\,.\,1-\beta x\,.\,1-\gamma x,$$

that is,

$$\begin{aligned} -b &= \alpha \;\;+\beta \;\;+\gamma, \\ +c &= \alpha\beta+\alpha\gamma+\beta\gamma, \\ -d &= \alpha\beta\gamma\,; \end{aligned}$$

and similarly if the number of roots be $=4$, or any larger number.

12. The symmetric functions of a given degree, say 4, in the roots, viz.

$$S\alpha^4,\quad S\alpha^3\beta,\quad S\alpha^2\beta^2,\quad S\alpha^2\beta\gamma,\quad S\alpha\beta\gamma\delta,$$

or

$$4,\quad 31,\quad 2^2,\quad 21^2,\quad 1^4,$$

or

$$E,\quad BD,\quad C^2,\quad B^2C,\quad B^4,$$

are equal in number to the combinations of the weight 4 in the coefficients, viz.

$$e,\quad bd,\quad c^2,\quad b^2c,\quad b^4\,;$$

and the terms of the one set are in fact linear combinations (with mere numerical multipliers) of the terms of the other set; but more than this, we have for instance

$e=\alpha\beta\gamma\delta+\ldots$, that is, $e=B^4$:

$bd=(\alpha+\beta+\gamma+\delta\ldots)(\alpha\beta\gamma+\alpha\beta\delta+\alpha\gamma\delta+\beta\gamma\delta\ldots)$ contains only terms $\alpha^2\beta\gamma$ and $\alpha\beta\gamma\delta$, that is, bd is a linear function of B^2C and B^4:

$c^2=(\alpha\beta+\alpha\gamma+\alpha\delta+\beta\gamma+\beta\delta+\gamma\delta\ldots)^2$ contains only terms $\alpha^2\beta^2$, $\alpha^2\beta\gamma$ and $\alpha\beta\gamma\delta$, that is, c^2 is a linear function of C^2, B^2C and B^4; and so on.

13. We have in fact the Table IV (a) which I quote from my paper "A Memoir on the Symmetric Functions of the Roots of an Equation," *Phil. Trans.* t. 147 (1857), pp. 489—496, [147],

‖	e	bd	c^2	b^2c	b^4
$S\alpha^4 = 4 = E$					$+1$
$S\alpha^3\beta = 31 = BD$				$+1$	$+4$
$S\alpha^2\beta^2 = 2^2 = C^2$			$+1$	$+2$	$+6$
$S\alpha^2\beta\gamma = 21^2 = B^2C$		$+1$	$+2$	$+5$	$+12$
$S\alpha\beta\gamma\delta = 1^4 = B^4$	$+1$	$+4$	$+6$	$+12$	$+24$

inserting on the left-hand outside margin the new symbols E, BD, &c., with their explanations: the || indicates that the table is to be read according to the columns, $e = +1B^4$, $bd = +1B^2C + 4B^4$, &c. This table gives conversely a Table IV (b), read according to the lines and serving to express the symmetric functions E, BD, &c., as linear functions of the combinations e, bd, c^2, b^2c, b^4 of the coefficients.

14. The (a) and (b) tables are given in the Memoir up to X (a) and X (b): it is proper to quote here the (b) tables up to VI (b) with only the change of substituting on the outside left-hand margins the literal terms such as E, BD, &c., instead of the symbols 4, 31, &c., originally used to denote these symmetric functions—it is to be observed that the left-hand symbols are in AO, the upper symbols in CO, this distinction first manifesting itself in the Table VI (b), so that it was necessary to go as far as this in order to put in evidence the true form of the tables.

II (b).

=	c	b^2
C	-2	$+1$
B^2	$+1$	

III (b).

=	d	bc	b^3
D	-3	$+3$	-1
BC	$+3$	-1	
B^3	-1		

IV (b).

=	e	bd	c^2	b^2c	b^4
E	-4	$+4$	$+2$	-4	$+1$
BD	$+4$	-1	-2	$+1$	
C^2	$+2$	-2	$+1$		
B^2C	-4	$+1$			
B^4	$+1$				

V (b).

=	f	be	cd	b^2d	bc^2	b^3c	b^5
F	-5	$+5$	$+5$	-5	-5	$+5$	-1
BE	$+5$	-1	-5	$+1$	$+3$	-1	
CD	$+5$	-5	$+1$	$+2$	-1		
B^2D	-5	$+1$	$+2$	-1			
BC^2	-5	$+3$	-1				
B^3C	$+5$	-1					
B^5	-1						

VI (b).

=	g	bf	ce	b^2e	d^2	bcd	b^3d	c^3	b^2c^2	b^4c	b^6
G	-6	$+6$	$+6$	-6	$+3$	-12	$+6$	-2	$+9$	-6	$+1$
BF	$+6$	-1	-6	$+1$	-3	$+7$	-1	$+2$	-4	$+1$	
CE	$+6$	-6	$+2$	$+2$	-3	$+4$	-2	-2	$+1$		
D^2	$+3$	-3	-3	$+3$	$+3$	-3	0	$+1$			
B^2E	-6	$+1$	$+2$	-1	$+3$	-3	$+1$				
BCD	-12	$+7$	$+4$	-3	-3	$+1$					
C^3	-2	$+2$	-2	0	$+1$						
B^3D	$+6$	-1	-2	$+1$							
B^2C^2	$+9$	-4	$+1$								
B^4C	-6	$+1$									
B^6	$+1$										

It is hardly necessary to remark in relation to these tables that if there are only two roots, then $d=0$, &c., viz. Table II. is not affected but all the subsequent tables assume a simplified form; if there are only three roots, then $e=0$, &c., viz. Tables II. and III. are not affected but all the subsequent tables assume a simplified form; and so on.

15. We have between the differential symbols ∂_b, ∂_c, ∂_d, ... and ∂_α, ∂_β, ∂_γ, ... certain relations which it is interesting to develop: it will be convenient to consider successively the cases, three roots, four roots, &c.

In the case of three roots, starting from

$$-b=\alpha+\beta+\gamma,$$

$$c=\alpha\beta+\alpha\gamma+\beta\gamma,$$

$$-d=\alpha\beta\gamma,$$

we have

$$\partial_\alpha=-\partial_b+(\beta+\gamma)\,\partial_c-\beta\gamma\partial_d,$$

$$\partial_\beta=-\partial_b+(\gamma+\alpha)\,\partial_c-\gamma\alpha\partial_d,$$

$$\partial_\gamma=-\partial_b+(\alpha+\beta)\,\partial_c-\alpha\beta\partial_d,$$

equations which give conversely ∂_b, ∂_c, ∂_d as linear functions of ∂_α, ∂_β, ∂_γ: I write down the three equations thus obtained together with a fourth equation which I will explain. The four equations are

$$-\partial_a+\delta'=\frac{\alpha^3}{\alpha-\beta\,.\,\alpha-\gamma}\partial_\alpha+\frac{\beta^3}{\beta-\gamma\,.\,\beta-\alpha}\partial_\beta+\frac{\gamma^3}{\gamma-\alpha\,.\,\gamma-\beta}\partial_\gamma,$$

$$-\partial_b\quad=\frac{\alpha^2}{\alpha-\beta\,.\,\alpha-\gamma}\partial_\alpha+\frac{\beta^2}{\beta-\gamma\,.\,\beta-\alpha}\partial_\beta+\frac{\gamma^2}{\gamma-\alpha\,.\,\gamma-\beta}\partial_\gamma,$$

$$-\partial_c\quad=\frac{\alpha}{\alpha-\beta\,.\,\alpha-\gamma}\partial_\alpha+\frac{\beta}{\beta-\gamma\,.\,\beta-\alpha}\partial_\beta+\frac{\gamma}{\gamma-\alpha\,.\,\gamma-\beta}\partial_\gamma,$$

$$-\partial_d\quad=\frac{1}{\alpha-\beta\,.\,\alpha-\gamma}\partial_\alpha+\frac{1}{\beta-\gamma\,.\,\beta-\alpha}\partial_\beta+\frac{1}{\gamma-\alpha\,.\,\gamma-\beta}\partial_\gamma.$$

In verification of the last three equations, observe that they give

$$-\partial_b+(\beta+\gamma)\,\partial_c-\beta\gamma\partial_d$$

$$=\frac{\alpha^2-\alpha\,(\beta+\gamma)+\beta\gamma}{\alpha-\beta\,.\,\alpha-\gamma}\partial_\alpha+\frac{\beta^2-\beta\,(\beta+\gamma)+\beta\gamma}{\beta-\gamma\,.\,\beta-\alpha}\partial_\beta+\frac{\gamma^2-\gamma\,(\beta+\gamma)+\beta\gamma}{\gamma-\alpha\,.\,\gamma-\beta}\partial_\gamma,$$

that is, $-\partial_b+(\beta+\gamma)\,\partial_c-\beta\gamma\partial_d=\partial_\alpha$: and similarly from the same three equations we deduce the values of ∂_β and ∂_γ; the three equations are thus equivalent to the foregoing three equations for ∂_α, ∂_β, ∂_γ.

As to the first equation, to avoid confusion with a root δ, I have written therein δ' (afterwards replaced by δ) to denote the degree of a function homogeneous in (a, b, c, d), upon which the symbols are supposed to operate; this is also the degree in the roots α, β, γ. The four equations give

$$-a\,(\partial_a-\delta')-b\partial_b-c\partial_c-d\partial_d=\frac{\alpha^3+b\alpha^2+c\alpha+d}{\alpha-\beta\,.\,\alpha-\gamma}\partial_\alpha+\&c.,\ =0,$$

since

$$\alpha^3+b\alpha^2+c\alpha+d=0,\quad \beta^3+b\beta^2+c\beta+d=0,\quad \gamma^3+b\gamma^2+c\gamma+d=0.$$

The equations thus give

$$a\partial_a+b\partial_b+c\partial_c+d\partial_d=\delta',$$

which is right, and the first equation is thus verified.

16. From the last three equations for ∂_b, ∂_c, ∂_d, we deduce

$$-3\partial_b-2b\partial_c-c\partial_d=\frac{3\alpha^2+2b\alpha+c}{\alpha-\beta\,.\,\alpha-\gamma}\partial_\alpha+\frac{3\beta^2+2b\beta+c}{\beta-\gamma\,.\,\beta-\alpha}\partial_\beta+\frac{3\gamma^2+2b\gamma+c}{\gamma-\alpha\,.\,\gamma-\beta}\partial_\gamma,$$

$$=\partial_\alpha+\partial_\beta+\partial_\gamma,$$

a result more easily deducible from the first set of three equations for ∂_α, ∂_β, ∂_γ: but I have preferred to obtain it in this manner for the sake of the remark that it is a peculiarity of this combination of ∂_b, ∂_c, ∂_d that the coefficients of ∂_α, ∂_β, ∂_γ

become integral functions of the roots (in the actual case constants and $=1$): for a somewhat similar form

$$-(c\partial_b + d\partial_c), \quad = \frac{c\alpha^2 + d\alpha}{\alpha - \beta \,.\, \alpha - \gamma}\partial_\alpha + \frac{c\beta^2 + d\beta}{\beta - \gamma \,.\, \beta - \alpha}\partial_\beta + \frac{c\gamma^2 + d\gamma}{\gamma - \alpha \,.\, \gamma - \beta}\partial_\gamma,$$

the coefficients are fractional.

We at once have

$$\alpha\partial_\alpha + \beta\partial_\beta + \gamma\partial_\gamma = b\partial_b + 2c\partial_c + 3d\partial_d,$$

viz. these symbols operating upon a function of the roots of the degree ω, or what is the same thing, a function of the coefficients of the weight ω, are each of them equivalent to a constant factor ω.

Again, we have

$$\begin{aligned}\alpha^2\partial_\alpha + \beta^2\partial_\beta + \gamma^2\partial_\gamma &= -(b^2 - 2c)\,\partial_b - (bc - 3d)\,\partial_c - bd\partial_d,\\ &= -b\,(b\partial_b + c\partial_c + d\partial_d) + 2c\partial_b + 3d\partial_c,\end{aligned}$$

or since $a\partial_a + b\partial_b + c\partial_c + d\partial_d = \delta'$ (if as before δ' is the degree of the function operated upon) and therefore $b\partial_b + c\partial_c + d\partial_d = \delta' - a\partial_a$ or say $= \delta' - \partial_a$, this is

$$\alpha^2\partial_\alpha + \beta^2\partial_\beta + \gamma^2\partial_\gamma = -b\delta' + b\partial_a + 2c\partial_b + 3d\partial_c,$$

so that we have here another form $-b\delta' + b\partial_a + 2c\partial_b + 3d\partial_c$, for which the coefficients of ∂_α, ∂_β, ∂_γ are integral functions of the roots.

17. In the case of four roots, the corresponding equations are

$$\begin{aligned}-b &= \alpha \quad + \beta \quad + \gamma \quad + \delta,\\ +c &= \alpha\beta \ + \alpha\gamma \ + \alpha\delta \ + \beta\gamma + \beta\delta + \gamma\delta,\\ -d &= \alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta,\\ +e &= \alpha\beta\gamma\delta,\end{aligned}$$

and we then have

$$\begin{aligned}\partial_\alpha &= -\partial_b + (\beta + \gamma + \delta)\,\partial_c - (\beta\gamma + \beta\delta + \gamma\delta)\,\partial_d + \beta\gamma\delta\partial_e,\\ \partial_\beta &= -\partial_b + (\gamma + \delta + \alpha)\,\partial_c - (\gamma\delta + \gamma\alpha + \delta\alpha)\,\partial_d + \gamma\delta\alpha\partial_e,\\ \partial_\gamma &= -\partial_b + (\delta + \alpha + \beta)\,\partial_c - (\delta\alpha + \delta\beta + \alpha\beta)\,\partial_d + \delta\alpha\beta\partial_e,\\ \partial_\delta &= -\partial_b + (\alpha + \beta + \gamma)\,\partial_c - (\alpha\beta + \alpha\gamma + \beta\gamma)\,\partial_d + \alpha\beta\gamma\partial_e,\end{aligned}$$

and the converse set of equations, which for shortness I write in the form

$$-\partial_a + \delta', \ -\partial_b, \ -\partial_c, \ -\partial_d, \ -\partial_e$$

$$= \frac{\alpha^{4,\,3,\,2,\,1,\,0}}{\alpha - \beta \,.\, \alpha - \gamma \,.\, \alpha - \delta}\partial_\alpha + \frac{\beta^{4,\,3,\,2,\,1,\,0}}{\beta - \gamma \,.\, \beta - \delta \,.\, \beta - \alpha}\partial_\beta + \frac{\gamma^{4,\,3,\,2,\,1,\,0}}{\gamma - \delta \,.\, \gamma - \alpha \,.\, \gamma - \beta}\partial_\gamma + \frac{\delta^{4,\,3,\,2,\,1,\,0}}{\delta - \alpha \,.\, \delta - \beta \,.\, \delta - \gamma}\partial_\delta.$$

We have, in like manner as in the former case,

$$-4\partial_b - 3b\partial_c - 2c\partial_d - d\partial_e = \partial_\alpha + \partial_\beta + \partial_\gamma + \partial_\delta,$$
$$b\partial_b + c\partial_c + d\partial_d + e\partial_e = \alpha\partial_\alpha + \beta\partial_\beta + \gamma\partial_\gamma + \delta\partial_\delta, = \omega,$$
$$-b\delta' + b\partial_a + 2c\partial_b + 3d\partial_c + 4e\partial_d = \alpha^2\partial_\alpha + \beta^2\partial_\beta + \gamma^2\partial_\gamma + \delta^2\partial_\delta;$$

and similarly in the case of five or more roots.

18. In the case of σ' roots, I write $m = a_{\sigma'}$, and for shortness

$$\Theta_{\sigma'} = \sigma'\partial_b + (\sigma'-1)\, b\partial_c + \ldots \quad \ldots + l\partial_m,$$
$$P = b\partial_a + \qquad 2c\partial_b + 3d\partial_c + \ldots + \sigma' m\partial_l,$$

so that, besides the equation $b\partial_b + c\partial_c \ldots + m\partial_m = S\alpha\partial_\alpha = \omega$, the foregoing investigations show that we have

$$\Theta_{\sigma'} = -S\partial_\alpha,$$
$$P - b\delta = \quad S\alpha^2\partial_\alpha.$$

The operand for these symbols is a symmetric function of the roots, which is thus also a function of the coefficients: it is of the degree ω in the roots, and consequently of the weight ω in the coefficients, and its degree in the coefficients is taken to be $=\delta$. It is sometimes convenient to represent this operand, *quà* function of the roots, by Υ and, *quà* function of the coefficients, by U, so that we have in general $\Upsilon = U$. If Υ be a non-unitary function of the roots, then we may say that Υ, $=U$, is a non-unitariant.

19. I give some illustrations of the equation $\Theta_{\sigma'} = -S\partial_\alpha$. Suppose

$$\Upsilon = U = S\alpha^4 = E = -4e + 4bd + 2c^2 - 4b^2c + b^4$$

(Table IV (b)); σ' must be $=4$ at least and I take it to be 4 and 5 successively; we thus have

$$\Theta_4 = 4\partial_b + 3b\partial_c + 2c\partial_d + \ d\partial_e,$$
$$\Theta_5 = 5\partial_b + 4b\partial_c + 3c\partial_d + 2d\partial_e,$$

omitting from Θ_5 the term $e\partial_f$ which is obviously inoperative. For any number whatever of roots, we have

$$-S\partial_\alpha . S\alpha^4 = -4S\alpha^3 = -4(-3d + 3bc - b^3), \quad = 12d - 12bc + 4b^3,$$

and this should therefore be the value as well of $\Theta_4 E$ as of $\Theta_5 E$. The calculations may be arranged as follows:

$\Theta_4 E$

$4\ .$	$4d - 8bc + 4b^3$	$d + 16$	-4	12
$3b\ .$	$4c\ - 4b^2$	$bc - 32 + 12 + 8$		-12
$2c\ .$	$4b$	$b^3 + 16 - 12$		$+\ 4,$
$d\ . -4$				

$$\Theta_5 E$$

$$\begin{array}{lll|l}
5\ . & 4d-8bc+4b^3 & d-20 \qquad -8 & 12 \\
4b\ . & 4c\ -4b^2 & bc-40+16+12 & -12 \\
3c\ . & 4b & b^3+20-16 & +\ 4, \\
d\ . & -4 & &
\end{array}$$

giving in each case the right result.

20. In the foregoing example, $S\alpha^4$ was a non-unitary function of the roots, but I take the case of a unitary function. Suppose

$$\Upsilon = U = S\alpha^3\beta = BD = 4e - bd - 2c^2 + b^2c.$$

Here $-S\partial_a . S\alpha^3\beta$ is not independent of the number of the roots; in the case of 4 roots, we have

$$-S\partial_a . S\alpha^3\beta = -3S\alpha^2\beta - 3S\alpha^3, \quad = -3(3d-bc) - 3(-3d+3bc-b^3), \quad = 0d - 6bc + 3b^3;$$

and in the case of 5 roots, we have

$$-S\partial_a . S\alpha^3\beta = -3S\alpha^2\beta - 4S\alpha^3, \quad = -3(3d-bc) - 4(-3d+3bc-b^3), \quad = 3d - 9bc + 4b^3;$$

and these should therefore be the values of $\Theta_4 BD$ and $\Theta_5 BD$ respectively. The calculations are

$$\Theta_4 BD$$

$$\begin{array}{lll|l}
4\ . & -\ d+2bc & d-4 \qquad +4 & 0 \\
3b\ . & -4c+\ b^2 & bc+8-12-2 & -6 \\
2c\ . & -\ b & b^3 \qquad +\ 3 & +3, \\
d\ . & +4 & &
\end{array}$$

$$\Theta_5 BD$$

$$\begin{array}{lll|l}
5\ . & -\ d+2bc & d-5 \qquad +8 & +3 \\
4b\ . & -4c+\ b^2 & bc+10-16-3 & -9 \\
3c\ . & -\ b & b^3 \qquad +\ 4 & +4, \\
2d\ . & +\ b & &
\end{array}$$

giving in each case the correct result. We have $\Theta_5 - \Theta_4 = \partial_b + b\partial_c + c\partial_d + d\partial_e$, and the examples show that performing this operation on the non-unitariant $S\alpha^4$, $= E$, we obtain a result $= 0$; whereas for the unitary function $S\alpha^3\beta$, $= BD$, the result is not $= 0$.

21. Considering the question generally, I take the highest coefficient in U to be $q = a_\sigma$ (σ equal to or less than ω), or what is the same thing, the extent of U to be $= \sigma$; this implies that σ' is at least $= \sigma$; and taking it to be first $= \sigma$, and then to be any number greater than σ, we have

$$\Theta_\sigma = -S\partial_a, \quad \Theta_{\sigma'} = -S\partial_a,$$

where the function U operated upon by Θ_σ and $\Theta_{\sigma'}$ respectively is in each case the same function of the coefficients. It is easy to see that, if Υ is a non-unitary function of the roots, then whatever be the number of the roots we have $S\partial_\alpha . \Upsilon =$ a determinate symmetric function of the roots, and consequently = a determinate function of the coefficients. We thus have $\Theta_{\sigma'} U$ and $\Theta_\sigma U$ equal to each other; that is,

$$(\Theta_{\sigma'} - \Theta_\sigma)\, U = 0\,;$$

we may write

$$\Theta_\sigma = \sigma\,\partial_b + (\sigma\; - 1)\, b\partial_c + \ldots + \qquad\qquad p\partial_q,$$

$$\Theta_{\sigma'} = \sigma'\partial_b + (\sigma' - 1)\, b\partial_c + \ldots + (\sigma' - \sigma + 1)\, p\partial_q,$$

for the subsequent terms of $\Theta_{\sigma'}$, as involving ∂_r, ∂_s, &c., are inoperative; hence writing

$$\Delta = \partial_b + b\partial_c + c\partial_b + \ldots + p\partial_q,$$

or as we may more simply express it

$$\Delta = \partial_b + b\partial_c + c\partial_b + \ldots,$$

we have $\Theta_{\sigma'} - \Theta_\sigma = (\sigma' - \sigma)\,\Delta$, and consequently $\Delta U = 0$; Δ is thus an annihilator of any function U of the coefficients which is equal to a non-unitary function of the roots; or more shortly Δ is an annihilator of any non-unitariant.

22. Similarly, from the two equations $\Theta_\sigma = - S\partial_\alpha$, and $\Theta_{\sigma'} = S\partial_\alpha$ regarded as operating upon a non-unitary function, we deduce $\sigma'\Theta_\sigma - \sigma\Theta_{\sigma'} = (\sigma - \sigma')\, S\partial_\alpha$: the left-hand side is here $= (\sigma - \sigma')\,\Delta_1$, if

$$\Delta_1 = b\partial_c + 2c\partial_b + 3c\partial_d + \ldots + (\sigma - 1)\, p\partial_q,$$

or say

$$\Delta_1 = b\partial_c + 2c\partial_b + 3c\partial_d + \ldots,$$

viz. we have $\Delta_1 = S\partial_\alpha$; for instance, if as before

$$\Upsilon = U = S\alpha^4 = - 4e + 4bd + 2c^2 - 4b^2c + b^4,$$

then

$$(b\partial_c + 2c\partial_d + 3d\partial_e)\,(- 4e + 4bd + 2c^2 - 4b^2c + b^4) = S\partial_\alpha . S\alpha^4, \quad = 4S\alpha^3, \quad = 4\,(- 3d + 3bc - b^3),$$

as can be at once verified. It is to be noticed, however, that $S\partial_\alpha$ operating upon a non-unitary function of the roots does not in every case give a non-unitary function; and thus successive operations with Δ_1 will not give a succession of non-unitariants.

23. I investigate the foregoing result in regard to Δ in a different manner; suppose, for instance, that $\Upsilon = U$ is the non-unitary function $S\alpha^4$ of the roots,

$$(= - 4e + 4bd + 2c^2 - 4b^2c + b^4).$$

The number of roots is at least $= 4$, and I take it to be $= 4$, say the roots are α, β, γ, δ. Consider a fifth root θ, and let $\Upsilon_1 = U_1 = S\alpha^4$ be the like function for the five roots, we have $\Upsilon_1 = \Upsilon + \theta^4$, or say $U_1 = U + \theta^4$. Write $-b_1$, c_1, $-d_1$, e_1, $-f_1$

for the symmetric functions of the five roots; U_1 will not involve f_1 and it will be the same function of b_1, c_1, d_1, e_1 that U is of b, c, d, e, say we have

$$U_1 = U(b_1, c_1, d_1, e_1).$$

But we have

$$b_1 = b - \theta, \quad c_1 = c - b\theta, \quad d_1 = d - c\theta, \quad e_1 = e - d\theta;$$

and thus the foregoing equation $U_1 = U + \theta^4$ becomes

$$U(b-\theta,\ c - b\theta,\ d - c\theta,\ e - d\theta) = U + \theta^4;$$

it is in fact easy to verify that, for the foregoing value of U, the terms in θ, θ^2, θ^3 all vanish, and that the expression on the left-hand becomes $= U + \theta^4$. But attending only to the term in θ, this is $= -\theta(\partial_b + b\partial_c + c\partial_d + d\partial_e)\,U$, $= -\theta\Delta U$; viz. this term vanishing we have $\Delta U = 0$, the result which was to be proved.

In the case of a unitary function, for instance $\Upsilon = U = S\alpha^3\beta$, here introducing the new root θ we have $U_1 = U + \theta S\alpha^3 + \theta^3 S\alpha$; or there is here a term in θ, and instead of $\Delta U = 0$, we have $\Delta U = S\alpha^3$, or the unitary function is not annihilated by Δ.

The foregoing investigation is really quite general, and establishes the conclusion that Δ is an annihilator of every non-unitariant.

It is to be noticed that Θ_σ and Δ are operators, which leave each of them the degree unaltered but diminish the weight by unity: the operator $P - b\delta$, and another operator $\frac{1}{2}Q - b\omega$ which will be considered, increase each of them the degree by unity and also the weight by unity.

24. Coming now to the equation

$$P - b\delta = S\alpha^2\partial_\alpha,$$

it is to be remarked that, if $\sigma' = \sigma$, the expression for P ends in $q\partial_p$ where, as before, $q = a_\sigma$ is the highest coefficient in the operand; since the operand thus contains q, the next succeeding term in $r\partial_q$ would be not inoperative, and in order to include it in the expression of P we may take $\sigma' = \sigma + 1$; we thus have

$$P = b\partial_a + 2c\partial_b + 3d\partial_c + \ldots + (\sigma + 1)\, r\partial_q,$$

or as we may more simply write it

$$P = b\partial_a + 2c\partial_b + 3d\partial_c + \ldots;$$

the operation thus increases the extent by unity. The symbol $S\alpha^2\partial_\alpha$ operating upon a symmetric function of the roots, gives, whatever may be the number of roots, the same symmetric function of the roots: and we see further that, operating upon a non-unitary function, it gives a non-unitary function of the roots. Hence $P - b\delta$ operating upon a non-unitariant gives a non-unitariant. I give an example.

25. Suppose, as before,

$$\Upsilon = U = S\alpha^4 = E = -4e + 4bd + 2c^2 - 4b^2c + b^4;$$

here $\delta = 4$, and therefore

$$P - b\delta = b\partial_a + 2c\partial_b + 3d\partial_c + 4e\partial_d + 5f\partial_e - 4b.$$

We have

$$S\alpha^2\partial_\alpha \,.\, S\alpha^4 = 4S\alpha^5, \quad = 4\,(-\,5f + 5be + 5cd - 5b^2d - 5bc^2 + 5b^3c - b^5),$$

and this should therefore be the result of the operation $P - b\delta$: the calculation is

$b\,.\,-12c + 8bd + 4c^2 - 4b^2c$	f					-20		-20
$2c\,.\quad 4d - 8bc + 4b^3$	be	-12			$+16$		$+16$	$+20$
$3d\,.\quad 4c - 4b^2$	cd		$+8$	$+12$				$+20$
$4e\,.\quad 4b$	b^2d	$+8$		-12			-16	-20
$5f\,.-\;4$	bc^2	$+4$	-16				-8	-20
$-4b\,.-\;4e + 4bd + 2c^2 - 4b^2c + b^4$	b^3c	-4	$+8$				$+16$	$+20$
	b^5						-4	-4,

which is the right result.

We have seen that every non-unitariant is annihilated by Δ; it at once appears that conversely every function of the coefficients which is annihilated by Δ is a non-unitariant: it is, in fact, a symmetric function of the roots, and unless it were a non-unitary function of the roots it would not be annihilated by Δ. Non-unitariants are analogous to seminvariants; the precise relation between them will be shown further on.

26. We can, by an investigation similar to that for seminvariants, show that $P - b\delta$ operating upon a non-unitariant gives a non-unitariant. In fact, considering the two operations Δ and $P - b\delta$, we have

$$\Delta\,(P - b\delta)\,\dagger = \Delta\,(P - b\delta) + \Delta\,.\,(P - b\delta),$$

the meaning being that, if upon any operand U we perform first the operation $P - b\delta$ and then the operation Δ, this is equivalent to operating on U with the sum of the two operations $\Delta\,(P - b\delta)$, and $\Delta\,.\,P - b\delta$, the first of these symbols denoting the mere algebraical product of Δ and $P - b\delta$, the second of them the result of the operation Δ performed upon $P - b\delta$. We have similarly

$$(P - b\delta)\,\Delta\dagger = (P - b\delta)\,\Delta + (P - b\delta)\,.\,\Delta.$$

Hence observing that $\Delta\,(P - b\delta)$ and $(P - b\delta)\,\Delta$ are equal to each other, and subtracting, we have

$$\Delta\,(P - b\delta)\,\dagger - (P - b\delta)\,\Delta\dagger = \Delta\,.\,(P - b\delta) - (P - b\delta)\,.\,\Delta.$$

But from the values

$$\Delta = a\partial_b + b\partial_c + c\partial_d + \ldots\quad,$$

and

$$P - b\delta = b\partial_a + c\partial_b + d\partial_c + \ldots - b\delta,$$

we find

$$\Delta . (P - b\delta) = a\partial_a + 2b\partial_b + 3c\partial_c + \ldots - \delta,$$

$$(P - b\delta) . \Delta = \qquad b\partial_b + 2c\partial_c + \ldots,$$

and thence

$$\Delta . (P - b\delta) - (P - b\delta) . \Delta = a\partial_a + b\partial_b + c\partial_c \ldots - \delta, \ = 0,$$

since δ is the degree in the coefficients. Hence writing down the operand U,

$$\Delta . (P - b\delta)\, U - (P - b\delta) . \Delta U = 0,$$

where for greater clearness I have inserted the dots, to show that Δ operates on $(P - b\delta)\, U$, and $(P - b\delta)$ on ΔU. Taking U to be a non-unitariant, we have $\Delta U = 0$; and this being so, the equation gives $\Delta . (P - b\delta)\, U = 0$, viz. this shows that $(P - b\delta)\, U$ is a non-unitariant.

27. There is another symbol $\frac{1}{2}Q - b\omega$, which is precisely analogous to $P - b\delta$, viz. operating upon a non-unitariant, it gives a non-unitariant: ω is, as before, the weight of the function operated upon, and the expression of Q is

$$\tfrac{1}{2}Q = c\partial_b + 3d\partial_c + 6e\partial_d + \ldots + \tfrac{1}{2}\sigma(\sigma + 1)\, r\partial_q,$$

or say

$$\tfrac{1}{2}Q = c\partial_b + 3d\partial_c + 6e\partial_d + \ldots.$$

The proof is exactly similar, viz. we have to show that

$$\Delta . (\tfrac{1}{2}Q - b\omega) - (\tfrac{1}{2}Q - b\omega) . \Delta = 0.$$

We have

$$\Delta . (\tfrac{1}{2}Q - b\omega) = b\partial_b + 3c\partial_c + 6d\partial_d + \ldots - \omega,$$

$$(\tfrac{1}{2}Q - b\omega) . \Delta = \qquad c\partial_c + 3d\partial_d + \ldots,$$

and the difference of the two expressions is

$$b\partial_b + 2c\partial_c + 3d\partial_d + \ldots - \omega, \ = 0,$$

since ω is the weight of the function operated upon. Hence, as before, if U be a non-unitariant and therefore $\Delta U = 0$, we have $\Delta . (\frac{1}{2}Q - b\omega)\, U = 0$, that is, $(\frac{1}{2}Q - b\omega)\, U$ is also a non-unitariant.

28. The symbol $\frac{1}{2}Q - b\omega$ has no simple expression in terms of $\partial_\alpha, \partial_\beta, \partial_\gamma, \ldots$, and the form varies with the number of the roots: thus for 3 roots, it is

$$= -\left\{\left(\frac{c\alpha^2 + 3d\alpha}{\alpha - \beta . \alpha - \gamma} + b\alpha\right)\partial_\alpha + \&\text{c.}\right\},$$

for 4 roots it is

$$= -\left\{\left(\frac{c\alpha^3 + 3d\alpha^2 + 6e\alpha}{\alpha - \beta . \alpha - \gamma . \alpha - \delta} + b\alpha\right)\partial_\alpha + \&\text{c.}\right\},$$

for 5 roots it is

$$= -\left\{\left(\frac{c\alpha^4 + 3d\alpha^3 + 6e\alpha^2 + 10f\alpha}{\alpha - \beta . \alpha - \gamma . \alpha - \delta . \alpha - \epsilon} + b\alpha\right)\partial_\alpha + \&\text{c.}\right\},$$

and so on. It is not easy to find the effect of such a symbol upon a given symmetric function of the roots, nor in particular when the function is non-unitary is it easy to show generally that the result is non-unitary.

It is to be remarked that, if the function operated upon is of the degree δ in the roots, then we must for $\frac{1}{2}Q - b\omega$ take the expression with $\delta + 1$ roots; for instance, if the function be of the degree 5 in the roots, then, *quà* function of the coefficients, this contains f, and it must be operated on with

$$\tfrac{1}{2}Q - b\omega, \; = c\partial_b + 3d\partial_c + 6e\partial_d + 10f\partial_e + 15g\partial_f - \omega b,$$

viz. this expression, as containing g, gives the 6-root expression for $\frac{1}{2}Q - b\omega$.

29. Suppose for instance the function operated upon is $F = S\alpha^5$; here taking the 6-root expression, this gives

$$-5\left\{\left(\frac{c\alpha^5 + 3d\alpha^4 + 6e\alpha^3 + 10f\alpha^2 + 15g\alpha}{\alpha-\beta\,.\,\alpha-\gamma\,.\,\alpha-\delta\,.\,\alpha-\epsilon\,.\,\alpha-\zeta} + b\alpha\right)\alpha^4 + \&\text{c.}\right\},$$

or omitting for the moment the outside factor -5, the expression in $\{\ \}$ is easily seen to be

$$= cH_4 + 3dH_3 + 6eH_2 + 10fH_1 + 15g + bS\alpha^5,$$

where H_4, H_3, H_2, H_1 denote the homogeneous functions of the degrees 4, 3, 2, 1 respectively: the values of these are obtained by adding together all the lines of the Table IV (b), all the lines of the Table III (b), &c.: the terms exclusive of $bS\alpha^5$ thus are

$$\begin{aligned} & c\,(-e + 2bd + c^2 - 3b^2c + b^4) \\ +\ & 3d\,(-d + 2bc - b^3) \\ +\ & 6e\,(-c + b^2) \\ +\ & 10f\,(-b) \\ +\ & 15g\,.\,1, \end{aligned}$$

and these are $= S\alpha^5\beta + S\alpha^4\beta^2 + S\alpha^3\beta^3$, as appears by the following calculation:

							$S\alpha^5\beta$	$S\alpha^4\beta^2$	$S\alpha^3\beta^3$	
g					+ 15	+ 15	+ 6	+ 6	+ 3	+ 15
bf				− 10		− 10	− 1	− 6	− 3	− 10
ce	− 1		− 6			− 7	− 6	+ 2	− 3	− 7
b^2e			+ 6			+ 6	+ 1	+ 2	+ 3	+ 6
d^2		− 3				− 3	− 3	− 3	+ 3	− 3
bcd	+ 2	+ 6				+ 8	+ 7	+ 4	− 3	+ 8
b^3d		− 3				− 3	− 1	− 2	0	− 3
c^3	+ 1					+ 1	+ 2	− 2	+ 1	+ 1
b^2c^2	− 3					− 3	− 4	+ 1		− 3
b^4c	+ 1					+ 1	+ 1			+ 1
b^6										

The omitted term $bS\alpha^5$, that is, $-S\alpha . S\alpha^5$, is $-S\alpha^6 - S\alpha^5\beta$; the addition hereof destroys therefore the non-unitary term $S\alpha^5\beta$, and thus the required expression, restoring the omitted factor -5, is $-5\,(-S\alpha^6 + S\alpha^4\beta^2 + S\alpha^3\beta^3)$, or say $= 5G - 5CE - 5D^2$. a non-unitary form: this then should be the result of the operation

$$\tfrac{1}{2}Q - b\omega, = c\partial_b + 3d\partial_c + 6e\partial_d + 10f\partial_e + 15g\partial_f - 5b,$$

performed upon

$$S\alpha^5 = F = -5f + 5be + 5cd - 5b^2d - 5bc^2 + 5b^3c = b^5.$$

Performing the calculation so as to omit on each side a factor 5, it is to be shown that $G - CE - D^2$ is

$$\begin{aligned} = \quad & c\,(e - 2bd - c^2 + 3b^2c - b^4) \\ + \quad & 3d\,(d - 2bc + b^3) \\ + \quad & 6e\,(c - b^2) \\ + \quad & 10f\,(b) \\ + \quad & 15g\,(-1) \\ - \quad & 5b\,(-f + be + cd - b^2d - bc^2 + b^3c - \tfrac{1}{5}b^5). \end{aligned}$$

Collecting the terms, and comparing the result with the expression for $G - CE - D^2$, we have

								$G - CE - D^2$		
g					-15		-15	-6	-6	-3
bf				$+10$		$+5$	$+15$	$+6$	$+6$	$+3$
ce	$+1$		$+6$				$+7$	$+6$	-2	$+3$
b^2e			-6			-5	-11	-6	-2	-3
d^2		$+3$					$+3$	$+3$	$+3$	-3
bcd	-2	-6				-5	-13	-12	-4	$+3$
b^3d		$+3$				$+5$	$+8$	$+6$	$+2$	0
c^3	-1						-1	-2	$+2$	-1
b^2c^2	$+3$					$+5$	$+8$	$+9$	-1	
b^4c	-1					-5	-6	-6		
b^6						$+1$	$+1$	$+1$		

and the two expressions are thus identical.

30. Suppose again, 6 roots as before, and that the function operated upon is $S\alpha^3\beta^2$; we find $\partial_a S\alpha^3\beta^2 = 3\alpha^2 S\alpha^2 + 2\alpha S\alpha^3 - 5\alpha^4$, and the general term is

$$\begin{aligned} & -3\left(\frac{c\alpha^5 + 3d\alpha^4 + 6e\alpha^3 + 10f\alpha^2 + 15g\alpha}{\alpha - \beta . \alpha - \gamma . \alpha - \delta . \alpha - \epsilon . \alpha - \zeta} + b\alpha\right)\alpha^2 . S\alpha^2 \\ & -2\left(\frac{c\alpha^5 + 3d\alpha^4 + 6e\alpha^3 + 10f\alpha^2 + 15g\alpha}{\alpha - \beta . \alpha - \gamma . \alpha - \delta . \alpha - \epsilon . \alpha - \zeta} + b\alpha\right)\alpha\ . S\alpha^3 \\ & +5\left(\frac{c\alpha^5 + 3d\alpha^4 + 6e\alpha^3 + 10f\alpha^2 + 15g\alpha}{\alpha - \beta . \alpha - \gamma . \alpha - \delta . \alpha - \epsilon . \alpha - \zeta} + b\alpha\right)\alpha^4. \end{aligned}$$

This gives

$$\begin{aligned}&-3\,(CH_2+3dH_1+6e+bS\alpha^3)\,S\alpha^2\\&-2\,(CH_1+3d \qquad +bS\alpha^2)\,S\alpha^3\\&+5\,(CH_4+3dH_3+6eH_2+10fH_1+15g+bS\alpha^5),\end{aligned}$$

which is found to be

$$\begin{aligned}=&-3\,(BD+C^2\ +bS\alpha^3)\,S\alpha^2\\&-2\,(BC \qquad +bS\alpha^2)\,S\alpha^3\\&+5\,(BF+CE+D^2+bS\alpha^5).\end{aligned}$$

Here

$$bS\alpha^3=-S\alpha S\alpha^3=-S\alpha^4-S\alpha^3\beta,\ =-E-BD,$$

$$bS\alpha^2=-S\alpha S\alpha^2=-S\alpha^3-S\alpha^2\beta,\ =-D-BC,$$

and

$$bS\alpha^5=-S\alpha S\alpha^5=-S\alpha^6-S\alpha^5\beta,\ =-G-BF;$$

the expression thus is

$$\begin{aligned}=&-3\,(-E+C^2)\,.\,C &&\text{that is, } -3\,(-S\alpha^4+S\alpha^2\beta^2)\,S\alpha^2\\&-2\,(-D \qquad)\,.\,D &&-2\,(-S\alpha^3 \qquad)\,S\alpha^3\\&+5\,(-G+CE+D^2), &&+5\,(-S\alpha^6+S\alpha^4\beta^2+S\alpha^3\beta^3).\end{aligned}$$

Here

$$\begin{aligned}S\alpha^2S\alpha^4\ &=S\alpha^6\ +\ S\alpha^4\beta^2, &&=G+CE,\\S\alpha^3S\alpha^3\ &=S\alpha^6\ +2S\alpha^3\beta^3, &&=G+2D^2,\\S\alpha^2S\alpha^2\beta^2&=S\alpha^4\beta^2+3S\alpha^2\beta^2\gamma^2, &&=CE+3C^3;\end{aligned}$$

and the whole is

$$\begin{aligned}&-3\,\{-G-CE+(CE+3C^2)\}\\&-2\,(-G-2D^2)\\&+5\,(-G+CE+D^2),\end{aligned}$$

which is $=5CE+9D^2-9C^3$ (a non-unitary form). This then should be the value of

$$\tfrac{1}{2}Q-b\omega,\ =c\partial_b+3d\partial_c+6e\partial_d+10f\partial_e+15g\partial_f-5b,$$

operating upon

$$S\alpha^3\beta^2,\ =CD=5f-5be+cd+2b^2d-bc^2.$$

31. There is for non-unitariants a theorem which is a much more simple form than the transformation of it afterwards obtained for seminvariants: viz. for any non-unitariant we have $\Delta U=0=(\partial_b+b\partial_c+c\partial_d+\ldots)\,U$; attending only to the portion U' of U which is of the highest degree, it is clear that we have $(b\partial_c+c\partial_d+\ldots)\,U'=0$, and if we herein diminish the letters, then $(\partial_b+b\partial_c+\ldots)\,U''=0$, where U'' is what U' becomes by a diminution of the letters; that is, U'' is a non-unitariant, viz. in any seminvariant, the terms of highest degree U' are obtained from a non-unitariant U'' by a mere augmentation of the letters: e.g. $2e-2bd+c^2$ is a non-unitariant weight 4; augmenting the letters, we have $2bf-2ce+d^2$ which with a change of sign is the portion of highest degree of the non-unitariant $2g-2bf+2ce-d^2$.

The MacMahon Form of Equation. Art. Nos. 32 to 34.

32. The equation connecting the coefficients and the roots is here taken to be

$$1+\frac{b}{1}x+\frac{c}{1.2}x^2+\frac{d}{1.2.3}x^3+\ldots=1-\alpha x.1-\beta x.1-\gamma x\ldots.$$

As to this it may be remarked that, if we had started with a form of the nth order with binomial coefficients,

$$1+\frac{n}{1}bx+\frac{n.n-1}{1.2}cx^2+\frac{n.n-1.n-2}{1.2.3}dx^3+\ldots=1-\alpha x.1-\beta x.1-\gamma x\ldots(n \text{ factors}),$$

then writing herein $\frac{x}{n}$ for x, and also $n\alpha$, $n\beta$, $n\gamma$, ..., for α, β, γ, ... and putting ultimately $n=\infty$, we have the form in question.

We pass from the ordinary form to the MacMahon form, by writing for

$$b,\ c,\ d,\ e,\ \ldots,\ \frac{b}{1},\ \frac{c}{1.2},\ \frac{d}{1.2.3},\ \frac{e}{1.2.3.4},\ \ldots \text{ or say } b,\ \frac{c}{2},\ \frac{d}{6},\ \frac{e}{24},\ \frac{f}{120},\ \frac{g}{720},\ \ldots.$$

All the results obtained for the ordinary form will, after making therein this change, apply to the new form. We thus find

$$\Theta_\sigma=\sigma\partial_b+(\sigma-1)\,2b\partial_c+(\sigma-2)\,3c\partial_d+\ldots+\qquad 1\ \sigma p\partial_q,$$

$$\Theta_{\sigma'}=\sigma'\partial_b+(\sigma'-1)\,2b\partial_c+(\sigma'-2)\,3c\partial_d+\ldots+(\sigma'-\sigma+1)\,\sigma p\partial_q,$$

$$\Theta_\sigma-\Theta_{\sigma'}=(\sigma'-\sigma)\,\Delta,$$

where

$$\Delta=\partial_b+2b\partial_c+3c\partial_d+\ldots+\sigma p\partial_q,$$

or say

$$=\partial_b+2b\partial_c+3c\partial_d+\ldots.$$

Also

$$P=b\partial_a+\ c\partial_b+\ d\partial_c+\ldots+\ r\partial_q,$$

or say

$$=b\partial_a+\ c\partial_b+\ d\partial_c+\ldots,$$

$$Q=\qquad c\partial_b+2d\partial_c+\ldots+\sigma r\partial_q,$$

or say

$$=\qquad c\partial_b+2d\partial_c+\ldots.$$

The change α, β, γ, ... into $n\alpha$, $n\beta$, $n\gamma$, ... would change $S\partial_\alpha$, $S\alpha\partial_\alpha$, $S\alpha^2\partial_\alpha$ into $n^{-1}S\partial_\alpha$, $S\alpha\partial_\alpha$, $nS\alpha^2\partial_\alpha$ respectively ($n=\infty$): but this change is, in fact, compensated for by the introduction into the formulæ of the binomial coefficients as above; it is $-S\alpha$, $S\alpha\beta$, ... not $-nS\alpha$, $n^2S\alpha\beta$, ... which are equal to b, $\frac{1}{2}c$, ...; and the conclusion is that we have to retain without alteration the symbols $S\partial_\alpha$, $S\alpha\partial_\alpha$, $S\alpha^2\partial_\alpha$: thus in the new form as in the old one, we have $\Theta_4S\alpha^4=-S\partial_\alpha.S\alpha^4=-4S\alpha^3$, see the example *ante* No. 23.

33. In the new form, a non-unitariant is annihilated by the operator

$$\Delta,\ =\partial_b+2b\partial_c+3c\partial_d+\ldots,$$

and conversely any function annihilated by Δ is a non-unitariant; comparing herewith the subsequent theory of seminvariants, this is in fact the theorem that a non-unitariant is the same thing as a seminvariant; or to state this more explicitly: for the MacMahon form of equation, a function of the coefficients which is a non-unitary symmetric function of the roots is a seminvariant.

I consider for instance the Table VI (b), but attend only to the non-unitary portions thereof, viz. the lines G, CE, D^2, C^3: I convert these into columns, at the same time changing the arrangement of the headings g, bf, ce, &c., from CO to AO: and then making the foregoing change b, c, d, e, f, g into $b, \frac{c}{2}, \frac{d}{6}, \frac{e}{24}, \frac{f}{120}, \frac{g}{720}$, but to avoid fractions multiplying the whole by 720, I form the table

$\div\ 720$ $\Vert$		C^3	D^2	CE	G
1	g	-2	$+3$	$+6$	-6
6	bf	$+2$	-3	-6	$+6$
15	ce	-2	-3	$+2$	$+6$
20	d^2	$+1$	$+3$	-3	$+3$
30	b^2e		$+3$	$+2$	-6
60	bcd		-3	$+4$	-12
90	c^3		$+1$	-2	-2
120	b^3d			-2	$+6$
180	b^2c^2			$+1$	$+9$
360	b^4c				-6
720	b^6				$+1$
		$[d^2]$	$[c^3]$	$[b^2c^2]$	$[b^6]$

which is to be read according to the columns: and observe that the outside left-hand numbers are to be multiplied into the numbers of each column: thus the first column is to be read

$$C^3 = S\alpha^2\beta^2\gamma^2 = \frac{1}{720}(-2g + 12bf - 30ce + 20d^2),$$

the second column is to be read

$$D^2 = S\alpha^3\beta^3 = \frac{1}{720}(3g - 18bf \ldots + 90c^3),$$

and so on.

By what precedes, the columns are seminvariants,—as afterwards explained, "blunt" seminvariants; and they are named as such by the outside bottom line of symbols with a [ ]; viz.

$$[d^2] = (-2g + 12bf - 30ce + 20d^2), \quad [c^3] = (3g - 18bf \ldots + 90c^3), \text{ \&c.,}$$

where it will be observed that the symbol within the [ ] is, in fact, the power-ender which is in AO the lowest term of the column; and further that this is also the conjugate of the capital letter symbol at the head of the column.

The (b) Tables I to X, with only the change $b, c, d, e, \ldots$ into $b, \frac{c}{2}, \frac{d}{6}, \frac{e}{24}, \ldots$ are given in my paper, "Tables of the Symmetric Functions of the Roots to the degree 10, for the form

$$1 + bx + \frac{cx^2}{1.2} + \ldots = (1 - \alpha x)(1 - \beta x)(1 - \gamma x)\ldots,"$$

American Mathematical Journal, t. VII. (1885), pp. 47—56, [829].

34. By what precedes, it appears that $P - b\delta$ operating on a seminvariant gives a seminvariant, and that $Q - 2b\omega$ operating on a seminvariant gives a seminvariant: these operators will be further considered in the development of the theory of seminvariants. We see further that $\frac{1}{2}\Delta, = b\partial_c + 3c\partial_d + 6d\partial_e + \ldots$, operating on a seminvariant gives sometimes but not always a seminvariant, e.g.

$$(b\partial_c + 3c\partial_d + 6d\partial_e)(e - 4bd - 3c^2 + 12b^2c - 6b^4) = 6(d - 3bc + 2b^3).$$

Seminvariants—the I-and-F Problem, and Solution by Square Diagrams.
Art. Nos. 35 to 47.

35. Writing

$$\begin{aligned}
1 &= 1,\\
b_1 &= b + \theta,\\
c_1 &= c + 2b\theta + \theta^2,\\
d_1 &= d + 3c\theta + 3b\theta^2 + \theta^3,\\
e_1 &= e + 4d\theta + 6c\theta^2 + 4b\theta^3 + \theta^4,\\
&\&c.,
\end{aligned}$$

then there are functions of the unsuffixed letters which remain unaltered if for these we substitute the suffixed letters: any such function is termed a seminvariant. We have for instance

$$\begin{aligned}
c_1 &= c + 2b\theta + \theta^2, & \text{i.e.,}\quad c_1 - b_1^2 &= c - b^2,\\
-b_1^2 &= -b^2 - 2b\theta - \theta^2, & &\\
d_1 &= d + 3c\theta + 3b\theta^2, & d_1 - 3b_1c_1 + 2b_1^3 &= d - 3bc + 2b^3,\\
-3b_1c_1 &= -3bc - 6b^2\theta - 3b\theta^2, & &\\
&\quad\;\; - 3c\theta - 6b\theta^2 - 3\theta^3, & &\\
+2b_1^3 &= 2b^3 + 6b^2\theta + 6b\theta^2 + 2\theta^3, & &
\end{aligned}$$

and thus $c-b^2$, $d-3bc+2b^3$ are seminvariants; they are, in fact, the first and second terms of the series

$$\begin{aligned}
&c - b^2,\\
&d - 3bc + \ 2b^3,\\
&e - 4bd + \ 6b^2c - \ 3b^4,\\
&f - 5bc + 10b^2d - 10b^3c + \ 4b^5,\\
&g - 6bf + 15b^2e - 20b^3d + 15b^4c - 5b^6,\\
&\ \vdots
\end{aligned}$$

where the law is obvious; the numbers in each line are binomial coefficients except the last number, which is the next binomial coefficient diminished by unity. The successive terms are, in fact, what c_1, d_1, e_1, f_1, g_1, ... become upon writing therein $\theta=-b$.

36. Any rational and integral function of these forms is a seminvariant, and it is to be observed that we can form functions for which (by the destruction of terms of a higher degree) there is a diminution of degree; for instance,

$$(e-4bd+6b^2c-3b^4)+3(c-b^2)^2$$

gives a seminvariant $e-4bd+3c^2$.

It is important to remark that a seminvariant is completely determined by its non-unitary terms; thus for $e-4bd+3c^2$, the non-unitary terms are $e+3c^2$, and for this writing $e_1+3c_1^2$, and for e_1, c_1 substituting their above values for $\theta=-b$, we reproduce the original value $e-4bd+3c^2$.

37. It is at once seen that a seminvariant is reduced to zero by the operation $\Delta, =\partial_b+2b\partial_c+3c\partial_d+\ldots$, or say that Δ is an annihilator of a seminvariant; in fact, if in any function of b, c, d, ... we write for these the suffixed letters b_1, c_1, d_1, ... then the coefficient of θ herein is at once found by operating on the function of $(b, c, d, \ldots)$ with Δ, and therefore in the case of a seminvariant the result of this operation must be $=0$. And conversely, every function of $(b, c, d, \ldots)$ which is reduced to zero by the operation Δ is a seminvariant.

38. For a given weight, the number of seminvariants is equal to the excess of the number of terms of that weight above the number of terms of the next preceding weight, or what is the same thing, it is equal to the number of power-enders of the given weight. More definitely, considering the terms of a seminvariant as arranged in AO, we have seminvariants the finals whereof are the several power-enders of the given weight: and we arrange the seminvariants *inter se* by taking these power-enders in AO: thus for the weight 6, we have seminvariants $[d^2]$, $[c^3]$, $[b^2c^2]$, $[b^6]$ ending in these terms respectively. We may, if we please, consider all these seminvariants as beginning with g, or say the forms may be taken to be

$g\,(\text{ao})\,d^2$, $g\,(\text{ao})\,c^3$, $g\,(\text{ao})\,b^2c^2$, $g\,(\text{ao})\,b^6$. Such forms are, in fact, furnished by the MacMahon equation: viz. up to the weight 6, we thus have for the present purpose

$\div 2$ ‖		C
1	c	-2
2	b^2	$+1$
		$[b^2]$

$\div 6$ ‖		D
1	d	-3
3	bc	$+3$
6	b^3	-1
		$[b^3]$

$\div 24$ ‖		C^2	E
1	e	$+2$	-4
4	bd	-2	$+4$
6	c^2	$+1$	$+2$
12	b^2c		-4
24	b^4		$+1$
		$[c^2]$	$[b^4]$

$\div 120$ ‖		CD	F
1	f	$+5$	-5
5	be	-5	$+5$
10	cd	$+1$	$+5$
20	b^2d	$+2$	-5
30	bc^2	-1	-5
60	b^3c		$+5$
120	b^5		-1
		$[bc^2]$	$[b^5]$

$\div 720$ ‖		C^3	D^2	CE	G
1	g	-2	$+3$	$+6$	-6
6	bf	$+2$	-3	-6	$+6$
15	ce	-2	-3	$+2$	$+6$
20	d^2	$+1$	$+3$	-3	$+3$
30	b^2e		$+3$	$+2$	-6
60	bcd		-3	$+4$	-12
90	c^3		$+1$	-2	-2
120	b^3d			-2	$+6$
180	b^2c^2			$+1$	$+9$
360	b^4c				-6
720	b^6				$+1$
		$[d^2]$	$[c^3]$	$[b^2c^2]$	$[b^6]$

read for instance

$$[d^2] = -2g + 12bf - 30ce + 20d^2,$$

$$[c^3] = 3g - 18bf - 45ce + 60d^2 + 90b^2e - 180bcd + 90c^3,$$

&c.

I say that $[d^2]$, $[c^3]$, $[b^2c^2]$, $[b^6]$ are "specific" when they are regarded as standing for these tabulated functions; but in general I take them to be "indefinite," that is, I regard them as denoting (as above) any seminvariants ending in d^2, c^3, b^2c^2, b^6 respectively.

39. The seminvariant $[d^2]$ is of the form $(g \infty d^2)$, including those terms which are in CO not superior to g and in AO not inferior to d^2: by a combination of $[d^2]$ and $[c^3]$, we obtain a seminvariant $(ce \infty c^3)$ containing terms which are in CO not superior to ce and in AO not inferior to c^3: similarly, from $[d^2]$, $[c^3]$, $[b^2c^2]$ we obtain a seminvariant $(d^2 \infty b^2c^2)$; and from the four forms a seminvariant $(c^3 \infty b^6)$: these four seminvariants

	$(g \infty d^2)$	$(ce \infty c^3)$	$(d^2 \infty b^2c^2)$	$(c^3 \infty b^6)$
g	$+1$			
bf	-6			
ce	$+15$	$+1$		
d^2	-10	-1	$+1$	
b^2e		-1		
bcd		$+2$	-6	
c^3		-1	$+4$	$+1$
b^3d			$+4$	
b^2c^2			-3	-3
b^4c				$+3$
b^6				-1

are said to be "sharp" seminvariants: viz. considering the final as given, a sharp seminvariant is one having an initial which is in CO as low as possible; or considering the initial as given, it is one having a final which is in AO as high as possible. A seminvariant which is not sharp is said to be "blunt."

40. The sharp seminvariants are in general designated as above, $(g \infty d^2)$, &c.: but it is sometimes convenient to give the numerical coefficients of the initial and final terms respectively: as to this, it is to be noticed that the coefficient of the initial term is in most cases, but not always, $=1$,—we might of course take it to be always $=1$, but we should then in the excepted cases have fractional coefficients, and it is better to avoid this by giving a proper value to the numerical coefficient of the initial term; the numerical coefficient of the final term is in general different from ± 1, and it is not in general a multiple of the numerical coefficient of the initial term. As an instance take $dh \infty b^2e^2$, the more complete expression of which is $4dh \infty -35b^2e^2$. The sharp seminvariants up to the weight 12 are designated in this more complete form in the table *post* No. 62.

41. In the calculation of the sharp seminvariants by elimination as above, it will be noticed how unitary terms disappear: thus in combining $[d^2]$ and $[c^3]$ so as to get rid of g, the term bf disappears of itself, and we have as above the form

$(ce \infty c^3)$ beginning with the non-unitary term ce. We may, in fact, write $b=0$; we thus have

$$[d^2] = -2g - 30ce + 20d^2,$$
$$[c^3] = 3g - 45ce + 60d^2 + 90c^3,$$

giving $3[d^2] + 2[c^3] = -180(ce - d^2 - c^3)$, and then $ce - d^2 - c^3$, putting therein for c, d, e the values $c - b^2$, $d - 3bc + 2b^3$, $e - 4bd + 6b^2c - 3b^4$, gives the complete value *ut supra*, $ce - d^2 - b^2e + 2bcd - c^3$, and we thus see *à priori* that this contains no term bf, but in fact begins with ce. And in carrying out this process for any higher given weight, it is proper also to arrange the non-unitary terms not in AO but in CO, and then in each case beginning with the terms highest in CO and eliminating as many as possible of these terms we obtain the sharp seminvariant. Consider for instance the weight 12: taking the finals in AO, we have here

$$(m \infty g^2),\ (m \infty cf^2),\ (m \infty e^3),\ (m \infty b^2f^2),\ \ldots$$

the initials in CO are m, ck, dj, ei, ... and it might at first sight appear that the foregoing process of elimination would lead to the forms $(m \infty g^2)$, $(ck \infty cf^2)$, $(dj \infty e^3)$, $(ei \infty b^2f^2)$, ...; we in fact have the form $(m \infty g^2)$; and if from $(m \infty g^2)$ and $(m \infty cf^2)$ we eliminate m, we obtain the form $(ck \infty cf^2)$; but we cannot have a form $(dj \infty e^3)$ (for a form beginning with dj is of necessity of the degree 4 at least); what happens is that when from $(m \infty g^2)$, $(m \infty cf^2)$ and $(m \infty e^3)$ we eliminate m and ck, the next term dj disappears of itself, and (the following term ei not disappearing) the resulting form is $(ei \infty e^3)$: to obtain a form beginning with dj we must use the fourth form $(m \infty b^2f^2)$, and we thence obtain $(dj \infty b^2f^2)$. Arranging the initials in CO and the finals in AO, we thus have

$$\begin{array}{ccc} m & \text{———} & g^2 \\ ck & \text{———} & cf^2 \\ dj & \diagdown\ \diagup & e^3 \\ & \times & \\ ei & \diagup\ \diagdown & b^2f^2 \end{array}$$

that is, we have the sharp seminvariants $m \infty g^2$, $ck \infty cf^2$, $ei \infty e^3$, $dj \infty b^2f^2$, ...; these are the results given by the MacMahon linkage as will be explained further on, but I will first approach the question from a different side.

42. It has been seen that we have $\Delta, = \partial_b + 2b\partial_c + 3c\partial_d + \ldots$, as the annihilator of a seminvariant. Considering in the first place the entire set of terms, say for the weight 6, $g(\text{ao})b^6$, we assume for a seminvariant the sum of these each multiplied by an arbitrary coefficient; the number of coefficients is equal to the number of terms of $g(\text{ao})b^6$. Operating with Δ, we obtain a function of the next inferior weight 5, containing all the terms of $Dg(\text{ao})b^6$, that is, of $f(\text{ao})b^5$, each

term multiplied by a linear function (with mere numerical factors) of the arbitrary coefficients: the expression thus obtained must be identically $=0$; and we thus find between the arbitrary coefficients a number of linear relations equal to the number of terms $f(\text{ao})\,b^5$: these relations are independent; for it is only on the supposition that they are so, that the number of coefficients which remain arbitrary will be $11-7$, $=4$, agreeing with the number of the seminvariants $[d^2]$, $[c^3]$, $[b^2c^2]$, $[b^6]$; whereas if the relations were not independent, there would be a larger number of seminvariants.

But if, instead of the whole set $g(\text{ao})\,b^6$, we consider a set $(g \infty d^2)$ or say $(ce \infty c^3)$ and assume for a seminvariant the sum of these terms each multiplied by an arbitrary coefficient, then operating as before with Δ we obtain between the arbitrary coefficients a number of relations equal to that of the terms $D\,(ce \infty c^3)$, and if this be less by unity than the number of the terms of $ce \infty c^3$, say if we have $(1-D)\,(ce \infty c^3)=1$, then there will be a single seminvariant $ce \infty c^3$. We, in fact, find $(1-D)$: $(g \infty d^2)$, $(ce \infty c^3)$, $(d^2 \infty b^2c^2)$, $(c^3 \infty b^6)$, each $=1$, and thus establish the existence of the foregoing seminvariants $g \infty d^2$, $ce \infty c^3$, $d^2 \infty b^2c^2$, $c^3 \infty b^6$. And similarly if in any case we have $(1-D)\,(I \infty F)=2$ or any larger number, then we have 2 or more seminvariants $I \infty F$.

43. It will be convenient to write down at once the system of square diagrams for the several weights 2 to 16; each of these may theoretically be obtained by a direct process of calculation such as I exhibit for the weight 10, but the labour would be very great indeed, and I have in fact formed the squares for the weights 11 to 16, not in this manner but by the MacMahon linkage.

$w=2$

	b^2
c	1

$w=3$

	b^3
d	1

$w=4$

	c^2	b^4
e	1	
c^2		1

$w=5$

	bc^2	b^5
f	1	
cd		1

$w=6$

	d^2	c^3	b^2c^2	b^6
g	1			
ce		1		
d^2			1	
c^3				1

$w=7$

	bd^2	bc^3	b^3c^2	b^7
h	1			
cf		1		
de			1	
c^2d				1

$w = 8$

i	1						
cg		1					
df			1				
e^2				1			
c^2e					1		
cd^2						1	
c^4							1
	e^2	cd^2	b^2d^2	c^4	b^2c^3	b^4c^2	b^8

$w = 9$

j	1							
ch		1						
dg			1					
ef				1				
c^2f					1			
cde						1		
d^3							1	
c^3d								1
	be^2	d^3	bcd^2	b^3d^2	bc^4	b^3c^3	b^5c^2	b^9

$w = 10$

k	1											
ci		1										
dh			1									
eg				1								
c^2g						2						
f^2					1							
cdf								2				
ce^2							1					
d^2e									1			
c^3e										1		
c^2d^2											1	
c^5												1
	f^2	ce^2	b^2e^2	bd^3	c^2d^2	b^2cd^2	c^5	b^4d^2	b^2c^4	b^4c^3	b^6c^2	b^{10}

The subsequent squares $w=11$ to 16 are, for convenience, given at the end of the present memoir (pp. 331 et seq.).

44. It is to be observed that in each square the outside left-hand terms are the non-unitaries in CO and the outside bottom terms are the power-enders in AO. I have inside each square written down only the significant numbers, but we might fill up the whole square. For instance, when $w=7$, the filled-up square would be

h	1	2	3	4
cf	0	1	2	3
de	-1	0	1	2
c^2d	0	0	0	1
	bd^2	bc^3	b^3c^2	b^7

where in the first column the numbers relate to the sets $h \infty bd^2$, $cf \infty bd^2$, $de \infty bd^2$ and $c^2d \infty bd^2$ (this last set $c^2d \infty bd^2$ is non-existent since c^2d is in AO inferior to bd^2, i.e. as well for the set as for the diminished set, number of terms is $=0$, and we have for the compartment $0-0$, $=0$). And similarly for the remaining three columns. The process of thus filling up the whole square is a direct and non-tentative one, and the conclusions to which the numbers lead are as follows: col. 1, the final being bd^2, the initial cannot be c^2d, de or cf, but taking it to be h, we have the seminvariant $h \infty bd^2$. Col. 2, the final being bc^3 the initial cannot be c^2d or de, but taking it to be cf we have the seminvariant $cf \infty bc^3$: it may be added that the top number 2 shows that there are two seminvariants $h \infty bc^3$, these are of course the foregoing ones $h \infty bd^2$ and $cf \infty bc^3$. Similarly, col. 3, the final being b^3c^2, the initial cannot be c^2d, but taking it to be de, we have the seminvariant $de \infty b^3c^2$, and col. 4, we have the seminvariant $c^2d \infty b^7$.

For the several weights up to 9, we have simply units in the dexter diagonal of each square, viz. the non-unitaries in CO correspond to the power-enders in AO, or the sharp seminvariants are $c \infty b^2$, $d \infty b^3$, &c. See *post*, Table of Reductions, No. 62, which exhibits these correspondences.

45. For the weight 10, we have deviations: the figures 1 and 2 denote as follows:

$$
\begin{array}{rlll}
1-D & k & \infty f^2 & =1 \\
 & ci & \text{,, } ce^2 & \text{,, } 1 \\
 & dh & \text{,, } b^2e^2 & \text{,, } 1 \\
 & eg & \text{,, } bd^3 & \text{,, } 1 \\
 & f^2 & \text{,, } c^2d^2 & \text{,, } 1 \\
 & c^2g & \text{,, } b^2cd^2 & \text{,, } 2 \\
 & ce^2 & \text{,, } c^5 & \text{,, } 1 \\
 & cdf & \text{,, } b^4d^2 & \text{,, } 2 \\
 & d^2e & \text{,, } b^2c^4 & \text{,, } 1 \\
 & c^3e & \text{,, } b^4c^3 & \text{,, } 1 \\
 & c^2d^2 & \text{,, } b^6c^2 & \text{,, } 1 \\
 & c^5 & \text{,, } b^{10} & \text{,, } 1,
\end{array}
$$

and they indicate the sharp seminvariants $k \propto f^2$, $ci \propto ce^2$, &c.: where observe that the power-enders being in AO as before, the non-unitaries are not in CO, but we have inversions (c^2g, f^2) and (cdf, ce^2).

In particular, $(1-D)(f^2 \propto c^2d^2) = 1$ indicates the seminvariant $f^2 \propto c^2d^2$;

$$(1-D)(c^2g \propto b^2cd^2) = 2,$$

means in the first instance that there are 2 seminvariants $c^2g \propto b^2cd^2$, but here the set $c^2g \propto b^2cd^2$ includes as part of itself the set $f^2 \propto c^2d^2$; so that, if $c^2g \propto b^2cd^2$ is used to denote any particular form, then the general form is $c^2g \propto b^2cd^2$ plus an arbitrary multiple of $f^2 \propto c^2d^2$, and we have thus virtually a single form $c^2g \propto b^2cd^2$. And similarly, the set $cdf \propto b^4d^2$ includes as part of itself the set $ce^2 \propto c^5$; and thus the general form $cdf \propto b^4d^2$ is = particular form plus an arbitrary multiple of $ce^2 \propto c^5$, or we have virtually a single form $cdf \propto b^4d^2$.

I remark that it would be allowable to take as a standard form of $c^2g \propto b^2cd^2$, a form not containing any term in f^2, and similarly for the standard form of $cdf \propto b^4d^2$ a form not containing any term in ce^2; but this is not done in the tables.

46. The diagram for weight 10 is constructed by the following calculation; viz. in col. 1 we calculate $(1-D)(k \propto f^2)$ and for this purpose write down the terms of $k \propto f^2$, and $D(k \propto f^2)$ in CO: in col. 2 we calculate $(1-D)(ci \propto ce^2)$, and for this purpose write down the terms of $k \propto ce^2$ and $D(k \propto ce^2)$ in CO, the terms of $ci \propto ce^2$ and $D(ci \propto ce^2)$ being thence found by rejecting the terms k, bj and the term j at the head of the two halves of the column. So in col. 3 we calculate $(1-D)(dh \propto b^2e^2)$, and for this purpose write down the terms of $(k \propto b^2e^2)$ and $D(k \propto b^2e^2)$ in CO, and for $dh \propto b^2e^2$ and $D(dh - b^2e^2)$ reject the terms k, bj, ci, b^2i and j, bi at the head of the two halves of the column. And so for the remaining columns. It is to be remarked that there is in each successive column a continually increasing number of terms to be rejected; by a properly devised variation of the algorithm it would have been possible to avoid writing down these terms at all, but for greater clearness I have inserted them.

1		2		3		4		5		6		7		8		9		10		11		12	
k	j	k	j	k	j	k	j	k	j	k	j	k	j	k	j	k	j	k	j	k	j	k	
bj	bi	bj	bi	bj	bi	bj	bi	bj	bi	bj	bi	bj	bi	bj	bi	bj	bi	bj	bi	bj	bi	bj	b
ci	ch	ci	ch	ci	ch	ci	ch	ci	ch	ci	ch	ci	ch	ci	ch	ci	ch	ci	ch	ci	ch	ci	c
dh	dg	b^2i	b^2h	b^2i	b^2h	b^2i	b^2h	b^2i	b^2h	b^2i	b^2h	b^2i	b^2h	b^2i	b^2h	b^2i	b^2h	b^2i	b^2h	b^2i	b^2h	b^2i	b^2
eg	ef	dh	dg	dh	dg	dh	dg	dh	dg	dh	dg	dh	dg	dh	dg	dh	dg	dh	dg	dh	dg	dh	d
f^2		bch	bcg	bch	bcg	bch	bcg	bch	bcg	bch	bcg	bch	bcg	bch	bcg	bch	bcg	bch	bcg	bch	bcg	bch	bc
		eg	ef	b^3h	b^3g	b^3h	b^3g	b^3h	b^3g	b^3h	b^3g	b^3h	b^3g	b^3h	b^3g	b^3h	b^3g	b^3h	b^3g	b^3h	b^3g	b^2h	b^2
		bdg	bdf	eg	ef	eg	ef	eg	ef	eg	ef	eg	ef	eg	ef	eg	ef	eg	ef	eg	ef	eg	e
		c^2g	c^2f	bdg	bdf	bdg	bdf	bdg	bdf	bdg	bdf	bdg	bdf	bdg	bdf	bdg	bdf	bdg	bdf	bdg	bdf	bdg	bd
		f^2	be^2	c^2g	c^2f	c^2g	c^2f	c^2g	c^2f	c^2g	c^2f	c^2g	c^2f	c^2g	c^2f	c^2g	c^2f	c^2g	c^2f	c^2g	c^2f	c^2g	c^2
		bef	cde	b^2cg	b^2cf	b^2cg	b^2cf	b^2cg	b^2cf	b^2cg	b^2cf	b^2cg	b^2cf	b^2cg	b^2cf	b^2cg	b^2cf	b^2cg	b^2cf	b^2cg	b^2cf	b^2cg	b^2c
		cdf		f^2	be^2	f^2	be^2	f^2	be^2	b^4g	b^4f	b^4g	b^4f	b^4g	b^4f	b^4g	b^4f	b^4g	b^4f	b^4g	b^4f	b^4g	b^4
		ce^2		bef	cde	bef	cde	bef	cde	f^2	be^2	f^2	be^2	f^2	be^2	f^2	be^2	f^2	be^2	f^2	be^2	f^2	be
				cdf	b^2de	cdf	b^2de	cdf	b^2de	bef	cde	bef	cde	bef	cde	bef	cde	bef	cde	bef	cde	bef	cd
				b^2df	d^3	b^2df	bc^2e	b^2df	bc^2e	cdf	b^2de	cdf	b^2de	cdf	b^2de	cdf	b^2de	cdf	b^2de	cdf	b^2de	cdf	b^2d
				ce^2		bc^2f	d^3	bc^2f	d^3	b^2df	bc^2e	b^2df	bc^2e	b^2df	bc^2e	b^2df	bc^2e	b^2df	bc^2e	b^2df	bc^2e	b^2df	bc^2
				d^2e		ce^2	bcd^2	ce^2	bcd^2	bc^2f	b^3ce	bc^2f	b^3ce	bc^2f	b^3ce	bc^2f	b^3ce	bc^2f	b^3ce	bc^2f	b^3ce	bc^2f	b^3c
				b^2e^2		b^2e^2		b^2e^2	c^3d	b^3cf	d^3	b^3cf	d^3	b^3cf	b^5e	b^3cf	b^5e	b^3cf	b^5e	b^3cf	b^5e	b^3cf	b^5
						d^2e		d^2e		ce^2	bcd^2	ce^2	bcd^2	b^5f	d^3	b^5f	d^3	b^5f	d^3	b^5f	d^3	b^5f	d
						bcde		bcde		b^2e^2	b^3d^2	b^2e^2	b^3d^2	ce^2	bcd^2	ce^2	bcd^2	ce^2	bcd^2	ce^2	bcd^2	ce^2	bcd
						bd^3		c^3e		d^2e	c^3d	d^2e	c^3d	b^2e^2	b^3d^2	b^2e^2	b^3d^2	b^2e^2	b^3d^2	b^2e^2	b^3d^2	b^2e^2	b^3d
								bd^3		bcde	b^2c^2d	bcde	b^2c^2d	d^2e	c^3d	d^2e	c^3d	d^2e	c^3d	d^2e	c^3d	d^2e	c^3
								c^2d^2		b^3de		b^3de	bc^4	bcde	b^2c^2d	bcde	b^2c^2d	bcde	b^2c^2d	bcde	b^2c^2d	bcde	b^2c^2
										c^3e		c^3e		b^3de	b^4cd	b^2de	b^4cd	b^3de	b^4cd	b^3de	b^4cd	b^3de	b^4c
										b^2c^2e		b^2c^2e		c^3e	bc^4	c^3e	bc^4	c^3e	b^6d	c^3e	b^6d	c^3e	b^6
										bd^3		bd^3		b^2c^2e		b^2c^2e	b^3c^3	b^2c^2e	bc^4	b^2c^2e	bc^4	b^2c^2e	bc
										c^2d^2		c^2d^2		b^4ce		b^4ce		b^4ce	b^3c^3	b^4ce	b^3c^3	b^4ce	b^3c
										b^2cd^2		b^2cd^2		bd^3		bd^3		b^6e	b^5c^2	b^6e	b^5c^2	b^6e	b^5c
												bc^3d		c^2d^2		c^2d^2		bd^3		bd^3	b^7c	bd^3	b^7
												c^5		b^2cd^2		b^2cd^2		c^2d^2		c^2d^2		c^2d^2	b
														b^4d^2		b^4d^2		b^2cd^2		b^2cd^2		b^2cd^2	
														bc^3d		bc^3d		b^4d^2		b^4d^2		b^4d^2	
														c^5		b^3c^2d		bc^3d		bc^3d		bc^3d	
																c^5		b^2c^2d		b^3c^2d		b^3c^2d	
																b^2c^4		b^5cd		b^5cd		b^5cd	
																		c^5		b^7d		b^7d	
																		b^2c^4		c^5		c^5	
																		b^4c^3		b^2c^4		b^2c^4	
																				b^4c^3		b^4c^3	
																				b^6c^2		b^6c^2	
																						b^8c	
																						b^6	
$6-5=$		$11-10=$		$14-13=$		$14-13=$		$12-11=$		$19-17=$		$12-11=$		$19-17=$		$14-13=$		$14-13=$		$11-10=$		$6-5=$	
1		1		1		1		1		2		1		2		1		1		1		1	

47. As to the first of the foregoing inversions c^2g, f^2, it is proper to remark, that filling up two compartments of the square we have

c^2g		1	2
f^2		1	1
		c^2d^2	b^2cd^2

where the meaning of the numbers (1, 1) has to be considered: the first (1) seems to indicate a seminvariant $c^2g \propto c^2d^2$, but there is in fact no such form, what it really indicates is a form $0c^2g + f^2 \propto c^2d^2$, that is, $f^2 \propto c^2d^2$; and similarly, the second (1) seems to indicate a seminvariant $f^2 \propto b^2cd^2$, but there is in fact no such form, what it really indicates is $f^2 \propto c^2d^2 + 0b^2cd^2$, that is, $f^2 \propto c^2d^2$. The explanation is correct, but to make it perfectly clear some further developments would be required. The like remarks apply to the inversion cdf, ce^2.

The MacMahon Linkage. Art. Nos. 48 to 52.

48. We require the two theorems:

The first is: if a seminvariant S has q for its highest letter, then $\partial_q S$ is also a seminvariant.

The second has presented itself for unitariants (*ante* No. 31); for seminvariants the form is less simple, viz. If in any seminvariant, attending only to the terms of the highest degree, we therein change $b, c, d, e, \ldots$ into $b, 2c, 6d, 24e, \ldots$ and then diminish the letters (that is, replace each letter by the next preceding letter) and in the result so obtained change $b, c, d, e, \ldots$ into $b, \frac{c}{2}, \frac{d}{6}, \frac{e}{24}, \ldots$ we obtain a seminvariant. For instance $g - 6bf + 15ce - 10d^2$, in the terms of degree 2, making the numerical change we have $-720bf + 720ce - 360d^2$, and then diminishing the letters and making the numerical change, we obtain $-720\frac{e}{24} + 720\frac{bd}{6} - 360\frac{c^2}{4}$, that is,

$$-30(e - 4bd + 3c^2),$$

a seminvariant.

For the proof, observe that the equation $\Delta S = 0$, attending therein only to the terms of the highest degree, gives $(2b\partial_c + 3c\partial_d + \ldots)S' = 0$, if S' denote the terms of the highest degree: making the numerical change, this is $(b\partial_c + c\partial_d + \ldots)S''$, if S'' is what S' becomes thereby; diminishing the letters, this is $(\partial_b + b\partial_c + \ldots)S''' = 0$, if S'''

is the diminished value of S'', and finally making the numerical change, if T be what S''' becomes on writing therein $b, \frac{c}{2}, \frac{d}{6}, \ldots$ for $b, c, d, \ldots$, this gives

$$(\partial_b + 2b\partial_c + \ldots)\, T = 0,$$

viz. T is a seminvariant.

49. Assume that, for the weights up to a certain weight w, the forms of the sharp seminvariants are known: and for the weight w consider a seminvariant $I\,(\text{ca})\,F$: here if I be given, the first theorem establishes a limit F' such that F is in AO not higher than F'. For instance, when $w = 12$, if $I = dj$, the coefficient of j as being a seminvariant can only be $d \propto b^3$, and thus the seminvariant contains a term b^3j, or the final term F must be in AO not higher than b^3j; the degree is thus $= 4$ at least.

Similarly, if F be given, then the second theorem determines a limit I' such that I is in CO not lower than I'. Thus when $w = 12$, as before, if $F = b^4cd^2$, then diminishing the letters we have bc^2, a term belonging to $f \propto bc^2$; the diminished form has thus terms $a^4(a^2f,\ bc^2)$, so that augmenting these the seminvariant has terms $b^4(b^2g,\ cd^2)$ and thus the initial term I is in CO not lower than b^6g.

50. A limit for I or F, when the other is given, can also in some cases be found as follows: Considering a seminvariant of the weight w as before, and denoting its extent and degree by σ and δ respectively, then we have $\sigma\delta - 2w = 0$ or positive; that is, $\sigma\delta = 2w$ at least; here given I, we have σ, and then $\delta = \frac{2w}{\sigma}$ at least; and given F we have δ, and then $\sigma = \frac{2w}{\delta}$ at least.

51. We may now explain the MacMahon linkage; for a given weight, we write down in two columns the initials or non-unitaries in CO, and the finals or power-enders in AO: by what precedes, it appears that we cannot combine the terms of the one column each with the term opposite to it in the other column; what we do is: beginning with the top of the column of initials, we combine successively each term with the highest admissible term in the column of finals: or beginning with the bottom of the column of finals, we combine successively each term with the lowest admissible term in the column of initials.

52. For the weight 12, the linkage is

read downwards.

shown by	not in AO higher than			lower than not in CO	shown by
		m	g^2	bl	$k \infty f^2$
$(c \infty b^2)\, k$	b^2k	ck	cf^2	b^2k	j ,, be^2
$(d$,, $b^3)\, j$	b^3j	dj	e^3	bdi	ch ,, d^3
$(e$,, $c^2)\, i$	c^2i	ei	b^2f^2	b^3j	i ,, e^2
$(c^2$,, $b^4)\, i$	b^4i	c^2i	bde^2	b^2dh	cg ,, cd^2
$(f$,, $bc^2)\, h$	bc^2h	fh	c^2e^2	b^2eg	df ,, b^2d^2
$(cd$,, $b^5)\, h$	b^5h	cdh	d^4	b^2f^2	e^2 ,, c^4
$(g$,, $d^2)\, g$	d^2g	g^2	b^2ce^2	b^4i	h ,, bd^2
$(ce$,, $c^3)\, g$	c^3g	ceg	bcd^3	b^3dg	cf ,, bc^3
$(d^2$,, $b^2c^2)\, g$	b^2c^2g	d^2g	c^3d^2	b^3ef	de ,, b^3
$(c^3$,, $b^6)\, g$	b^6g	c^3g	b^4e^2	b^5h	g ,, d^2
$(cf$,, $bc^3)\, f$	bc^3f	cf^2	b^3d^3	b^4df	ce ,, c^3
$(de$,, $b^3c^2)\, f$	b^3c^2f	def	$b^2c^2d^2$	b^4e^2	d^2 ,, b^2c^2
$(c^2d$,, $b^7)\, f$	b^7f	c^2df	c^6	b^3d^3	c^3 ,, b^6
$(e^2$,, $c^4)\, e$	c^4e	e^3	b^4cd^2	b^6g	f ,, bc^2
$(c^2e$,, $b^2c^3)\, e$	b^2c^3e	c^2e^2	b^2c^5	b^5de	cd ,, b^5
$(cd^2$,, $b^4c^2)\, e$	b^4c^2e	cd^2e	b^6d^2	b^7f	e ,, c^2
$(c^4$,, $b^8)\, e$	b^8e	c^4e	b^4c^4	b^6d^2	c^2 ,, b^4
$(d^3$,, $b^5c^2)\, d$	b^5c^2d	d^4	b^6c^3	b^8e	d ,, b^3
$(c^2d$,, $b^9)\, d$	b^9d	c^3d^2	b^8c^2	b^9d	c ,, b^2
$(c^5$,, $b^{10})\, c$	$b^{10}c$	c^6	b^{12}		

read upwards.

Thus, beginning at the top of the column of initials, m is to be linked with g^2, that is, we have $(m \infty g^2)$; ck with cf^2, that is, we have $(ck \infty cf^2)$; dj cannot be linked with e^3, for the final must be in AO not higher than b^3j, but it is linked with the highest term b^2f^2 for which this condition is satisfied, that is, we have $(dj \infty b^2f^2)$; ei is then linked with the highest admissible term e^3, that is, we have $(ei \infty e^3)$; and so on.

Or beginning at the bottom of the column of finals, b^{12} is linked with c^6, that is, we have $(c^6 \propto b^{12})$, b^8c^2 with c^3d^2, that is, we have $(c^3d^2 \propto b^8c^2)$; b^6c^3 cannot be linked with d^4, for the initial must be in CO not lower than b^8e, but it is linked with the lowest term c^4e for which this condition is satisfied, that is, we have $(c^4e \propto b^6c^3)$; and so on.

The Umbral Notation. Stroh's Theory. Art. Nos. 53 to 56.

53. Employing the umbræ α, β, γ, δ, ..., which are such that

$$\alpha = \beta = \gamma, \ \ldots, = b; \quad \alpha^2 = \beta^2 = \gamma^2, \ \ldots, = c; \quad \alpha^3 = \beta^3 = \gamma^3, \ \ldots, = d;$$

and so on, then for instance

$$(\alpha - \beta)^2 = \alpha^2 - 2\alpha\beta + \beta^2, \ = c - 2b^2 + c, \ = 2(c - b^2),$$

a seminvariant;

$$(\alpha - \beta)^2 (\alpha - \gamma) = \alpha^3 - 2\alpha^2\beta + \alpha\beta^2 - \alpha^2\gamma + 2\alpha\beta\gamma - \beta^2\gamma,$$

$$= d - 2bc + bc - bc + 2b^3 - bc, \ = d - 3bc + 2b^3,$$

a seminvariant: and so in general any rational and integral function of the differences of the umbræ developed and interpreted is a seminvariant. For the seminvariants of a given weight, e.g. $w = 6$, Dr Stroh* considers the function

$$\Omega^6 = (\alpha x + \beta y + \gamma z + \delta w + \epsilon t + \zeta u)^6,$$

where x, y, z, w, t, u are numbers the sum of which is $= 0$, or we may if we please have more than 6 such numbers: the expression is obviously a function of the differences of the umbræ and it is thus a seminvariant. To develop its value, observe that after expansion of the sixth power we have sets of similar terms, for instance $\alpha^6x^6 + \beta^6y^6 + \ldots$ which putting therein $\alpha^6 = \beta^6 = \gamma^6, \ \ldots = g$ become $= g \,.\, Sx^6$, and generally each set becomes equal to a literal term multiplied by a symmetric function of the x, y, z, w, ...; introducing capital letters to denote the elementary symmetric functions of these quantities, and recollecting that their sum is assumed to be $= 0$, say we have

$$1 + C\vartheta^2 + D\vartheta^3 + E\vartheta^4 + \ldots = 1 - x\vartheta \,.\, 1 - y\vartheta \,.\, 1 - z\vartheta \ldots,$$

(that is, $0 = Sx$, $+C = Sxy$, $-D = Sxyz$, &c.) then by aid of the Table VI (b) writing therein 0, C, D, E, F, G for b, c, d, e, f, g, we find

$$\Omega^6 = (\alpha x + \beta y + \gamma z \ldots)^6 = \alpha^6 Sx^6 + 6\alpha^5\beta Sx^5y + \text{\&c.},$$

* See the paper "Ueber die Symbolische Darstellung der Grundsyzyganten einer binären Form sechster Ordnung und eine Erweiterung der Symbolik von Clebsch," *Math. Ann.* t. XXXVI. (1890), pp. 262—303, in particular § 10, Das Formensystem einer Form unbegrenzt hoher Ordnung.

as shown in the following table:

				C^3	D^2	CE	G
1	g	Sx^6	=	-2	$+3$	$+6$	-6
$+6$	bf	Sx^5y	=	$+2$	-3	-6	$+6$
$+15$	ce	Sx^4y^2	=	-2	-3	$+2$	$+6$
$+20$	d^2	Sx^3y^3	=	$+1$	$+3$	-3	$+3$
$+30$	b^2e	Sx^4yz	=		$+3$	$+2$	-6
$+60$	bcd	Sx^3y^2z	=		-3	$+4$	-12
$+90$	c^3	$Sx^2y^2z^2$	=		$+1$	-2	-2
$+120$	b^3d	Sx^3yzw	=			-2	$+6$
$+180$	b^2c^2	Sx^2y^2zw	=			$+1$	$+9$
$+360$	b^4c	Sx^2yzwt	=				-6
$+720$	b^6	$Sxyzwtu$	=				$+1$
				$[d^2]$	$[c^3]$	$[b^2c^2]$	$[b^6]$

the numbers whereof are, it will be observed, identical with those of the foregoing table No. 33, relating to the MacMahon equation.

This is to be read

$$\Omega^6 = C^3[d^2] + D^2[c^3] + CE[b^2c^2] + G[b^6],$$

viz. Ω^6 is a linear function of C^3, D^2, CE and G, the coefficients of these, being given functions of (b, c, d, e, f, g), which given functions are the specific blunt seminvariants which have been already called $[d^2]$, $[c^3]$, $[b^2c^2]$ and $[b^6]$. And so in general, the developed value of Ω^w affords a complete definition of these specific blunt seminvariants of the weight w. Observe that α, β, γ, δ, ... are umbræ in nowise connected with the roots α, β, γ, δ, ... before made use of, and that B, C, D, ... are actual quantities in nowise connected with the symbolic capitals B, C, D, ... before made use of.

54. The capital and small letter symbols are conjugate to each other. It will be convenient to give here, in reference to subsequent investigations, a table of these conjugate forms up to the degree 6 and weight 15.

Table of Conjugates.

2	3	4	5	6	7	8	9	10	11	12	13	14	15
$C \quad b^2$	$D \quad b^3$	$C^2 \quad c^2$	$CD \quad bc^2$	$C^3 \quad d^2$	$C^2D \quad bd^2$	$C^4 \quad e^2$	$C^3D \quad be^2$	$C^5 \quad f^2$	$C^4D \quad bf^2$	$C^6 \quad g^2$	$C^5D \quad bg^2$	$C^7 \quad h^2$	$C^6D \quad bh^2$
		$E \quad b^4$	$F \quad b^5$	$D^2 \quad c^3$	$DE \quad bc^3$	$CD^2 \quad cd^2$	$D^3 \quad d^3$	$C^2D^2 \quad ce^2$	$CD^3 \quad de^2$	$C^3D^2 \quad cf^2$	$C^2D^3 \quad df^2$	$C^4D^2 \quad cg^2$	$C^3D^3 \quad dg^2$
				$CE \quad b^2c^2$	$CF \quad b^3c^2$	$C^2E \quad b^2d^2$	$CDE \quad bcd^2$	$C^3E \quad b^2e^2$	$C^2DE \quad bce^2$	$D^4 \quad e^3$	$C^3DE \quad bcf^2$	$CD^4 \quad ef^2$	$D^5 \quad f^3$
				$G \quad b^6$		$E^2 \quad c^4$	$C^2F \quad b^3d^2$	$D^2E \quad bd^3$	$DE^2 \quad cd^3$	$C^4E \quad b^2f^2$	$D^3E \quad be^3$	$C^5E \quad b^2g^2$	$C^4DE \quad bcg^2$
						$DF \quad b^2c^3$	$EF \quad bc^4$	$CE^2 \quad c^2d^2$	$C^2F \quad b^3e^2$	$CD^2E \quad bde^2$	$CDE^2 \quad cde^2$	$C^2D^2E \quad bdf$	$CD^3E \quad bef^2$
						$CG \quad b^4c^2$	$DG \quad b^3c^3$	$CDF \quad b^2cd^2$	$D^2F \quad b^2d^3$	$C^2E^2 \quad c^2e^2$	$C^4F \quad b^3f^2$	$C^3E^2 \quad c^2f^2$	$C^2DE^2 \quad cdf^2$
								$F^2 \quad c^5$	$CEF \quad bc^2d^2$	$E^3 \quad d^4$	$CD^2F \quad b^2de^2$	$D^2E^2 \quad ce^3$	$DE^3 \quad de^3$
								$C^2G \quad b^4d^2$	$CDG \quad b^3cd^2$	$C^2DF \quad b^2ce^2$	$C^2EF \quad bc^2e^2$	$CE^3 \quad d^2e^2$	$C^5F \quad b^3g^2$
								$EG \quad b^2c^4$	$FG \quad bc^5$	$DEF \quad bcd^3$	$E^2F \quad bd^4$	$C^3DF \quad b^2cf^2$	$C^2D^2F \quad b^2df^2$
										$CF^2 \quad c^3d^2$	$DF^2 \quad c^2d^3$	$D^3F \quad b^2e^3$	$C^3EF \quad bc^2f^2$
										$C^3G \quad b^4e^2$	$C^2DG \quad b^3ce^2$	$CDEF \quad bcde^2$	$D^2EF \quad bce^3$
										$D^2G \quad b^3d^3$	$DEG \quad b^2cd^3$	$C^2F^2 \quad c^3e^2$	$CE^2F \quad bd^2e^2$
										$CEG \quad b^2c^2d^2$	$CFG \quad bc^3d^2$	$EF^2 \quad cd^4$	$CDF^2 \quad c^2de^2$
										$G^2 \quad c^6$		$C^4G \quad b^4f^2$	$F^3 \quad d^5$
												$CD^2G \quad b^3de^2$	$C^3DG \quad b^3cf^2$
												$C^2EG \quad b^2c^2e^2$	$D^3G \quad b^3e^3$
												$E^2G \quad b^2d^4$	$CDEG \quad b^2cde^2$
												$DFG \quad bc^2d^3$	$C^2FG \quad bc^3e^2$
												$CG^2 \quad c^4d^2$	$EFG \quad cd^4$
													$DG^2 \quad c^3d^3$

55. We can, by means of the umbral notation, write down for the blunt seminvariants of a given weight (indefinite forms, not the above-mentioned specific forms) expressions far more simple than those which are given by the foregoing theories: we can, in fact, find without difficulty *monomial* umbral expressions; and in many cases obtain also the sharp forms. To illustrate this, I consider the weight 10: I write down for convenience the symbols of the sharp forms (though the knowledge of these is in nowise required) and I form a table as follows:

Sharp forms, finals in AO.		
$k \quad \infty f^2$	1	$(\alpha-\beta)^{10}$
ci „ ce^2	2	$(\alpha-\beta)^8(\alpha-\gamma)^2$
dh „ b^2e^2	3	$(\alpha-\beta)^8(\alpha-\gamma)(\alpha-\delta)$
eg „ bd^3	4	$(\alpha-\beta)^6(\alpha-\gamma)^3(\alpha-\delta)$
f^2 „ c^2d^2	5	$(\alpha-\beta)^6(\alpha-\gamma)^2(\alpha-\delta)^2$
c^2g „ b^2cd^2	6	$(\alpha-\beta)^6(\alpha-\gamma)^2(\alpha-\delta)(\alpha-\epsilon)$
ce^2 „ c^5	7	$(\alpha-\beta)^4(\alpha-\gamma)^2(\alpha-\delta)^2(\alpha-\epsilon)^2$
cdf „ b^4d^2	8	$(\alpha-\beta)^6(\alpha-\gamma)(\alpha-\delta)(\alpha-\epsilon)(\alpha-\zeta)$
d^2e „ b^2c^4	9	$(\alpha-\beta)^4(\alpha-\gamma)^2(\alpha-\delta)^2(\alpha-\epsilon)(\alpha-\zeta)$
c^3e „ b^4c^3	10	$(\alpha-\beta)^4(\alpha-\gamma)^2(\alpha-\delta)(\alpha-\epsilon)(\alpha-\zeta)(\alpha-\eta)$
c^2d^2 „ b^6c^2	11	$(\alpha-\beta)^4(\alpha-\gamma)(\alpha-\delta)(\alpha-\epsilon)(\alpha-\zeta)(\alpha-\eta)(\alpha-\theta)$
c^5 „ b^{10}	12	$(\alpha-\beta)^2(\alpha-\gamma)(\alpha-\delta)(\alpha-\epsilon)(\alpha-\zeta)(\alpha-\eta)(\alpha-\theta)(\alpha-\iota)(\alpha-\kappa)$

It will be observed that all the differences used are $\alpha-\beta$, $\alpha-\gamma$, ... containing each of them an α; hence in all the forms we have $\alpha^{10}, =k$; in $(\alpha-\beta)^{10}$, the lowest term (in AO) is $\alpha^5\beta^5, =f^2$; in $(\alpha-\beta)^8(\alpha-\gamma)^2$, the lowest term is $\alpha^4\beta^4.\gamma^2, =ce^2$; and so on, viz. in each case the lowest term is the final term of the sharp form set down in the same line.

56. The form $(\alpha-\beta)^{10}$ gives at once the sharp form $k \infty f^2$; we thus develop it:

α^{10}	$\alpha^9\beta$	$\alpha^8\beta^2$	$\alpha^7\beta^3$	$\alpha^6\beta^4$	$\alpha^5\beta^5$
β^{10}	$\alpha\beta^9$	$\alpha^2\beta^8$	$\alpha^3\beta^7$	$\alpha^4\beta^6$	
1	-10	$+45$	-120	$+210$	-252
$+1$	-10	$+45$	-120	$+210$	
$=2(k$	$-10bj$	$+45ci$	$-120dh$	$+210eg$	$-126f^2)$;

$(\alpha-\beta)^8(\alpha-\gamma)^2$ contains a term $\alpha^{10}, =k$ and thus gives a blunt form $k a o c e^2$; if instead of it we employ the form $(\alpha-\beta)^8(\alpha-\gamma)(\beta-\gamma)$, then here as before the lowest term is $\alpha^4\beta^4.\gamma^2, =ce^2$, but there is no term α^{10}: there is a term $\alpha^9\beta, =bj$, but as this cannot appear, we must have terms of this form destroying each other. The simplest mode of effecting the development is to write

$$(\alpha-\beta)^8(\alpha-\gamma)(\beta-\gamma)=(\alpha-\beta)^8\{\alpha\beta-\gamma(\alpha+\beta)+\gamma^2\}:$$

we may herein put at once $\gamma=b$, $\gamma^2=c$, and thus the form is

$$(\alpha-\beta)^8\{\alpha\beta-b(\alpha+\beta)+c\};$$

I develop thus:

$$\begin{array}{lrrrrrrrrrr}
(\alpha-\beta)^8 & 1, & -8, & +28, & -56, & +70, & -56, & +28, & -8, & +1, & \\
 & & +1, & -8, & +28, & -56, & +70, & -56, & +28, & -8, & +1 \\
\hline
(\alpha-\beta)^8(\alpha+\beta) & 1, & -7, & +20, & -28, & +14, & +14, & -28, & +20, & -7, & +1
\end{array}$$

$$\begin{array}{l}
1bj-8ci+28dh-56eg+70f^2 \\
+1\quad -8\quad +28\quad -56 \\[1ex]
-b\begin{pmatrix} 1j & -7bi & +20ch & -28dg & +14ef \\ +1 & -7 & +20 & -28 & +14 \end{pmatrix} \\[1ex]
+c\begin{pmatrix} 1i & -8bh & +28cg & -56df & +70e^2 \\ +1 & -8 & +28 & -56 & \end{pmatrix}
\end{array}$$

÷ − 14

k					
bj	+ 2	− 2		0	
ci	− 16		+ 2	− 14	+ 1
dh	+ 56			+ 56	− 4
eg	− 112			− 112	+ 8
f^2	+ 70			+ 70	− 5
b^2i		+ 14		+ 14	− 1
bch		− 40	− 16	− 56	+ 4
bdg		+ 56		+ 56	− 4
bef		− 28		− 28	+ 2
c^2g			+ 56	+ 56	− 4
cdf			− 112	− 112	+ 8
ce^2			+ 70	+ 70	− 5

± 23,

which, in fact, exhibits the calculation of the sharp form $ci\propto ce^2$. The disappearance of the term in bj will be noticed.

Instead of $(\alpha-\beta)^8(\alpha-\gamma)(\beta-\delta)$ which contains α^{10}, that is, k, we may take

$$(\alpha-\beta)^8(\gamma-\delta)^2,$$

that is,

$$(i-8bh+28cg-56df+35e^2)(c-b^2):$$

this is $ciaob^2e^2$, a blunt form; by subtracting from it $ci \infty ce^2$, we could obtain the next sharp form $dh \infty b^2e^2$; but this in passing; it does not appear that there is any monomial umbral expression for the last-mentioned form.

I do not stop to examine the next following forms, but pass on at once to the last of them; instead of the expression given, we may take the expression

$$(\alpha-\beta)^2(\gamma-\delta)^2(\epsilon-\zeta)^2(\eta-\theta)^2(\iota-\kappa)^2,$$

that is, $(c-b^2)^5$, which is in fact the sharp form $c^5 \infty b^{10}$.

Seminvariants of a given Degree: Generating Functions. Art. Nos. 57 to 59.

57. We may consider the seminvariants of a given degree, and arrange them according to their weights: thus in each case writing down the series of finals, and for a reason that will appear also the conjugates of these finals (see Table of Conjugates, *ante* No. 54).

For degree 2, or quadric seminvariants, we have

2	3	4	5	6 ...
$C,\ b^2$	$-$	$C^2,\ c^2$	$-$	$C^3,\ d^2$

there is here for every even weight (beginning with 2) a single form, and for every odd weight no form: the number of forms of the weight w is thus $=$ coeff. of x^w in $x^2\div(1-x^2)$, or writing for shortness $\mathbf{2}$ to denote $1-x^2$ (and similarly $\mathbf{3}, \mathbf{4}, \ldots$ to denote $1-x^3$, $1-x^4, \ldots$), say that for degree 2, Generating Function, *G.F.*, is $=x^2\div\mathbf{2}$.

For degree 3, or cubic seminvariants, we have

3	4	5	6	7 ...
$D,\ b^3$	$-$	$CD,\ bc^2$	$D^2,\ c^3$	$C^2D,\ bc^2$

the counting is most easily effected by means of the conjugate forms; these contain all of them the factor D, and omitting this factor we have all the combinations of C, D which make up the weight $w-3$, viz. for weight w, we have number of ways in which $w-3$ can be made up with the parts 2, 3: that is,

for degree 3, *G.F.* is $=x^3\div\mathbf{2}.\mathbf{3}$.

Similarly for degree 4 or quartic seminvariants, we have terms each containing E, and removing this factor, we have all the combinations of C, D, E which make up the weight $w-4$, viz.

for degree 4, *G.F.* is $=x^4\div\mathbf{2}.\mathbf{3}.\mathbf{4}$.

Thus for degrees

$$2, \qquad 3, \qquad 4, \qquad 5, \qquad 6, \qquad \ldots$$

the $G.F.$'s are

$$= x^2 \div \mathbf{2}, \quad x^3 \div \mathbf{2.3}, \quad x^4 \div \mathbf{2.3.4}, \quad x^5 \div \mathbf{2.3.4.5}, \quad x^6 \div \mathbf{2.3.4.5.6}, \ldots$$

58. We may analyse these results by separating the finals into classes. I use the expression $b, c, d, \ldots$ are discrete letters, meaning thereby that they are distinct letters, not of necessity consecutive but with any intervals between them. Thus deg. 3, if (b, c) are discrete letters, then the finals are b^3, and bc^2; deg. 4, if b, c, d are discrete letters, then the finals are b^4, bc^3, b^2c^2, and bcd^2; and so on, the number of classes being doubled at each step, as will presently appear for the weights 5 and 6 respectively.

I notice also a property of the conjugates of these classes; for b^3 and bc^2 themselves the conjugates are D and CD, and these occur as factors, D in the conjugate of every form of the class b^3 (for instance conjugates of c^3, d^3 are D^2, D^3), and CD in the conjugate of every form of the class bc^2 (for instance, the conjugates of bd^2, ce^2 are C^2D, C^2D^2); and the like in other cases, viz. for any class whatever the conjugate of the first or representative form occurs as a factor in the conjugates of the several other forms belonging to the same class.

59. With these explanations, the expressions for the several $G.F.$'s are obtained without difficulty, and we have

$$\begin{array}{llll} \text{deg. } 2, & \text{class } C, & b^2 & G.F. = x^2 \div \mathbf{2}, \\ \text{deg. } 3, & \quad ,, \; D, & b^3 & \quad ,, \; = x^3 \div \mathbf{3}, \\ & \quad ,, \; CD, & bc^2 & \quad ,, \; = x^5 \div \mathbf{2.3}; \end{array}$$

we ought here to have

$$\begin{array}{lll} x^3 \div \mathbf{2.3} = & x^3 \div \mathbf{3} + x^5 \div \mathbf{2.3}, & \text{viz. in verification} \\ \hline x^3 & = x^3 . \mathbf{2} = x^3 - x^5 & \\ & \;\; + x^5 \qquad + x^5 & \\ & \qquad\qquad = x^3; & \end{array}$$

$$\begin{array}{llll} \text{deg. } 4, & \text{class } E, & b^4 & G.F. = x^4 \div \mathbf{4}, \\ & \quad ,, \; DE, & bc^3 & \quad ,, \; = x^7 \div \mathbf{3.4}, \\ & \quad ,, \; CE, & b^2c^2 & \quad ,, \; = x^6 \div \mathbf{2.4}, \\ & \quad ,, \; CDE, & bcd^2 & \quad ,, \; = x^9 \div \mathbf{2.3.4}; \end{array}$$

we ought here to have

$$\begin{array}{lll} x^4 \div \mathbf{2.3.4} = & x^4 \div \mathbf{4} + x^7 \div \mathbf{3.4} + x^6 \div \mathbf{2.4} + x^9 \div \mathbf{2.3.4}, & \text{viz. in verification} \\ \hline x^4 & = x^4 . \mathbf{2.3} = x^4 - x^6 - x^7 + x^9 & \\ & \;\; + x^7 . \mathbf{2} \qquad\qquad + x^7 - x^9 & \\ & \;\; + x^6 . \mathbf{3} \qquad + x^6 \qquad - x^9 & \\ & \;\; + x^9 \qquad\qquad\qquad + x^9 & \\ & \qquad\qquad = x^4; & \end{array}$$

deg. 5, class	F,	b^5	$G.F. = x^5 \div 5$,
	EF,	bc^4	$x^9 \div 4.5$,
	DF,	b^2c^3	$x^8 \div 3.5$,
	CF,	b^3c^2	$x^7 \div 2.5$,
	DEF,	bcd^3	$x^{12} \div 3.4.5$,
	CEF,	bc^2d^2	$x^{11} \div 2.4.5$,
	CDF,	b^2cd^2	$x^{10} \div 2.3.5$,
	$CDEF$,	$bcde^2$	$x^{14} \div 2.3.4.5$;

and for the sum of the eight terms

$$G.F. = x^5 \div 2.3.4.5,$$

which may be verified as before.

deg. 6, class	G,	b^6	$G.F. = x^6 \div 6$,
	FG,	bc^5	$x^{11} \div 5.6$,
	EG,	b^2c^4	$x^{10} \div 4.6$,
	DG,	b^3c^3	$x^9 \div 3.6$,
	CG,	b^4c^2	$x^8 \div 2.6$,
	EFG,	bcd^4	$x^{15} \div 4.5.6$,
	DFG,	bc^2d^3	$x^{14} \div 3.5.6$,
	CFG,	bc^3d^2	$x^{13} \div 2.5.6$,
	DEG,	b^2cd^3	$x^{13} \div 3.4.6$,
	CEG,	$b^2c^2d^2$	$x^{12} \div 2.4.6$,
	CDG,	b^3cd^2	$x^{11} \div 2.3.6$,
	$DEFG$,	$bcde^3$	$x^{18} \div 3.4.5.6$,
	$CEFG$,	bcd^2e^2	$x^{17} \div 2.4.5.6$,
	$CDFG$,	bc^2de^2	$x^{16} \div 2.3.5.6$,
	$CDEG$,	b^2cde^2	$x^{15} \div 2.3.4.6$,
	$CDEFG$,	$bcdef^2$	$x^{20} \div 2.3.4.5.6$;

and for the sum of the sixteen terms

$$G.F. = x^6 \div 2.3.4.5.6,$$

which may be verified as before.

Reducible Seminvariants—Perpetuants. Art. Nos. 60 to 64.

60. Seminvariants of the degrees 2 and 3 are irreducible—or say they are perpetuants. Hence by what precedes, as regards perpetuants

$$\text{for degree } 2,\ G.F. = x^2 \div 2;$$
$$\text{for degree } 3,\ G.F. = x^3 \div 2.3.$$

For the degree 4 (if as before b, c, d denote discrete letters), then the finals are b^4, bc^3, b^2c^2 and bcd^2. For a final $b^4 = b^2.b^2$ or $b^2c^2 = b^2.c^2$, we have evidently a product of two quadric seminvariants ending in b^2 and b^2, or in b^2 and c^2, with the same final term as the quartic seminvariant; so that, considering the quartic seminvariants arranged with their finals in AO, and adding to such quartic seminvariant a proper numerical multiple of the product in question, we obtain a quartic seminvariant the final term whereof is in AO higher than the original final term b^4 or b^2c^2, and such quartic seminvariant is thus said to be reducible; a quartic seminvariant not thus reducible is a perpetuant. The quartic perpetuants are consequently those which end in bc^3 or bcd^2. The lowest form is that ending in bc^3, of the weight 7. Taking the sum of the $G.F.$'s for the forms bc^3 and bcd^2 respectively, the $G.F.$ for quartic perpetuants is

$$x^7 \div \mathbf{3 . 4} + x^9 \div \mathbf{2 . 3 . 4},$$

viz. this is

$$x^7 (1 - x^2) + x^9 \div \mathbf{2 . 3 . 4},$$

or finally

$$G.F. = x^7 \div \mathbf{2 . 3 . 4}.$$

As an instance of a reduction, we have

$$(d^2 \propto b^2c^2) - (c \propto b^2)(e \propto c^2) = (ce \propto c^3),$$

viz. this is

$$(d \propto b^2c^2) = (c - b^2)(e - 4bd + 3c^2) - (ce - d^2 - b^2e + 2bcd - c^3).$$

We have also

$$(d^2 \propto b^2c^2) = (d \propto b^3)^2 + 4 (c \propto b^2)^3,$$

viz.

$$(d \propto b^2c^2) = (d - 3bc + 2b^3)^2 + 4 (c - b^2)^3,$$

but this is *not* a reduction, there are on the right-hand side terms of the degree 6, which is higher than the degree of the seminvariant $d^2 \propto b^2c^2$. In general, we say that a seminvariant of any given degree is reducible when we can, by adding to it products *of its own degree* of seminvariants of inferior degrees, reduce it to a seminvariant the final of which is in AO higher than the original final.

61. For the degree 5 (taking b, c, d, e to denote discrete letters), if the final be b^5, bc^4, b^2c^3, b^3c^2, bc^2d^2 or b^2cd^2, then the seminvariant will be reducible; a perpetuant must have therefore a final bcd^3 or $bcde^2$. But it is not true that every quintic seminvariant with either of these finals is a perpetuant. To explain this, observe that the first mentioned six finals are some of them in one way only, some of them in two ways, expressible as a product of power-enders, or say they are singly, or else doubly, composite: viz. we have

$$b^5 = b^2 . b^3 ; \quad bc^4 = c^2 . bc^2 ; \quad b^2c^3 = b^2 . c^3 ; \quad b^3c^2 = c^2 . b^3 = b^2 . bc^2 ;$$

$$bc^2d^2 = c^2 . bd^2 = d^2 . bc^2 ; \quad b^2cd^2 = b^2 . cd^2.$$

For a doubly composite form, for instance b^3c^2, forming first the product of the quadric and cubic seminvariants ending in c^2, b^3 respectively, and secondly the product of the quadric and cubic seminvariants ending in b^2 and bc^2 respectively, we have two products each with the final b^3c^2, and forming a linear combination so as to eliminate this term b^3c^2, we have thus it may be a quintic seminvariant with a final such as bcd^3 or $bcde^2$, and the process then furnishes a reduction of such a quintic seminvariant. Or on the other hand, it may be that the finals of the degree 5 all of them disappear, and we have a relation between products of the form in question (i.e. of a quadric and a cubic seminvariant) and seminvariants of a degree inferior to 5, say this is a quintic syzygy.

In particular, a non-composite final first presents itself for the weight 12, viz. here the finals are b^2ce^2, bcd^3, c^3d^2, the last of these is doubly composite, and it furnishes a reduction of bcd^3. For the weight 13, the finals are b^3f^2, b^2de^2, bc^2e^2, bd^4, c^2d^3 which are each of them singly or doubly composite: for the weight 14, they are b^2cf^2, b^2c^3, $bcde^2$, c^3e^2 and cd^4, and here the doubly composite form furnishes a reduction of $bcde^2$. For the weight 15, we have a final bce^3 which gives a quintic perpetuant. I have, in fact, in my paper "A Memoir on Seminvariants," *American Journal of Mathematics*, vol. VII. (1885), pp. 1—25, [828], worked out the theory of quintic syzygies and perpetuants, and subsequently connecting this with the present theory of finals, I succeeded in showing that, when the doubly composite final contains a b, then there is not a reduction but a syzygy; we thus have

$$G.F. \text{ for finals } b^3c^2,\ b^3d^2,\ \ldots = x^7 \div 2,$$

$$\text{,,} \qquad \text{,,} \qquad bc^2d^2, \qquad \ldots = x^{11} \div 2\,.\,4,$$

whence for the two forms

$$G.F. \text{ is } x^7 \div 2 + x^{11} \div 2\,.\,4 = \{x^7(1-x^4) + x^{11}\} \div 2\,.\,4,$$

or say for S_5, the number of quintic syzygies $G.F.$ is $= x^7 \div 2\,.\,4$.

I further satisfied myself that the finals for the quintic perpetuants are $bc0e^3$, and $bc0ef^2$, viz. the b, c, e, f being discrete letters, the interposed 0 denotes that the c and e are not consecutive letters. The conjugates of these forms contain the factors D^2EF and CD^2EF respectively, and it hence appears that the $G.F.$'s are $= x^{15} \div 3\,.\,4\,.\,5$ and $x^{17} \div 2\,.\,3\,.\,4\,.\,5$; adding these, we find

$$\text{for quintic perpetuants } G.F. \text{ is } = x^{15} \div 2\,.\,3\,.\,4\,.\,5,$$

which expression was given in the memoir just referred to: the result was obtained by investigating in the first instance an expression for S_5, the number of quintic syzygies of a given weight. The course of Stroh's investigation to be presently given is different; he determines directly the number of perpetuants, and we may if we please use conversely this result to obtain the number of syzygies.

62. The foregoing theory of reduction is independent of the form of the seminvariants, which may be blunt or sharp at pleasure: the actual formulæ will of course be different, and they are very much more simple for the sharp seminvariants, viz. here in many cases a seminvariant is found to be equal to a product of seminvariants of inferior degrees. I subjoin the following table of the reduction of the several sharp seminvariants up to the weight 12; the forms referred to are the tabulated forms, and to mark that this is so I write down in each case the numerical coefficients of the initial and final terms, viz. instead of $c \infty b^2$, $d \infty b^3$, &c., I write $c \infty -b^2$, $d \infty 2b^3$, &c. As appears by the table, these are for shortness denoted by C, D respectively, and so for weight 4, the forms are called E, E_2, for weight 5, F, F_2, for weight 6, G, G_2, G_3, G_4, and so on, the unsuffixed letters having thus an implied suffix, not 0 but 1. The table is

Table of Reductions.

$w=$			
2	$c \infty -b^2$	C	
3	$d \infty 2b^3$	D	
4	$e \infty 3c^2$	E	
	c^2 ,, b^4	E_2	$= C_2$
5	$f \infty -6bc^2$	F	
	cd ,, $-2b^5$	F_2	$= CD$
6	$g \infty -10d^2$	G	
	ce ,, $-c^3$	G_2	
	d^2 ,, $-3b^2c^2$	G_3	$= CE-G_2$
	c^3 ,, $-b^6$	G_4	$= C^3$
7	$h \infty 20bd^2$	H	
	cf ,, $3bc^3$	H_2	
	de ,, $6b^3c^2$	H_3	$= DE$
	c^2d ,, $2b^7$	H_4	$= C^2D$
8	$i \infty 35e^2$	I	
	cg ,, $2cd^2$	I_2	
	$3df$,, $10b^2d^2$	I_3	$= CG-I_2$
	e^2 ,, $9c^4$	I_4	$= E^2$
	c^2e ,, b^2c^3	I_5	$= CG_2$
	cd^2 ,, $3b^4c^2$	I_6	$= CG_3$
	c^4 ,, b^8	I_7	$= C_4$
9	$j \infty -70be^2$	J	
	$2ch$,, $-20d^3$	J_2	
	dg ,, $-4bcd^2$	J_3	
	ef ,, $-20b^3d^2$	J_4	$= \frac{1}{2}(2CH-J_2-7J_3)$
	$2c^2f$,, $-3bc^4$	J_5	$= \frac{1}{6}(EF-J_4)$
	cde ,, $-2b^3c^3$	J_6	$= DG_2$
	d^3 ,, $-6b^5c^2$	J_7	$= DG_3$
	c^3d ,, $-2b^9$	J_8	$= C^3D$
10	$k \infty -12bf^2$	K	
	ci ,, $-5ce^2$	K_2	
	$4dh$,, $-35b^2e^2$	K_3	$= CI-K_2$
	$16eg$,, $-80bd^3$	K_4	
	f^2 ,, $-32c^2d^2$	K_5	$= \frac{1}{15}(16EG-K_4)$
	c^2g ,, $-2b^2cd^2$	K_6	$= CI_2$
	ce^2 ,, $-3c^5$	K_7	$= EG_2$
	$3cdf$,, $-10b^4d^2$	K_8	$= CI_3$
	d^2e ,, $-9b^2c^4$	K_9	$= EG_3$
	c^3e ,, $-b^4c^3$	K_{10}	$= C^2G_2$
	c^2d^2 ,, $-3b^6c^2$	K_{11}	$= C^2G_3$
	c^5 ,, $-b^{10}$	K_{12}	$= C_5$

$w=$			
11	$l \infty 252bf^2$	L	
	$2cj$,, $35de^2$	L_2	
	di ,, $10bce^2$	L_3	
	eh ,, $20cd^3$	L_4	
	$16c^2h$,, $-70b^3e^2$	L_5	$= -DI+L_3+2L_4$
	fg ,, $160b^2d^3$	L_6	$= 8CJ_2-L_5$
	cef ,, $-2bc^2d^2$	L_7	$= -\frac{1}{30}(FG-L_6)$
	cdg ,, $4b^3cd^2$	L_8	$= DI_2$
	d^2f ,, $3bc^5$	L_9	$= \frac{1}{2}(FG_2-L_7)$
	$12c^3f$,, $-20b^5d^2$	L_{10}	$= -DI_3+3L_9$
	de^2 ,, $18b^3c^4$	L_{11}	$= DE^2$
	c^2de ,, $2b^5c^3$	L_{12}	$= CDG_2$
	cd^3 ,, $6b^7c^2$	L_{13}	$= CDG_3$
	c^4d ,, $2b^{11}$	L_{14}	$= C^4D$
12	$m \infty 462g^2$	M	
	$3ck$,, $42cf^2$	M_2	
	ei ,, $15e^3$	M_3	
	$15dj$,, $378b^2f^2$	M_4	$= 3CK-M_2$
	$25fh$,, $175bde^2$	M_5	
	g^2 ,, $125c^2e^2$	M_6	$= \frac{1}{21}(25EI-25M_3-4M_5)$
	ceg ,, d^4	M_7	$= \frac{1}{100}(G^2-M_6)$
	c^2i ,, $5b^2ce^2$	M_8	$= CK_2$
	$5d^2g$,, $20bcd^3$	M_9	$= -3GG_2+5EI_2-2M_7$
	cf^2 ,, $20c^3d^2$	M_{10}	$= GG_2-M_7$
	$4cdh$,, $35b^4e^2$	M_{11}	$= CK_3$
	$18def$,, $80b^3d^3$	M_{12}	$= \frac{1}{10}(10CK_4-160M_7-32M_9+54M_{10})$
	e^3 ,, $36b^2c^2d^2$	M_{13}	$= \frac{1}{8}(9CK_5-9M_{10}-M_{12})$
	c^2e^2 ,, c^6	M_{14}	$= G_2{}^2$
	c^3g ,, $2b^4cd^2$	M_{15}	$= C^2I_2$
	cd^2e ,, $3b^2c^5$	M_{16}	$= EI_5-M_{14}$
	$3c^2df$,, $10b^6d^2$	M_{17}	$= CK_8$
	d^4 ,, $9b^4c^4$	M_{18}	$= G_2{}^3$
	c^4e ,, $-3b^6c^3$	M_{19}	$= C^3G_2$
	c^3d^2 ,, $3b^8c^2$	M_{20}	$= C^3G$
	c^6 ,, b^{12}	M_{21}	

Where no reduction is given, the form is irreducible, i.e. it is a perpetuant.

63. As to these reductions, it may be observed that in very many cases we have the sharp seminvariant given as an actual product $E_2 = C^2$, $F_2 = CD$, $G_4 = C^3$, &c. We have next other reductions such as $G_3 = CE - G_2$, where on the right-hand side there is a single product; this has a final the same as that of the seminvariant which is to be reduced, so that, eliminating this term from the seminvariant and product in question, we have an expression which must be a linear combination (with numerical coefficients) of the preceding seminvariants of the same weight. To take a less simple example, $L_5 = -DI + L_3 + 2L_4$; here $L_5 = -fg + 16c^2h \ldots - 70b^3e^2$, and $DI = (d - 3bc + 2b^3)(i \ldots + 35e^2)$ has the final $+70b^3e^2$. The verification is

$$\begin{array}{rcl} -DI &=& -di \qquad\qquad\qquad\qquad \ldots - 70b^3e^2 \\ +L_3 &=& di - 2eh + fg \\ -2L_4 &=& \qquad\quad 2eh - 2fg \\ \hline L_5 &=& \qquad\qquad\quad - fg \ldots - 70b^3e^2. \end{array}$$

The only case in which we have on the right-hand side two products is $(d^2g \propto bcd^3)$, $M_9 = -3GG_2 + 5EI_2 - 2M_7$; viz. here the final of M_9 is bcd^3 which is incomposite (viz. it is not the product of two power-enders), this is in fact the first instance of a quintic seminvariant with an incomposite final and which is nevertheless reducible. For observe, the next seminvariant M_{10} has the final c^3d^2, which is a product in the two ways $c^2 . cd^2$ and $c^3 . d^2$; we have thus the two products $(e \propto c^2)(cg \propto cd^2)$ and $(ce \propto c^3)(g \propto d^2)$, that is, EI_2 and GG_2 with the same final c^3d^2, and combining them so as to eliminate this term we have an expression having the final bcd^3, and which is thus expressible in terms of M_9 and preceding seminvariants: the verification is

$$\begin{array}{rcl} -3GG_2 &=& -3ceg \qquad + 3d^2g \ldots + 60bcd^3 - 30c^3d^2 \\ +5EI_2 &=& +5ceg \qquad\qquad\qquad\quad - 40bcd^3 + 30c^3d^2 \\ -2M_7 &=& -2ceg + 2cf^2 + 2d^2g \\ \hline M_9 &=& \qquad\quad 2cf^2 + 5d^2g \ldots + 20bcd^3. \end{array}$$

64. I annex to this a table (taken from the square diagrams) for the initials and finals of the sharp seminvariants for the weights 13, 14, 15, and 16.

$w=$				
13	n	∞	bg^2	N
	cl	,,	df^2	N_2
	dk	,,	bcf^2	3
	ej	,,	be^3	4
	fi	,,	cde^2	5
	c^2j	,,	b^3f^2	6
	gh	,,	b^2de^2	7
	ceh	,,	bc^2e^2	8
	d^2h	,,	bd^4	9
	cfg	,,	c^2d^3	10
	cdi	,,	b^3ce^2	11
	deg	,,	b^2cd^3	12
	df^2	,,	bc^3d^2	13
	c^3h	,,	b^5e^2	14
	e^2f	,,	b^4d^3	15
	c^2ef	,,	b^3c^2d	16
	cd^2f	,,	bc^6	17
	c^2dg	,,	b^5cd^2	18
	cde^2	,,	b^3c^5	19
	c^4f	,,	b^7d^2	20
	d^3e	,,	b^5c^4	21
	c^3de	,,	b^7c^3	22
	c^2d^3	,,	b^9c^2	23
	c^5d	,,	b^{13}	24
14	o	∞	h^2	O
	cm	,,	cg^2	O_2
	ek	,,	ef^2	3
	dl	,,	b^2g^2	4
	fj	,,	bdf^2	5
	gi	,,	c^2f^2	6
	cei	,,	ce^3	7
	h^2	,,	d^2e^2	8
	c^2k	,,	b^2cf^2	9
	d^2i	,,	b^2e^3	10
	cfh	,,	$bcde^2$	11
	cg^2	,,	c^3e^2	12
	dfg	,,	cd^4	13
	cdj	,,	b^4f^2	14
	deh	,,	b^3de^2	15
	e^2g	,,	$b^2c^2e^2$	16
	c^2eg	,,	b^2d^4	17
	ef^2	,,	bcd^3	18
	c^2f^2	,,	c^4d^2	19
	c^3i	,,	b^4ce^2	20
	cd^2g	,,	b^3cd^3	21
	$cdef$	,,	$b^2c^3d^2$	22
	ce^3	,,	c^7	23
	c^2dh	,,	b^6e^2	24
	d^3f	,,	b^5d^3	25
	d^2e^2	,,	$b^4c^2d^2$	26
	c^3e^2	,,	b^2c^6	27
	d^4g	,,	b^6cd^2	28
	c^2d^2e	,,	b^4c^5	29
	c^3df	,,	b^8d^2	30
	cd^4	,,	b^6c^4	31
	c^5e	,,	b^8c^3	32
	c^4d^2	,,	$b^{10}c^2$	33
	c^7	,,	b^{14}	34

$w=$				
15	p	∞	ch^2	P
	cm	,,	dg^2	P_2
	el	,,	f^3	3
	dm	,,	bcg^2	4
	fk	,,	bef^2	5
	gj	,,	cdf^2	6
	cej	,,	de^3	7
	c^2l	,,	b^3g^2	8
	d^2j	,,	b^2df^2	9
	hi	,,	bc^2f^2	10
	cfi	,,	bce^3	11
	cgh	,,	bd^2e^2	12
	dfh	,,	c^2de^2	13
	e^2h	,,	d^5	14
	cdk	,,	b^3cf^2	15
	dei	,,	b^3e^2	16
	c^2eh	,,	b^2cde^2	17
	dg^2	,,	bc^3e^2	18
	efg	,,	bcd^4	19
	c^2fg	,,	c^3d^3	20
	c^3j	,,	b^5f^2	21
	e^4h	,,	b^4de^2	22
	cd^2h	,,	$b^3c^2e^2$	23
	$cdeg$	,,	b^3d^4	24
	f^3	,,	$b\ c^2d^3$	25
	cdf^2	,,	bc^4d^2	26
	ce^2f	,,	b^5ce^2	27
	d^3g	,,	b^4cd^3	28
	d^2ef	,,	$b^3c^3d^2$	29
	c^3ef	,,	bc^7	30
	c^4h	,,	b^7e^2	31
	c^2d^2f	,,	b^6d^3	32
	de^3	,,	$b^5c^2d^2$	33
	c^2de^2	,,	b^3c^6	34
	c^3dg	,,	b^7cd^2	35
	cd^3e	,,	b^5c^5	36
	c^5f	,,	b^9d^2	37
	d^5	,,	b^7c^4	38
	c^4de	,,	b^9c^3	39
	c^3d^3	,,	$b^{11}c^2$	40
	c^6d	,,	b^{15}	41

$w=$				
16	q	∞	i^2	Q
	co	,,	ch^2	Q_2
	em	,,	eg^2	3
	dn	,,	b^2h^2	4
	fl	,,	bdg^2	5
	gk	,,	bf^3	6
	cek	,,	c^2g^2	7
	hj	,,	cef^2	8
	i^2	,,	d^2f^2	9
	cgi	,,	e^4	10
	c^2m	,,	b^2cg^2	11
	c^3k	,,	b^2ef^2	12
	dej	,,	$bcdf^2$	13
	dfi	,,	bde^3	14
	e^2i	,,	c^3f^2	15
	ch^2	,,	c^2e^3	16
	dgh	,,	cd^2e^2	17
	cdl	,,	b^4g^2	18
	dej	,,	b^3df^2	19
	c^2ei	,,	$b^2c^2f^2$	20
	efh	,,	b^2ce^3	21
	eg^2	,,	$b^2d^2e^2$	22
	c^2fh	,,	bc^2de^2	23
	c^2g^2	,,	bd^5	24
	f^2g	,,	c^4e^2	25
	ce^2g	,,	c^2d^4	26
	c^3k	,,	b^4cf^2	27
	cd^2i	,,	b^4e^3	28
	$cdeh$	,,	b^3cde^2	29
	$cdfg$	,,	$b^2c^3e^2$	30
	d^2eg	,,	b^2cd^4	31
	cef^2	,,	bc^3d^3	32
	d^2f^2	,,	c^5d^2	33
	c^2dj	,,	b^6f^2	34
	d^3h	,,	b^5de^2	35
	c^3ej	,,	$b^4c^2e^2$	36
	c^3f^2	,,	b^4d^4	37
	de^2f	,,	$b^3c^2d^3$	38
	e^4	,,	$b^2c^4d^2$	39
	c^2e^3	,,	c^8	40
	c^4i	,,	b^6ce^2	41
	c^2d^2g	,,	b^5cd^3	42
	c^2def	,,	$b^4c^3d^2$	43
	cd^2e^2	,,	b^2c^7	44
	c^3dh	,,	b^8e^2	45
	cd^3f	,,	b^7d^3	46
	c^4e^2	,,	$b^6c^2d^2$	47
	d^4e	,,	b^4c^6	48
	c^5g	,,	b^8cd^2	49
	c^3d^2e	,,	b^6c^5	50
	c^4df	,,	$b^{10}d^2$	51
	c^2d^4	,,	b^8c^4	52
	c^6e	,,	$b^{10}c^3$	53
	c^5d^2	,,	$b^{12}c^2$	54
	c^8	,,	b^{16}	55

It would be interesting to complete this into a table of reductions as given for the weights 2 to 12.

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65. We can by means hereof establish, in regard to the specific blunt seminvariants, a general theory of reduction, or say a theory of the relations which exist between the seminvariants of a given degree and the powers and products of seminvariants of inferior degrees. To exhibit the form of these, it will be sufficient to take Ω a sum of two parts, $=\Omega'+\Omega''$, but the more general assumption is Ω a sum of any number of parts, $=\Omega'+\Omega''+\Omega'''+\ldots$. Taking then $\Omega=\Omega'+\Omega''$, where for the Ω' and Ω'' separately the sum of the $(x, y, z, \ldots)$ is $=0$, suppose that to the $(0, C, D, E, \ldots)$ of Ω there correspond $(0, C', D', E', \ldots)$ for Ω' and $(0, C'', D'', E'', \ldots)$ for Ω''. We have

$$\begin{aligned}
C &= C'+C'',\\
D &= D'+D'',\\
E &= E'+E''+C'C'',\\
F &= F'+F''+C'D''+C''D',\\
G &= G'+G''+C'E''+C''E'+D'D'',
\end{aligned}$$

the law of which is obvious.

66. We have, for instance,

$$\Omega^4=(\Omega'+\Omega'')^4,\quad =\Omega'^4+6\Omega'^2\Omega''^2+\Omega''^4,\ (\text{since } \Omega'=0,\ \Omega''=0),$$

that is,

$$\begin{aligned}
&(C'+C'')^2c^2 = C'^2c^2+6C'b^2\,.\,C''b^2+C''^2c^2\\
&+(E'+E''+C'C'')\,b^4+E'b^4 \qquad\qquad +E''b^4
\end{aligned}$$

where, and in what follows, c^2, b^4, b^2 are for shortness written instead of $[c^2]$, $[b^4]$, $[b^2]$ to denote the specific blunt seminvariants ending in c^2, b^4, b^2 respectively.

The terms in C'^2, C''^2, E', E'' are identical on each side of the equation and destroy each other: omitting these, we have only the terms in $C'C''$ which must be equivalent on the two sides of the equation, and comparing coefficients we find the relation

$$2c^2+b^4=6\,.\,b^2\,.\,b^2,$$

which of course means $2[c^2]+[b^4]=6[b^2][b^2]$, viz. this is

$$2(2e-8bd+6c^2)+(-4e+16bd+12c^2-48b^2c+24b^4)=6(-2c+2b^2)^2.$$

In like manner, for $\Omega^6=(\Omega'+\Omega'')^6$, we have

$$\begin{aligned}
&(C'+C'')^3 && .\,d^2\\
&+(D'+D'')^2 && .\,c^3\\
&+(C'+C'')(E'+E''+C'C'') && .\,b^2c^2\\
&+(G'+G''+C'E''+C''E'+D'D'') && .\,b^6
\end{aligned}$$

equal to

$$\left\{\begin{matrix} C'^3 & .\,d^2 \\ +D'^2 & .\,c^3 \\ +C'E' & .\,b^2c^2 \\ +G' & .\,b^6 \end{matrix}\right\} + 15\left\{\begin{matrix} C'^2.\,c^2 \\ +E'.\,b^4 \end{matrix}\right\} C''.\,b^2 + 20D'.\,b^3.\,D''.\,b^3 + 15C'^2.\,b^2\left\{\begin{matrix} C''^2.\,c^2 \\ +E''.\,b^4 \end{matrix}\right\} + \left\{\begin{matrix} C''^3 & .\,d^2 \\ +D''^2 & .\,c^3 \\ +C''E'' & .\,b^2c^2 \\ +G'' & .\,b^6 \end{matrix}\right\}.$$

Here omitting the terms which destroy each other and comparing the coefficients of the remaining terms, viz. $C'^2C'' + C''^2C'$, $D'D''$ and $C'E'' + C''E'$, we find the relations

$$\begin{aligned} 3d^2 + b^2c^2 &= 15\,.\,c^2\,.\,b^2, \\ 2c^3 + b^6 &= 20\,.\,b^3\,.\,b^3, \\ b^2c^2 + b^6 &= 15\,.\,b^4\,.\,b^2, \end{aligned}$$

which may be easily verified. There are on the right-hand side only products of two parts, but this is on account of the special assumption $\Omega = \Omega' + \Omega''$, a sum of two parts.

67. I write now

$$\begin{aligned} \Omega_2 &= \alpha x + \beta y &&, \ S_2 x = 0, \\ \Omega_3 &= \alpha x + \beta y + \gamma z &&, \ S_3 x = 0, \\ \Omega_4 &= \alpha x + \beta y + \gamma z + \delta w &&, \ S_4 x = 0, \\ \Omega_5 &= \alpha x + \beta y + \gamma z + \delta w + \epsilon t &&, \ S_5 x = 0, \\ \Omega_6 &= \alpha x + \beta y + \gamma z + \delta w + \epsilon t + \zeta u, && \ S_6 x = 0, \end{aligned}$$

and I say that Ω_2 and Ω_3 cannot break up: but that Ω_4 breaks up if it becomes a sum of $2 + 2$ terms (i.e. a sum of two parts Ω_2 for each of which $S_2 x = 0$, and so in other cases): that Ω_5 breaks up if it becomes a sum of $2 + 3$ terms, Ω_6 breaks up if it becomes a sum of $2 + 4$ or $2 + 2 + 2$ terms, or if it becomes a sum of $3 + 3$ terms: and similarly for any higher suffix.

The condition that Ω_4 may break up is $x + y = 0$, $x + z = 0$, or $y + z = 0$, or what is the same thing it is $\Pi_3(x + y) = 0$, where $\Pi_3(x + y)$ is the product of the three sums each containing x; this is a symmetric function, we in fact have

$$\Pi_3(x+y) = x^3 + x^2(y+z+w) + x(yz+yw+zw) + yzw, \ = xyz + xyw + xzw + yzw, \quad = -D.$$

The condition in order that Ω_5 may break up is $x + y = 0$, ..., or $w + t = 0$, say this is $\Pi_{10}(x+y) = 0$, where $\Pi_{10}(x+y)$ denotes the product of the ten sums $x + y$, ..., $w + t$. It will be shown that we have $\Pi_{10}(x+y) = -D^2E + CDF - F^2$.

The condition in order that Ω_6 may break up is, $x + y = 0$, ..., or $t + u = 0$, or again if $x + y + z = 0$, ..., or $x + t + u = 0$, viz. it is $\Pi_{15}(x+y)\,\Pi_{10}(x+y+z) = 0$, where $\Pi_{15}(x+y)$ is the product of the fifteen sums $x + y$, ..., $t + u$, and $\Pi_{10}(x+y+z)$ is the product of the ten sums $x + y + z$, ..., $x + t + u$, each containing x : $\Pi_{15}(x+y)$ and

$\Pi_{10}(x+y+z)$ are symmetric functions, the expressions for which will be given further on: the weights in the capital letters are 15 and 10 respectively. And similarly for Ω with any higher suffix, we have the condition that this may break up.

I introduce the factors $\Pi_4 x = E$, $\Pi_5 x = -F$, $\Pi_6 x = G$, ... respectively and write for

$$\begin{array}{ll} \Omega_4 & M_7 = \Pi_4 x \Pi_3 (x+y) = -DE \text{ as above,} \\ \Omega_5 & M_{15} = \Pi_5 x \Pi_{10}(x+y) = -F(-D^2E + CDF - F^2) \text{ as above,} \\ \Omega_6 & M_{31} = \Pi_6 x \Pi_{15}(x+y)\,\Pi_{10}(x+y+z), \\ & \vdots \end{array}$$

where observe that, for the even suffixes of Ω, the last factors $\Pi_3(x+y)$, $\Pi_{10}(x+y+z)$, ... denote the products of the sums $x+y$, $x+y+z$, ... which contain x, that is in each case the products of only half the whole number of such linear factors. The suffixes of M show the weights in the capital letters C, D, E, F, G, ... viz. these are $4+3$, $=7$, $5+10$, $=15$, $6+15+10$, $=31$, and so on; the law is obvious, and for Ω_n the weight is $=2^{n-1}-1$.

68. To explain the Strohian theory of perpetuants, I assume explicitly as presently appears. For perpetuants of any given degree δ, we consider in $\Omega_\delta{}^w$ ($w=\delta$ at least) the terms containing seminvariants of the given degree: for instance when $\delta = 4$, $w = 12$, these are

$$\begin{aligned} &C^4E \;.\, b^2f^2 \\ +\, &CD^2E \,.\, bde^2 \\ +\, &C^2E^2 \;.\, c^2e^2 \\ +\, &E^3 \quad .\, d^4, \end{aligned}$$

where the capital expressions all contain as factor the letter E of the weight 4. By making Ω to break up, it is assumed that *we obtain all the reductions of the seminvariants of the degree and weight in question; and every such seminvariant, if it be reducible, will be reduced by means of the resulting formulæ.* Now there are seminvariants which are not reducible by these formulæ: in the example just considered, the seminvariant bde^2 has the coefficient CD^2E containing the factor DE, $=xyzw(x+y)(x+z)(x+w)$ which vanishes when Ω_4 breaks up; so that, supposing Ω_4 to break up, the seminvariant bde^2 disappears from the formulæ, and we have no reduction of this seminvariant. And again it is assumed *that every seminvariant which does not in this way disappear from the equation is reducible.* The irreducible seminvariants are thus the seminvariants which, when Ω breaks up into a sum of two or more parts, disappear from the formulæ; viz. the seminvariants which thus disappear are the perpetuants.

69. In the case considered of quartic seminvariants, it has just been seen that, for the weight 12, bde^2 is a perpetuant; and so in general for the weight w, every quartic seminvariant, multiplied into a product of capitals which contains the factor DE, is a perpetuant: for the weight 7 the only term is $DE \,.\, bc^3$, viz. the product

of capitals is here $=DE$; and for any higher weight w we have products which are equal to DE multiplied into products of the weight $w-7$ in C, D, E: and we thus see that the $G.F.$ for quartic perpetuants is $=x^7 \div \mathbf{2}.\mathbf{3}.\mathbf{4}$.

70. For quintic perpetuants, we consider in $\Omega_5{}^w(w=5$ at least) the terms which contain quintic perpetuants; for instance, when $w=15$, the terms are

$$\begin{aligned} & C^5F \quad . b^3g^2 \\ + & C^2D^2F . b^2df^2 \\ + & C^3EF \; . bc^2f^2 \\ + & D^2EF \; . bce^3 \\ + & CE^2F \; . bd^2e^2 \\ + & CDF^2 \; . c^2de^2 \\ + & F^3 \qquad . d^5, \end{aligned}$$

where the functions of the capitals all contain the factor F; the finals b^3g^2, b^2df^2, ... are arranged in AO. Supposing Ω_5 to break up, we have an expression M, $=-D^2EF+CDF^2-F^3$, which is $=0$, and using this value of M to eliminate the term D^2EF which belongs to the seminvariant bce^3, the final whereof is highest in AO, viz. writing $D^2EF=-M+CDF^2-F^3$, the expression is

$$\begin{aligned} & C^5F \quad . b^3g^2 & \text{that is} \quad & C^5F \quad . b^3g^2 \\ + & C^2D^2F . b^2df^2 & + & C^2D^2F . b^2df^2 \\ + & C^3EF \; . bc^2f^2 & + & C^3EF \; . bc^2f^2 \\ + & (-M+CDF^2-F^3) . bce^3 & - & M \qquad . bce^3 \\ + & CE^2F \; . bd^2e^2 & + & CE^2F \; . bd^2e^2 \\ + & CDF^2 \; . c^2de^2 & + & CDF^2 \; . (c^2de^2+bce^3) \\ + & F^3 \qquad . d^5 & + & F^3 \qquad . (d^5-bce^3); \end{aligned}$$

and here when Ω_5 breaks up, we have $M=0$, that is, the seminvariant bce^3 disappears from the equation, and it is thus a perpetuant: but b^3g^2, b^2df^2, bc^2f^2 and the combinations $c^2de^2+bce^3$, and d^5-bce^3 are severally reducible.

The degree 15 is evidently the lowest degree for which there is an irreducible quintic seminvariant, and for any higher weight w the number of such seminvariants is equal to the number of capital terms which have the factor D^2EF, viz. this is equal to the number of terms weight $w-15$ which can be made up with C, D, E, F; and hence

for quintic perpetuants $G.F.=x^{15}\div \mathbf{2}.\mathbf{3}.\mathbf{4}.\mathbf{5}$.

71. For the degree 6, $M=\Pi_6 x\Pi_{15}(x+y)\,\Pi_{10}(x+y+z)$ is a function of the capitals of the weight 31, and we thence at once infer that

for sextic perpetuants $G.F.=x^{31}\div \mathbf{2}.\mathbf{3}.\mathbf{4}.\mathbf{5}.\mathbf{6}$.

But it is worth while to write down the expression for M: I do this, annexing to each term the seminvariant (i.e. final term) which belongs to it, arranging these final terms in AO; the value thus arranged is

$M =$		finals in AO
$+1$	D^4E^2FG	$bcei^3$
-2	CD^3EF^2G	$bdehi^2$
$+1$	$C^2D^2F^3G$	be^2gi^2
$+2$	D^2EF^3G	$befh^3$
-2	CDF^4G	bf^2gh^2
$+1$	F^5G	bg^5
-1	D^5EG^2	c^2di^3
$+1$	CD^4FG^2	cd^2hi^2
$+1$	$C^2D^2EFG^2$	$cdegi^2$
-4	$D^2E^2FG^2$	$cdfh^3$
-1	$C^3DF^2G^2$	ce^2fi^2
-1	$D^3F^2G^2$	ce^2h^3
$+4$	$CDEF^2G^2$	$cefgh^2$
$+1$	$C^2F^3G^2$	cf^3h^2
$+4$	EF^3G^2	cfg^4
-1	$C^2D^3G^3$	d^3g^2
$+4$	D^3EG^2	d^2eh^3.

It thus appears that the single sextic perpetuant of the weight 31 is $bcei^3$, and generally that, for any higher weight, the sextic perpetuants are such that the conjugate capital terms contain each of them the factor D^4E^2FG.

The like reasoning shows that

$$\text{for perpetuants of degree } n,\ G.F. \text{ is } = x^{2^{n-1}-1} \div \mathbf{2}.\mathbf{3}.\mathbf{4}\ldots n.$$

Investigation of the Values of the Foregoing Functions $\Pi_{10}(x+y)$, $\Pi_{15}(x+y)$ *and* $\Pi_{10}(x+y+z)$. Art. Nos. 72 to 74.

72. If x, y, z, w, t are the roots of a quintic equation, say

$$\lambda - x . \lambda - y . \lambda - z . \lambda - w . \lambda - t = (1, B, C, D, E, F \between \lambda, 1)^5 = 0,$$

we require the product $\Pi_{10}(x+y)$ of the sum of two roots in the particular case $B=0$. But in order to the determination of the expression for $\Pi_{10}(x+y+z)$, we require the value of $\Pi_{10}(x+y)$ in the general case, B any value whatever.

Writing

$$x = -\tfrac{1}{2}(\theta + \omega),$$

$$y = -\tfrac{1}{2}(\theta - \omega),$$

and therefore

$$\theta + x + y = 0,$$

we have

$$(\theta+\omega)^5 - 2B(\theta+\omega)^4 + 4C(\theta+\omega)^3 - 8D(\theta+\omega)^2 + 16E(\theta+\omega) - 32F = 0,$$

and the like equation with $-\omega$ for ω. Hence writing $\omega^2 = M$, we have

$$(\theta^5 - 2B\theta^4 + 4C\theta^3 - 8D\theta^2 + 16E\theta - 32F) + M(10\theta^3 - 12B\theta^2 + 12C\theta - 8D) + M^2(5\theta - 2B) = 0,$$

$$(5\theta^4 - 8B\theta^3 + 12C\theta^2 - 16D\theta + 16E) + M(10\theta^2 - 8B\theta + 4C) + M^2 . 1 = 0,$$

which are of the form $A + BM + CM^2 = 0$, $A' + B'M + C'M^2 = 0$, and give therefore by elimination of M the equation

$$-(CA' - C'A)^2 + (BC' - B'C)(AB' - A'B) = 0;$$

the left-hand side is here a function of θ of the degree 10 vanishing when $\theta + x + y = 0$, and which must therefore be, save as to a numerical factor, the product $\Pi_{10}(\theta + x + y)$. And we thus find

$$\Pi_{10}(\theta + x + y)$$

$$= -\left\{24\theta^5 - 48B\theta^4 + \begin{pmatrix} 56C \\ +16B^2 \end{pmatrix}\theta^3 + \begin{pmatrix} -72D \\ -24BC \end{pmatrix}\theta^2 + \begin{pmatrix} 64E \\ +32BD \end{pmatrix}\theta + \begin{pmatrix} -32BE \\ +32F \end{pmatrix}\right\}^2$$

$$+ \left\{40\theta^3 - 48B\theta^2 + \begin{pmatrix} 8C \\ +16B^2 \end{pmatrix}\theta + \begin{pmatrix} 8D \\ -8BC \end{pmatrix}\right\} . \left\{40\theta^7 - 112B\theta^6 + \begin{pmatrix} 136C \\ +80B^2 \end{pmatrix}\theta^5\right.$$

$$\left. + \begin{pmatrix} -120D \\ -200BC \end{pmatrix}\theta^4 + \begin{pmatrix} 0E \\ +192BD \\ +128C^2 \end{pmatrix}\theta^3 + \begin{pmatrix} 320F \\ -64BE \\ -526CD \end{pmatrix}\theta^2 + \begin{pmatrix} -256BF \\ +128CE \\ -128D^2 \end{pmatrix}\theta + \begin{pmatrix} 128CF \\ -128DE \end{pmatrix}\right\},$$

which is

$$= 1024\theta^{10} + \ldots + 1024(-F^2 + CDF + 2BEF - BC^2F - D^2E + BCDE),$$

and which therefore for $B = 0$ gives

$$\Pi_{10}(x + y) = -F^2 + CDF - D^2E.$$

73. Suppose now x, y, z, w, t, u are the roots of a sextic equation, say

$$\lambda - x . \lambda - y . \lambda - z . \lambda - w . \lambda - t . \lambda - u = (1, B, C, D, E, F, G \between \lambda, 1)^6 = 0.$$

Considering here the product $\Pi_{20}(x + y + z)$ of the sums of 3 roots, if $B = 0$, this will be a perfect square (for each sum $x + y + z$ is equal to $-$ a sum $(w + t + u)$) say it is the square of $\Pi_{10}(x + y + z)$, where the $x + y + z$ refers to the ten sums each containing x, and we wish to find this function $\Pi_{10}(x + y + z)$. Writing for the equation whose roots are y, z, w, t, u,

$$\lambda - y . \lambda - z . \lambda - w . \lambda - t . \lambda - u = (1, B', C', D', E', F' \between \lambda, 1)^5,$$

we have by what precedes $\Pi_{10}(\theta+y+z)=$ a function $(*\between\theta, 1)^{10}$, viz. this is the above-mentioned function with B', C', D', E', F' in place of the unaccented letters. Introducing a new root x and for λ writing as we may do θ, we have

$$\begin{aligned}\theta-x\,.\,\theta-y\,.\,\theta-z\,.\,\theta-w\,.\,\theta-t\,.\,\theta-u &= (\theta-x)\,.\,(1,\ B',\ C',\ D',\ E',\ F'\between\theta,\ 1)^5\\ &= (1,\ B,\ C,\ D,\ E,\ F,\ G\between\theta,\ 1)^6;\end{aligned}$$

that is, we have

$$\begin{array}{ll} B=B'-\ \ \theta \quad \text{or conversely,} & B'=B+\ \ \theta,\\ C=C'-B'\theta & C'=C+B\theta+\ \ \theta^2,\\ D=D'-C'\theta & D'=D+C\theta+B\theta^2+\ \ \theta^3,\\ E=E'-D'\theta & E'=E+D\theta+C\theta^2+B\theta^3+\ \ \theta^4,\\ F=F'-E'\theta & F'=F+E\theta+D\theta^2+C\theta^3+B\theta^4+\theta^5,\ \ =-\dfrac{G}{\theta},\\ G=\quad -F'\theta, & \end{array}$$

where I have retained B, but the value hereof is in fact $=0$. In the foregoing function $(*\between\theta, 1)^{10}$ with the accented letters, writing for these their values $B'=\theta$, $C'=C+\theta^2$, $D'=D+C\theta+\theta^3$, &c., which belong to $B=0$, we find

$$\begin{aligned}1024\Pi_{10}(\theta+y+z) = &-(48\theta^5+56C\theta^3+24D\theta^2+64E\theta+32F)^2\\ &+(16\theta^3+8C\theta+D)\,\{144\theta^7+264C\theta^5+72D\theta^4+128\,(C^2+E)\,\theta^3+192F\theta^2+128CE\theta\\ &\qquad\qquad +128\,(CF-DE)\},\end{aligned}$$

which equation divides by 64. Writing herein $\theta=x$, we have

$$\begin{aligned}16\Pi_{10}(x+y+z) = &-(6x^5+7Cx^3+3Dx^2+8Ex+4F)^2\\ &+(2x^3+Cx+D)\,\{18x^7+33Cx^5+9Dx^4+16\,(C^2+E)\,x^3+24Fx^2+16CEx+16\,(CF-DE)\},\end{aligned}$$

where $x^6+Cx^4+Dx^3+Ex^2+Fx+G$ is $=0$: the value ought, in virtue of this equation, to reduce itself to a mere function of the coefficients, and we in fact find that the equation is

$$16\Pi_{10}(x+y+z)=(16C^2-64E)\,(x^6+Cx^4+Dx^3+Ex^2+Fx)+16CDF-16D^2E-16F^2,$$

reducing itself to

$$-(16C^2-64E)\,G \qquad\qquad\qquad\qquad +16CDF-16D^2E-16F^2,$$

viz. dividing each side by 16, we have

$$\Pi_{10}(x+y+z)=4EG-C^2G-F^2+CDF-D^2E,$$

which is the required result. The equation $(\theta^2-1)^3=0$, for which

$$x,\ y,\ z,\ w,\ t,\ u=1,\ 1,\ 1,\ -1,\ -1,\ -1,$$

gives a numerical verification.

74. I find also, for the same value $B=0$, the function $\Pi_{15}(x+y)$. Writing, as before,

$$x=-\tfrac{1}{2}(\theta+\omega),$$
$$y=-\tfrac{1}{2}(\theta-\omega),$$

and therefore

$$\theta + x + y = 0,$$

we have

$$(\theta+\omega)^6 + 4C(\theta+\omega)^4 - 8D(\theta+\omega)^3 + 16E(\theta+\omega)^2 - 32F(\theta+\omega) + 64G = 0,$$

and the like equation with $-\omega$ for ω. Hence writing $\omega^2 = M$, we have

$$(\theta^6 + 4C\theta^4 - 8D\theta^3 + 16E\theta^2 - 32F\theta + 64G) + M(15\theta^4 + 24C\theta^2 - 24D\theta + 16E) + M^2(15\theta^2 + 4C) + M^3 = 0,$$

$$(6\theta^5 + 16C\theta^3 - 24D\theta^2 + 32E\theta - 32F) + M(20\theta^3 + 16C\theta - 8D) + M^2 . 6\theta = 0,$$

say these equations are $aM^3 + bM^2 + cM + d = 0$, $pM^2 + qM + r = 0$. Eliminating M, we have

$$\begin{array}{ll}
a^2 . r^3 & a = 1, \\
- ab . qr^2 & b = 15\theta^2 + \ \ 4C, \\
+ ac(-2pr^2 + q^2r) & c = 15\theta^4 + 24C\theta^2 - 24D\theta \ + 16E, \\
+ b^2 . pr^2 & d = \ \ \theta^6 + \ \ 4C\theta^4 - \ \ 8D\theta^3 + 16E\theta^2 - 32F\theta + 64G, \\
+ ad(3pqr - q^3) & \\
+ bc(-pqr) & p = \ \ 6\theta, \\
+ bd(-2p^2r + pq^2) & q = 20\theta^3 + 16C\theta \ - \ \ 8D, \\
+ c^2 . p^2r & r = \ \ 6\theta^5 + 16C\theta^3 - 24D\theta^2 + 32E\theta - 32F, \\
- cd . p^2q & \\
+ d^2 . p^3 = 0. &
\end{array}$$

The equation, as far as I have calculated it, is

$$-32768\theta^{15} - \ldots - 32768(-D^3G + F^3 - CDF^2 + D^2EF) = 0;$$

the left-hand side is here $= -32768\Pi_{15}(x+y)$; and we have therefore

$$\Pi_{15}(x+y) = -D^3G + F^3 - CDF^2 + D^2EF,$$

the required result. It may be remarked that, writing $G = 0$ and throwing out a factor $-F$, we have $-F^2 + CDF - D^2E$, which is the expression for $\Pi_{10}(x+y)$ in the quintic equation.

We have

$$\Pi_6 x \Pi_{15}(x+y)\Pi_{10}(x+y+z)$$
$$= G\{-D^3G + (F^2 - CDF + D^2E)F\}\{(4E - C^2)G - F^2 + CDF - D^2E\},$$

the developed expression whereof is the foregoing value

$$M = D^4E^2FG - 2CD^3EF^2G + \&c., \text{ ante No. 71.}$$

The Operators $P-\delta b$ *and* $Q-2\omega b$. Art. Nos. 75 to 84.

75. The analogous theory for non-unitariants is established, *ante* Nos. 24 *et seq.* For seminvariants, we have

$$P = b\partial_a + c\partial_b + d\partial_c + \dots,$$
$$Q = c\partial_b + 2d\partial_c + \dots,$$

or more definitely, if the seminvariant operated upon be of the degree δ, the weight ω and extent σ, say its highest letter is a_σ, $=p$, then

$$P = b\partial_a + c\partial_b + d\partial_c + \dots + q\partial_p,$$
$$Q = c\partial_b + 2d\partial_c + \dots + \sigma q\partial_p,$$

then we have

$$P-\delta b, \quad Q-2\omega b,$$

operators each of them of the deg. weight 1.1, viz. each of them operating upon a seminvariant S of the deg. weight $\delta.\omega$ gives a seminvariant S' of the deg. weight $\delta+1.\omega+1$; moreover, a new letter q is introduced, or say the extent is increased from σ to $\sigma+1$. For the proof, it is only necessary to show that $\Delta(P-b\delta)S$ and $\Delta(Q-2b\omega)$ are each $=0$, but it is unnecessary to do this, as the like proof has already been given for non-unitariants.

The two seminvariant operators were first considered in my paper "On a theorem relating to seminvariants," *Quart. Math. Journ.* t. XX. (1885), pp. 212, 213, [844].

76. We may, instead of $P-\delta b$ and $Q-2\omega b$, consider the linear combination $Y = 2\omega(P-\delta b) - \delta(Q-2\omega b)$, that is, $2\omega P - \delta Q$, which is of deg. weight 0.1, viz. it leaves the degree unaltered, while increasing as before the weight, and also the extent, each by unity. And again, the combination

$$Z = \sigma(P-\delta b) - (Q-2\omega b),$$

that is,

$$\sigma P - Q - (\sigma\delta - 2\omega)\,b,$$

where observe that $\sigma P - Q$, $=\sigma b\partial_a + (\sigma-1)\,c\partial_b + \dots + 1p\partial_o$ does not contain the new letter q; the operator Z is thus of the deg. weight 1.1 increasing the degree and also the weight each by unity, but leaving the extent unaltered.

There is a special case which it is important to attend to, we may have $\sigma\delta - 2\omega = 0$, viz. this is the case when the seminvariant operated upon is in regard to the letters comprised therein an invariant. Here the two combinations Y, Z are equivalent to each other, each of them is $=\sigma b\partial_a + (\sigma-1)\,c\partial_b + \dots + 1p\partial_o$, which is an annihilator of the seminvariant (invariant) operated upon. Hence in this case we cannot replace the original forms by the linear combinations, but must retain one (no matter which) of the original forms $P-\delta b$, $Q-2\omega b$.

77. We can, by means of the foregoing operators, starting from the quadric seminvariants $c-b^2$, &c., derive in order the seminvariants for the successive weights 3, 4, 5, ...

Thus writing down the series of finals (in AO as before),

$$\begin{array}{cccccccc}
b^2, & b^3, & c^2, & bc^2, & d^2, & bd^2, & e^2, & \&c., \\
 & & b^4 & b^5 & c^3 & bc^3 & cd^2 & \\
 & & & & b^2c^2 & b^3c^2 & b^2d^2 & \\
 & & & & b^6 & b^7 & c^4 & \\
 & & & & & & b^2c^3 & \\
 & & & & & & b^4c^2 & \\
 & & & & & & b^8, &
\end{array}$$

I proceed as follows, observing, however, that when the function operated upon is an invariant seminvariant we must instead of Z write $P - \delta b$.

$$\begin{array}{ccccccc}
b^2 \text{ emerges}, & b^3 = Zb^2, & c^2 \text{ emerges}, & bc^2 = Zc^2, & d^2 \text{ emerges}, & bd^2 = Zd^2, & e^2 \text{ emerges}, \\
 & & b^4 = Zb^3 & b^5 = Zb^4 & c^3 = Ybc^2 & bc^3 = Zc^3 & cd^2 = Ybd^2 \\
 & & & & b^2c^2 = Zbc^2 & b^3c^2 = Zb^2c^2 & b^2d^2 = Zbd^2 \\
 & & & & b^6 = Zb^5 & b^7 = Zb^6 & c^4 = Ybc^3 \\
 & & & & & & b^2c^3 = Zbc^2 \\
 & & & & & & b^4c^2 = Zb^3c^2 \\
 & & & & & & b^8 = Zb^7,
\end{array}$$

viz. whenever the seminvariant to be obtained has a final containing b, it is obtained by means of the operator Z (or it may be $P - \delta b$), but when there is no b then by the operator Y.

The seminvariants operated upon may be blunt or sharp, but there is an advantage in operating on the sharp forms as these are more simple, and we thereby obtain for the next superior weight forms more nearly approximating to the sharp forms. We do not however by thus operating on a sharp form obtain directly a sharp form; to do this, the form obtained must be modified by adding thereto a numerical multiple or multiples of a preceding sharp form: and thus the theory does not determine beforehand the forms of the sharp seminvariants. But making at each step the necessary modification (if any) we have thereby, when the sharp seminvariants of the next preceding weight are known, a very convenient process for the calculation of the sharp seminvariants of any given weight, in the AO arrangement of their final terms. Thus for the weight 10; $k \propto f^2$ is taken to be known, the next two forms $ci \propto ce^2$ and $dh \propto b^2e^2$ are calculated each from $j \propto be^2$, the expression for which is

$$= j - 9bi + 20ch - 28dg + 14ef + 16b^2h - 56bcg + 112bdf - 70be^2.$$

We have for $j \propto be^2$, $\delta = 3$, $\omega = 9$, $\sigma = 9$: and therefore

$$\tfrac{1}{3}Y = 6b\partial_a + 5c\partial_b + 4d\partial_c + 3e\partial_d + 2f\partial_e + g\partial_f - i\partial_h - 2j\partial_i - 3k\partial_j,$$

$$Z = 9b\partial_a + 8c\partial_b + 7d\partial_c + 6e\partial_d + 5f\partial_e + 4g\partial_f + 3h\partial_g + 2i\partial_h + j\partial_i - 9b.$$

78. I exhibit the calculation as follows:

$\frac{1}{3}Y(j \infty be^2)$

		1	2	3	4	5	6	7	8	9		†	÷ 70	*
k	+ 1									− 3	− 3	+ 3	0	
bj	− 10	+ 12							+ 18		+ 30	− 30	0	
ci	+ 45		− 45					− 20			− 65	+ 135	+ 70	+ 1
dh	− 120			+ 80							+ 80	− 360	− 280	− 4
eg	+ 210				− 84		+ 14				− 70	+ 630	+ 560	+ 8
f^2	− 126					+ 28					+ 28	− 378	− 350	− 5
b^2i		− 54						− 16			− 70		− 70	− 1
bch		+ 120	+ 160								+ 280		+ 280	+ 4
bdg		− 168		− 224			+ 112				− 280		− 280	− 4
bef		+ 84			+ 336	− 280					+ 140		+ 140	+ 2
c^2g			− 280								− 280		− 280	− 4
cdf			+ 560								+ 560		+ 560	+ 8
ce^2			− 350								− 350		− 350	− 5
d^2e														
b^3h														
b^2cg														
b^2df														
b^2e^2														
	± 256													± 23

$Z(j \infty be^2)$

	1	2	3	4	5	6	7	8	9	10		†	÷ − 18	*
k														
bj	+ 18								− 9	− 9	0		0	
ci		− 72						+ 40			− 32	+ 32	0	
dh			+ 140				− 84				+ 56	− 128	− 72	+ 4
eg				− 168		+ 56					− 112	+ 256	+ 144	− 8
f^2					+ 70						+ 70	− 160	− 90	+ 5
b^2i	− 81							+ 32		+ 81	+ 32	− 32	0	0
bch	+ 180	+ 256					− 168			− 180	+ 88	+ 128	+ 216	− 12
bdg	− 252		− 392			+ 448				+ 252	+ 56	− 128	− 72	+ 4
bef	+ 126			+ 672	− 700					− 126	− 28	+ 64	+ 36	− 2
c^2g		− 448									− 448	− 128	− 576	+ 32
cdf		+ 896									+ 896	+ 256	+ 1152	− 64
ce^2		− 560									− 560	− 160	− 720	+ 40
d^2e														
b^3h										− 144	− 144		− 144	+ 8
b^2cg										+ 504	+ 504		+ 504	− 28
b^2df										− 1008	− 1008		− 1008	+ 56
b^2e^2										+ 630	+ 630		+ 630	− 35
														± 149

The numbers (1, 2, ..., 9) and (1, 2, ..., 10) at the head of the columns refer to the nine terms $6b\partial_a$, $5c\partial_b$, ... of $\frac{1}{3}Y$, and the ten terms $9b\partial_a$, $8c\partial_b$, ... of Z respectively, these several operations being performed on $(j \infty be^2)$ the value of which is given above: the daggers † denote the additions which have to be made in order to obtain the proper initial term, viz. for the first † the added term is $+3\,(k \infty f^2)$ and for the second † the added term is $+32\,(ci \infty ce^2)$: the headings $\div 70$ and $\div -18$ explain themselves, and the columns headed with an asterisk * give the results, viz. the first of these is $(ci \infty ce^2)$ and the second of them is $(dh \infty b^2e^2)$. As appears above, the value of the first of these is used in the second † column for obtaining that of the second of them.

79. We may operate with $P - \delta b$ and $Q - 2\omega b$ on a product (deg. weight $\delta . \omega$) ST of two seminvariants S, T, deg. weights $\delta' . \omega'$ and $\delta'' . \omega''$ respectively, $\delta = \delta' + \delta''$, $\omega = \omega' + \omega''$. We have

$$(P - \delta b)\, ST = S . PT + T . PS - (\delta' + \delta'')\, bST, \quad = S\,(P - \delta'' b)\, T + T\,(P - \delta' b)\, S,$$

where $(P - \delta' b)\, S$ and $(P - \delta'' b)\, T$ are each of them a seminvariant. And similarly,

$$(Q - 2\omega b)\, ST = S . QT + T . QS - 2\,(\omega' + \omega'')\, bST = S\,(Q - 2\omega'' b)\, T + T\,(Q - 2\omega' b)\, S,$$

where $(Q - 2\omega' b)\, S$ and $(Q - 2\omega'' b)\, T$ are each of them a seminvariant. That is, operating either with $P - \delta b$ or $Q - 2\omega b$ on a product, we have a sum of products; and therefore also operating upon a sum of products (each product being of the deg. weight $\omega . \delta$) we have a sum of products, each product in such sum being of the deg. weight $\omega + 1 . \delta + 1$, and moreover of the extent $\sigma + 1$. And instead of binary products, we may, it is clear, consider ternary, quaternary, &c., products.

The like theorem applies to the derived operators Y and Z, but as to Y there is a specialty to be noticed. We have

$$\begin{aligned} Y . ST &= 2\omega\,(P - \delta b)\, ST - \delta\,(Q - 2\omega b)\, ST, \\ &= 2\omega\,\{S\,(P - \delta'' b)\, T + T\,(P - \delta' b)\, S\} - \delta\,\{S\,(Q - 2\omega'' b)\, T + T\,(Q - 2\omega' b)\, S\}, \\ &= S\,\{2\omega\,(P - \delta'' b)\, T - \delta\,(Q - 2\omega'' b)\, T\} + T\,\{2\omega\,(P - \delta' b)\, T - \delta\,(Q - 2\omega' b)\, S\}, \end{aligned}$$

where the whole of the right-hand side as being equal to $Y . ST$ is of the degree δ, but except in the particular case $\left(\frac{\delta}{\omega} = \frac{\delta'}{\omega'}, = \frac{\delta''}{\omega''}\right)$ the separate products $S\{\ \}$ and $T\{\ \}$ which occur on the right-hand side are each of them of the degree $\delta + 1$.

It is scarcely necessary, but it may be proper, to remark that we frequently combine by addition a seminvariant S of the deg. weight $\delta . \omega$ with a seminvariant T deg. weight $\delta - \epsilon . \omega$ of the same weight but of an inferior degree, but when this is done we regard the T as standing for $a^\epsilon T$, and as being thus of the same deg. weight $\delta . \omega$. We have

$$(P - \delta b)\, a^\epsilon T = a^\epsilon PT + TPa^\epsilon - (\epsilon + \delta - \epsilon)\, ba^\epsilon T, \quad = a^\epsilon\,\{P - (\delta - \epsilon)\, b\}\, T + T\,(P - \epsilon b)\, a^\epsilon,$$

where $(P-\epsilon b)\,a^{\epsilon}=(\epsilon-\epsilon)\,b=0$, and consequently $(P-b\delta)\,a^{\epsilon}T=\{P-(\delta-\epsilon)\,b\}\,T$; viz. for the operation upon T, we regard $P-\delta b$ as standing for $P-(\delta-\epsilon)\,b$. As regards Q, we have $(Q-2\omega b)\,a^{\epsilon}T=(Q-2\omega b)\,T$; viz. the degree of T does not here present itself.

80. We may write

$$(2\omega P-\delta Q)\,S=S',$$

the new seminvariant S' being of the weight $\omega+1$; hence also

$$\{(2\omega+2)\,P-\delta Q\}\,.\,\{2\omega P-\delta Q\}\,S=S'',$$

where S'' is of the weight $\omega+2$; viz. we have an operator

$$\{(2\omega+2)\,P-\delta Q\}\,.\,\{2\omega P-\delta Q\},$$

which, operating on a seminvariant of the deg. weight $\delta\,.\,\omega$, gives a seminvariant of the deg. weight $\delta\,.\,\omega+2$. This is

$$=(4\omega^2+4\omega)\,(P^2+P\,.\,P)-(2\omega+2)\,\delta\,(PQ+P\,.\,Q)-2\omega\delta\,(QP+Q\,.\,P)+\delta^2\,(Q^2+Q\,.\,Q),$$

where P^2, PQ, QP and Q^2 are the mere algebraical squares and products, while $P\,.\,Q$ and $Q\,.\,P$ denote respectively P operating on Q and Q operating on P; and since $PQ=QP$, this is

$$=(4\omega^2+4\omega)\,(P^2+P\,.\,P)-(4\omega+2)\,\delta PQ-2\,(\omega+2)\,\delta P\,.\,Q-2\omega\delta Q\,.\,P+\delta^2\,(Q^2+Q\,.\,Q).$$

Recollecting that

$$P=b\partial_a+c\partial_b+d\partial_c+\ldots,\quad Q=c\partial_b+2d\partial_c+\ldots,$$

we have

$$\begin{aligned}
P\,.\,P&=c\partial_a+\ d\partial_b+\ e\partial_c\quad+\ldots,\\
P\,.\,Q&=\qquad\quad\ d\partial_b+2e\partial_c\quad+\ldots,\\
Q\,.\,P&=c\partial_a+2d\partial_b+3e\partial_c\quad+\ldots,\ =P\,.\,P+P\,.\,Q,\\
Q\,.\,Q&=\quad 1\,.\,2d\partial_b+2\,.\,3e\partial_c+\ldots,
\end{aligned}$$

and attending to the relation just obtained $Q\,.\,P=P\,.\,P+P\,.\,Q$, we find that the operator may be written

$$\begin{aligned}
&(4\omega^2+4\omega)\,\{P^2-(\delta-1)\,P\,.\,P\}\\
&-(4\omega+2)\,\delta\,\{PQ-\omega P\,.\,P-\tfrac{1}{3}\,(\delta-3)\,P\,.\,Q\}\\
&+\delta^2\,\{Q^2+Q\,.\,Q-\tfrac{1}{3}\,(4\omega+2)\,P\,.\,Q\};
\end{aligned}$$

in fact, here the terms in P^2, PQ, Q^2 are in the original form, while those in $P\,.\,P$, $P\,.\,Q$, $Q\,.\,Q$ are

$$\begin{aligned}
(4\omega^2+4\omega)\,(1-\delta)\,P\,.\,P+(4\omega^2+2\omega)\,\delta P\,.\,P-\tfrac{1}{3}\,(4\omega+2)\,(\delta^2-3\delta)\,P\,.\,Q+\delta^2 Q\,.\,Q\\
+\tfrac{1}{3}\,(4\omega+2)\,\delta^2\qquad P\,.\,Q,
\end{aligned}$$

which are

$$=(4\omega^2+4\omega-2\omega\delta)\,P\,.\,P-(4\omega+2)\,\delta P\,.\,Q+\delta^2 Q\,.\,Q,$$

agreeing with the original form

$$(4\omega^2+4\omega)\,P\,.\,P-(2\omega+2)\,\delta P\,.\,Q-2\omega\delta\,(P\,.\,P+P\,.\,Q)+\delta^2 Q\,.\,Q.$$

81. I find that each of the three parts is separately an operator, viz. that we have

$$P^2-(\delta-1)\,P\,.\,P,$$

$$PQ-\omega P\,.\,P-\tfrac{1}{3}(\delta-3)\,P\,.\,Q,$$

$$Q^2+Q\,.\,Q-\tfrac{1}{3}(4\omega+2)\,P\,.\,Q,$$

each of them an operator which, operating on a seminvariant of deg. weight $\delta\,.\,\omega$, gives a seminvariant of deg. weight $\delta\,.\,\omega+2$.

I verify this for the first of the three operators, say

$$\Omega=P^2-(\delta-1)\,P\,.\,P=P^2+P\,.\,P-\delta\Theta,$$

if for a moment

$$P\,.\,P,\ =c\partial_a+d\partial_b+e\partial_c+\ldots$$

is put $=\Theta$.

Here for a seminvariant S, we have

$$\Omega S=(P^2+P\,.\,P-\delta\Theta)\,S=P\,(PS)-\delta\Theta S.$$

Writing $S'=(aP-b\delta)\,S$, then S' is a seminvariant, degree $=\delta+1$, and then if $S''=(aP-b\,(\delta+1))\,S'$, S'' is a seminvariant, degree $=\delta+2$. We have $PS=a^{-1}(S'+b\delta S)$, and thence

$$\Omega S=Pa^{-1}(S'+b\delta S)-\delta\Theta S,\ =-b\,(S'+b\delta S)+P\,(S'+b\delta S)-\delta\Theta S.$$

Here

$$P\,(S'+b\delta S)=PS'+c\delta S+b\delta PS,\ =S''+b\,(\delta+1)\,S'+c\delta S+b\delta\,(S'+b\delta S),$$

and hence

$$\Omega S=S''+2b\delta S'+\{c\delta+b^2(\delta^2-\delta)\}\,S-\delta\Theta S.$$

This will be a seminvariant if $\Delta\,.\,\Omega S=0$; we have

$$\begin{aligned}\Delta\,.\,\Omega S=\Delta S''+2b\delta\Delta S'&+\{c\delta+b^2(\delta^2-\delta)\}\,\Delta S-\delta\,(\Delta\Theta+\Delta\,.\,\Theta)\,S\\ +2\delta S'&+\{2b\delta+2b\,(\delta^2-\delta)\}\,S,\end{aligned}$$

or, omitting the terms in $\Delta S''$, $\Delta S'$, ΔS which respectively vanish, this is

$$=2\delta S'+2b\delta^2 S-\delta\,(\Delta\Theta+\Delta\,.\,\Theta)\,S.$$

But since $PS=S'+b\delta S$, and from $\Delta S=0$ we deduce $0=(\Theta\Delta+\Theta\,.\,\Delta)\,S$, the equation becomes

$$\Delta\,.\,\Omega S=2\delta PS-\delta\,(\Delta\,.\,\Theta-\Theta\,.\,\Delta)\,S,$$

and from

$$\begin{aligned}\Delta&=a\partial_b+2b\partial_c+3c\partial_d+\ldots,\\ \Theta&=c\partial_a+d\partial_b+e\partial_c+\ldots,\end{aligned}$$

we have

$$\begin{aligned}\Delta\,.\,\Theta&=2b\partial_a+3c\partial_b+4d\partial_c+\ldots,\\ \Theta\,.\,\Delta&=\phantom{2b\partial_a+{}}c\partial_b+2d\partial_c+\ldots,\end{aligned}$$

and thence

$$\Delta . \Theta - \Theta . \Delta = 2b\partial_a + 2c\partial_b + 2d\partial_c + \ldots, = 2P,$$

and we have thus the required equation $\Delta . \Omega S = 0$.

82. If instead of P, Θ, we write B, C, so that

$$B = b\partial_a + c\partial_b + d\partial_c + \ldots,$$

$$C = B . B = c\partial_a + d\partial_b + e\partial_c + \ldots,$$

and put further

$$D = B . C = d\partial_a + e\partial_b + f\partial_c + \ldots,$$

$$E = B . D = e\partial_a + f\partial_b + g\partial_c + \ldots,$$

then the foregoing operator is $B^2 - (\delta - 1) C$, or reversing the sign, say it is $(\delta - 1) C - B^2$, which is the first of a series of operators

$$(\delta - 1) C - B^2,$$

$$(\delta - 1)(\delta - 2) D - 3(\delta - 2) BC \quad + 2B^3,$$

$$(\delta - 1)(\delta - 2)(\delta - 3) E - 4(\delta - 2)(\delta - 3) BD + 6(\delta - 3) B^2C - 3B^4,$$

$$\vdots$$

which are of the deg. weights $0 . 2$, $0 . 3$, $0 . 4$, &c., respectively, viz. operating upon a seminvariant of deg. weight $\delta . \omega$ they leave the degree unaltered, but increase the weight by 2, 3, 4, ... respectively.

It is to be observed that B^2, BC, B^3, &c., denote the mere algebraical powers and products of the symbols B, C, D, &c., without any operation of one symbol on another.

As a simple illustration, take $(C - B^2)(ac - b^2)$: here,

$$\begin{array}{rl} C(ac - b^2) = & e - 2bd + c^2 \\ -B^2 = -(2bd\partial_a\partial_c + c^2\partial_b^2)(\quad ,, \quad) & \underline{ - 2bd + 2c^2} \\ \text{Value is} & e - 4bd + 3c^2; \end{array}$$

and similarly for $(C - B^2)(ae - 4bd + 3c^2)$, here:

$$\begin{array}{rl} C(ae - 4bd + 3c^2) = & g - 4bf + (6 + 1) ce - 4d^2 \\ -B^2 = -(2bf\partial_a\partial_e + 2ce\partial_b\partial_d + d^2\partial_c^2)(\qquad ,, \qquad) & \underline{ - 2bf \qquad + 8 \; ce - 6d^2} \\ \text{Value is} & g - 6bf \qquad + 15 \; ce - 10d^2. \end{array}$$

A direct proof may of course be obtained for any one of the foregoing operators; viz. calling it Ω, it may be shown that $\Delta\Omega S$ is $= 0$. I have not considered the like question of the derivation of series of operators from the other two forms

$$PQ - \omega P . P - \tfrac{1}{3}(\delta - 3) P . Q \quad \text{and} \quad Q^2 + Q . Q - \tfrac{1}{3}(4\omega + 2) P . Q$$

respectively.

83. I do not wish in the present paper to go into the theory of covariants, but it is nevertheless proper to point out the connexion which exists between the covariant theory of derivation and the operators P and Q.

Consider a quantic $(a, b, c, \ldots, a' = a_{\sigma'} \between x, y)^{\sigma'}$; any covariant hereof is $(A, B, C, \ldots \between x, y)^{\mu}$, where A is a seminvariant say of degree δ and weight, $\omega = \frac{1}{2}(\sigma'\delta - \mu)$, or $\mu = \sigma'\delta - 2\omega$, reduced to zero by the operation $\Delta = a\partial_b + 2b\partial_c + \ldots + \sigma' b' \partial_{a'}$: and if we write

$$\phi_{\sigma'} = \sigma' b\partial_a + (\sigma' - 1)\, c\partial_b + \ldots + a'\partial_{b'},$$

then

$$B = \phi_{\sigma'} A,\quad C = \tfrac{1}{2}\phi_{\sigma'} B,\quad D = \tfrac{1}{3}\phi_{\sigma'} C, \ldots$$

The derivative (f, F) is

$$\begin{aligned} &= \partial_x f \,.\, \partial_y F - \partial_y f \,.\, \partial_x F \\ &= (a,\ b, \ldots \between x,\ y)^{\sigma'-1} B x^{\mu-1} + \ldots \\ &\quad - (b,\ c,\ \ldots \between x,\ y)^{\sigma'-1} \mu A x^{\mu-1} + \ldots \\ &= (aB - \mu bA, \ldots \between x,\ y)^{\sigma'+\mu-2}, \end{aligned}$$

that is, A being a seminvariant, we have $aB - \mu bA$ a seminvariant, or say

$$(\phi_{\sigma'} - \mu' b)\, A = \text{sem.}\ \mu' = \sigma'\delta - 2\omega;$$

and similarly

$$(\phi_{\sigma} - \mu b)\, A = \text{sem.}\ \mu = \sigma\delta - 2\omega.$$

Hence

$$\{\phi_\sigma - \phi_{\sigma'} - (\mu - \mu')\, b\}\, A, \text{ and } \{\sigma'\phi_\sigma - \sigma\phi_{\sigma'} - (\sigma'\mu - \sigma\mu')\, b\}\, A$$

are each of them a seminvariant: but

$$\begin{aligned} \phi_\sigma &= \sigma b\partial_a + (\sigma - 1)\, c\partial_b + \ldots, \\ \phi_{\sigma'} &= \sigma' b\partial_a + (\sigma' - 1)\, c\partial_b + \ldots, \\ \phi_\sigma - \phi_{\sigma'} &= (\sigma - \sigma')\,(b\partial_a + c\partial_b + \ldots) = (\sigma - \sigma')\, P,\ \ \mu - \mu' = (\sigma - \sigma')\,\delta; \end{aligned}$$

the first form, omitting the factor $\sigma - \sigma'$, is $= (P - \delta b)\, A$: similarly

$$\sigma'\phi_\sigma - \sigma\phi_{\sigma'} = (\sigma - \sigma')\,(c\partial_b + 2d\partial_c + \ldots) = (\sigma - \sigma')\, Q$$

and

$$\sigma'\mu - \sigma\mu' = (\sigma - \sigma')\, 2\omega,$$

and the second form is

$$= (Q - 2\omega b)\, A.$$

We thus see that the operators $P - \delta b$ and $Q - 2\omega b$ upon a seminvariant A depend on the derivation of f upon a covariant which has A for its leading coefficient: the order of f is arbitrary, and we have thus two distinct forms.

84. As an illustration, consider the quantics

$$(1,\ b,\ c,\ d,\ e,\ f \between x,\ y)^5,$$

and

$$(1,\ b,\ c,\ d,\ e,\ f,\ g \between x,\ y)^6:$$

each of these has a covariant the leading coefficient of which is

$$A = f - 5be + 2cd + 8b^2d - 6bc^2,$$

viz. these are

$$\left(\begin{array}{c|c|c} f \; +1 & bf \; +5 & \ldots \\ be \; -5 & ce \; -16 & \\ cd \; +2 & d^2 \; +6 & \\ b^2d+8 & b^2e \; -9 & \\ bc^2 \; -6 & bcd+38 & \\ & c^3 \; -24 & \\ \hline \pm 11 & \pm 49 & \end{array}\right)(x, y)^5, \text{ and } \left(\begin{array}{c|c|c} f \; +1 & g \; +1 & \ldots \\ be \; -5 & bf \; +2 & \\ cd \; +2 & ce \; -19 & \\ b^2d+8 & d^2 \; +8 & \\ bc^2 \; -6 & b^2e \; -6 & \\ & bcd+44 & \\ & c^3 \; -30 & \\ \hline \pm 11 & \pm 55 & \end{array}\right)(x, y)^8,$$

and we find without difficulty

$$\begin{array}{lccc} & (g \infty d^2) & (ce \infty c^3) & (d^2 \infty b^2c^2) \\ \hline (f_1, F_1) \;\; = & & -16 & -10 \\ (f_2, F_2) \;\; = & 1 & -34 & -16 \\ (P - \;\;3b)\, A = & 1 & -18 & -\;\;6 \\ (Q - 10b)\, A = & 5 & -74 & -20 \end{array}$$

and thence

$$(P - \;\;3b)\, A = \;\;(f_2, F_2) - \;\;(f_1, F_1)$$

$$(Q - 10b)\, A = 5\,(f_2, F_2) - 6\,(f_1, F_1),$$

viz. we thus have $P - 3b$, and $Q - 10b$ upon $f \infty bc^2$ each given as a linear function of the derivatives (f_1, F_1) and (f_2, F_2), where f_1, f_2 are the quintic and the sextic function, and F_1, F_2 are like covariants of these functions respectively.

$w=11$

	bf^2	de^2	bce^2	cd^3	b^3e^2	b^2d^3	bc^2d^2	b^3cd^2	bc^5	b^5d^2	b^3c^4	b^5c^3	b^7c^2	b^{11}
l	1													
cj		1												
di			1											
eh				1										
c^2h					1									
fg						1								
cdg								2						
cef							1							
d^2f									1					
c^3f										1				
de^2											1			
c^2de												1		
cd^3													1	
c^4d														1

$w=12$

	g^2	cf^2	e^3	b^2f^2	bde^2	c^2e^2	d^4	b^2ce^2	bcd^3	c^3d^2	b^4e^2	b^3d^3	$b^2c^2d^2$	c^6	b^4cd^2	b^2c^5	b^6d^2	b^4c^4	b^6c^3	b^8c^2	b^{12}
m	1																				
ck		1																			
dj				2																	
ei			1																		
c^2i								3													
fh					1																
cdh											4										
g^2						1															
ceg							1														
d^2g									1												
c^3g															4						
cf^2										1											
def												1									
c^2df																	3				
e^3													1								
c^2e^2														1							
cd^2e																1					
c^4e																			2		
d^4																		1			
c^3d^2																				1	
c^6																					1

[The Tables for $w = 13$, 14, 15, 16 are given on the accompanying lithographed sheet.]