931.

ON SOME FORMULÆ OF CODAZZI AND WEINGARTEN IN RELATION TO THE APPLICATION OF SURFACES TO EACH OTHER.

[From the *Proceedings of the London Mathematical Society*, vol. XXIV. (1893), pp. 210—223.]

An extremely elegant theory of the application of surfaces one upon another is developed in the memoir, Codazzi, "Mémoire relatif à l'application des surfaces les unes sur les autres," *Mém. prés. à l'Académie des Sciences*, t. XXVII. (1883), No. 6, pp. 1—47; but the notation is not presented in a form which is easily comparable with that of the Gaussian notation in the theory of surfaces. I propose to reproduce the theory in the Gaussian notation.

Codazzi considers on a given surface two systems of curves depending on the parameters t, T respectively; the curves are in the memoir taken to be orthogonal to each other, but this restriction is removed in the general formula given p. 44, *Addition au chapitre premier*. For a curve of either system, he considers the tangent, the principal normal, or normal in the osculating plane, and the binormal, or line at right angles to the osculating plane (say these are tan, prn and bin). For a curve of the one system, that in which t is variable or say a t-curve, he denotes the cosine-inclinations of these lines to the axes by the letters a, b, c, thus:

	x	y	z
tan	a_x	a_y	a_z
prn	b_x	b_y	b_z
bin	c_x	c_y	c_z

and he writes also l for the inclination of the principal normal to the normal of the surface, $\frac{dm}{dt}dt$ for the angle of contingence, or inclination of the tangent at the point (t, T) to the tangent at the point $(t+dt, T)$, and $\frac{dn}{dt}dt$ for the angle of torsion, or inclination of the osculating plane at the point (t, T) to that at the point $(t+dt, T)$, or, what is the same thing, the inclination of the binormals at these points respectively. And he gives as known formulæ

$$\frac{da}{dt}=b\frac{dm}{dt}, \quad \frac{db}{dt}=-a\frac{dm}{dt}-c\frac{dn}{dt}, \quad \frac{dc}{dt}=b\frac{dn}{dt},$$

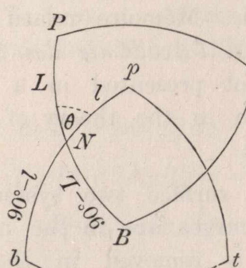
where a, b, c denote

$$(a_x,\ b_x,\ c_x), \quad (a_y,\ b_y,\ c_y), \quad \text{or} \quad (a_z,\ b_z,\ c_z).$$

He uses the capital letters

$$A_x,\ A_y,\ A_z,\ B_x,\ B_y,\ B_z,\ C_x,\ C_y,\ C_z,\ L,\ \frac{dM}{dT}dT,\ \frac{dN}{dT}dT$$

with the like significations in regard to the curve for which T is variable, or say the T-curve; for greater clearness I give the accompanying figure.



Codazzi writes further

$$\frac{dm}{dt}\cos l = u, \quad \frac{dm}{dt}\sin l = v, \quad \frac{dn}{dt}-\frac{dl}{dt}=w,$$

$$\frac{dM}{dT}\cos L = U, \quad \frac{dM}{dT}\sin L = V, \quad \frac{dN}{dT}-\frac{dL}{dT}=W;$$

also, if s, S are the arcs of the two curves respectively,

$$\frac{ds}{dt}=r, \quad \frac{dS}{dT}=R;$$

and he obtains a system of six formulæ, which in the *Addition*, p. 44, are presented in the following form—only I use therein θ, instead of his b, to denote the inclination of the two curves to each other:

$$\frac{du}{dT} = \frac{dU}{dt}\cos\theta + \frac{dW}{dt}\sin\theta + w\left(V - \frac{d\theta}{dT}\right) + (U\sin\theta - W\cos\theta)\left(v - \frac{d\theta}{dt}\right),$$

$$\frac{dU}{dt} = \frac{du}{dT}\cos\theta + \frac{dw}{dT}\sin\theta + W\left(v - \frac{d\theta}{dt}\right) + (u\sin\theta - w\cos\theta)\left(V - \frac{d\theta}{dT}\right),$$

$$\left(\frac{dv}{dT} + \frac{dV}{dt}\right)\frac{d^2\theta}{dt\,dT} + \sin\theta\,(uU - wW) - \cos\theta\,(uW + wU) = 0,$$

$$R\,(u\cos\theta + w\sin\theta) = r\,(U\cos\theta + W\sin\theta),$$

$$R\sin\theta\left(v - \frac{d\theta}{dt}\right) + \frac{dR}{dt}\cos\theta = \frac{dr}{dT},$$

$$r\sin\theta\left(V - \frac{d\theta}{dT}\right) + \frac{dr}{dT}\cos\theta = \frac{dR}{dt}.$$

In the Gaussian notation, taking p, q for the parameters, we have, with the slight variations presently referred to,

$$dx = a\,dp + a'dq + \tfrac{1}{2}\alpha\,dp^2 + \alpha' dp\,dq + \alpha'' dq^2,$$

$$dy = b\,dp + b'dq + \tfrac{1}{2}\beta\,dp^2 + \beta' dp\,dq + \beta'' dq^2,$$

$$dz = c\,dp + c'dq + \tfrac{1}{2}\gamma\,dp^2 + \gamma' dp\,dq + \gamma'' dq^2,$$

$$A,\ B,\ C = bc' - b'c,\ ca' - c'a,\ ab' - a'b,$$

$$E,\ F,\ G = a^2 + b^2 + c^2,\ aa' + bb' + cc',\ a'^2 + b'^2 + c'^2;$$

and therefore

$$dx^2 + dy^2 + dz^2 = E\,dp^2 + 2F\,dp\,dq + G\,dq^2;$$

$$\sqrt{EG - F^2} = V;$$

$$E',\ F',\ G' = A\alpha + B\beta + C\gamma,\quad A\alpha' + B\beta' + C\gamma',\quad A\alpha'' + B\beta'' + C\gamma'';$$

and I take further

$$\omega,\ \omega',\ \omega'' = a\alpha + b\beta + c\gamma,\quad a\alpha' + b\beta' + c\gamma',\quad a\alpha'' + b\beta'' + c\gamma'',$$

$$\varpi,\ \varpi',\ \varpi'' = a'\alpha + b'\beta + c'\gamma,\quad a'\alpha' + b'\beta' + c'\gamma',\quad a'\alpha'' + b'\beta'' + c'\gamma'',$$

$$\lambda,\ \lambda',\ \lambda'' = \alpha^2 + \beta^2 + \gamma^2,\quad \alpha'^2 + \beta'^2 + \gamma'^2,\quad \alpha''^2 + \beta''^2 + \gamma''^2,$$

$$\mu,\ \mu',\ \mu'' = \alpha'\alpha'' + \beta'\beta'' + \gamma'\gamma'',\quad \alpha''\alpha + \beta''\beta + \gamma''\gamma,\quad \alpha\alpha' + \beta\beta' + \gamma\gamma',$$

$$E\lambda - \omega^2 = \Delta,\quad G\lambda'' - \varpi''^2 = \Delta'',$$

where it is to be noticed that V^2, E', F', G', are written instead of Gauss's Δ, D, D', D'', and ω, ω', ω'', ϖ, ϖ', ϖ'' instead of his m, m', m'', n, n', n'': and that he gives for the last-mentioned quantities the values

$$m = \tfrac{1}{2}\frac{dE}{dp},\quad m' = \tfrac{1}{2}\frac{dE}{dq},\quad m'' = \frac{dF}{dq} - \tfrac{1}{2}\frac{dG}{dp},$$

$$n = \frac{dF}{dp} - \tfrac{1}{2}\frac{dE}{dq},\quad n' = \tfrac{1}{2}\frac{dG}{dp},\quad n'' = \tfrac{1}{2}\frac{dG}{dq},$$

or, say

$$\omega = \tfrac{1}{2}E_1,\quad \omega' = \tfrac{1}{2}E_2,\quad \omega'' = F_2 - \tfrac{1}{2}G_1,$$

$$\varpi = F_1 - \tfrac{1}{2}E_2,\quad \varpi' = \tfrac{1}{2}G_1,\quad \varpi'' = \tfrac{1}{2}G_2,$$

where the subscripts (1) and (2) denote differentiation in regard to p and q respectively.

Observing that the cosine-inclinations of the tangent to the p-curve are as a, b, c, and those of the binormal or perpendicular to the osculating plane are as $b\gamma - c\beta$, $c\alpha - a\gamma$, $a\beta - b\alpha$, we easily find

$$a_x,\ a_y,\ a_z = \frac{a}{\sqrt{E}},\quad \frac{b}{\sqrt{E}},\quad \frac{c}{\sqrt{E}},$$

$$c_x,\ c_y,\ c_z = \frac{b\gamma - c\beta}{\sqrt{\Delta}},\quad \frac{c\alpha - a\gamma}{\sqrt{\Delta}},\quad \frac{a\beta - b\alpha}{\sqrt{\Delta}},$$

$$b_x,\ b_y,\ b_z = \frac{E\alpha - a\omega}{\sqrt{E\Delta}},\quad \frac{E\beta - b\omega}{\sqrt{E\Delta}},\quad \frac{E\gamma - c\omega}{\sqrt{E\Delta}};$$

and for the cosine-inclinations of the normal of the surface

$$\Delta_x,\ \Delta_y,\ \Delta_z = \frac{A}{V},\ \frac{B}{V},\ \frac{C}{V},$$

we find

$$\cos l = \frac{A\,(b\gamma - c\beta) + B\,(c\alpha - a\gamma) + C\,(a\beta - b\alpha)}{V\sqrt{\Delta}}$$

$$= \frac{(a^2 + b^2 + c^2)(a'\alpha + b'\beta + c'\gamma) - (aa' + bb' + cc')(a\alpha + b\beta + c\gamma)}{V\sqrt{\Delta}}$$

$$= \frac{E\varpi - F\omega}{V\sqrt{\Delta}},$$

$$\sin l = \frac{A\,(E\alpha - a\omega) + B\,(E\beta - b\omega) + C\,(E\gamma - c\omega)}{V\sqrt{E\Delta}};$$

or, since $Aa + Bb + Cc = 0$, this is

$$\sin l = \frac{E\,(A\alpha + B\beta + C\gamma)}{V\sqrt{E\Delta}} = \frac{E'\sqrt{E}}{V\sqrt{\Delta}}.$$

We ought therefore to have

$$E'^2E + (E\varpi - F\omega)^2 = V^2\Delta,\quad = (EG - F^2)(E\lambda - \omega^2);$$

or, omitting the terms $F^2\omega^2$, which destroy each other, and throwing out a factor E, this is

$$E\varpi^2 - 2F\varpi\omega + G\omega^2 - \lambda\,(EG - F^2) = \lambda\,(EG - F^2) - E'^2;$$

viz. putting for $EG - F^2$ its value, $= A^2 + B^2 + C^2$, and for the other terms their values, this is

$$(a^2 + b^2 + c^2)(a'\alpha + b'\beta + c'\gamma)^2 - 2\,(aa' + bb' + cc')(a'\alpha + b'\beta + c'\gamma)(a\alpha + b\beta + c\gamma)$$
$$+ (a'^2 + b'^2 + c'^2)(\alpha^2 + \beta^2 + \gamma^2)$$
$$= (A^2 + B^2 + C^2)(\alpha^2 + \beta^2 + \gamma^2) - (A\alpha + B\beta + C\gamma)^2.$$

The right-hand side is here $= (B\gamma - C\beta)^2 + (C\alpha - A\gamma)^2 + (A\beta - B\alpha)^2$; the left-hand consists of three parts, the first whereof is

$$\{a\,(a'\alpha + b'\beta + c'\gamma) - a'\,(a\alpha + b\beta + c\gamma)\}^2 = (-B\gamma + C\beta)^2,$$

and similarly the other two parts are

$$(-C\alpha + A\gamma)^2 \quad \text{and} \quad (-A\beta + B\alpha)^2;$$

the equation is thus verified.

We require Codazzi's $\frac{dm}{dt}$ and $\frac{dn}{dt}$, or say $\frac{dm}{dp}$ and $\frac{dn}{dp}$; these are to be obtained from the equations

$$\frac{d}{dp}\frac{a}{\sqrt{E}} = \frac{E\alpha - a\omega}{\sqrt{E\Delta}}\frac{dm}{dp}, \quad \frac{d}{dp}\frac{b\gamma - c\beta}{\sqrt{\Delta}} = \frac{E\alpha - a\omega}{\sqrt{E\Delta}}\frac{dn}{dp},$$

and the values obtained should satisfy

$$\frac{d}{dp}\frac{E\alpha - a\omega}{\sqrt{E\Delta}} = -\frac{a}{\sqrt{E}}\frac{dm}{dp} - \frac{b\gamma - c\beta}{\sqrt{\Delta}}\frac{dn}{dp}.$$

I find

$$\frac{dm}{dp} = \frac{\sqrt{\Delta}}{E}, \quad \frac{dn}{dp} = -\frac{\sqrt{E}}{\Delta}\begin{vmatrix} a, & b, & c \\ \alpha, & \beta, & \gamma \\ \alpha_1, & \beta_1, & \gamma_1 \end{vmatrix},$$

where α_1, β_1, γ_1 are the derivatives of α, β, γ in regard to p; and the equation to be verified thus is

$$\frac{d}{dp}\frac{E\alpha - a\omega}{\sqrt{E\Delta}} = -\frac{a\sqrt{\Delta}}{E\sqrt{E}} + \frac{(b\gamma - c\beta)\sqrt{E}}{\Delta\sqrt{\Delta}}\begin{vmatrix} a, & b, & c \\ \alpha, & \beta, & \gamma \\ \alpha_1, & \beta_1, & \gamma_1 \end{vmatrix}.$$

First, for $\frac{dm}{dp}$: the derivatives of a, E are α and $2(a\alpha + b\beta + c\gamma)$, $= 2\omega$; we thus have

$$\frac{d}{dp}\frac{a}{\sqrt{E}}, = \frac{\alpha}{\sqrt{E}} - \frac{a\omega}{E\sqrt{E}} = \frac{E\alpha - a\omega}{E\sqrt{E}},$$

which is

$$= \frac{E\alpha - a\omega}{\sqrt{E\Delta}}\frac{dm}{dp};$$

viz. we have

$$\frac{dm}{dp} = \frac{\sqrt{\Delta}}{E}.$$

Next, for $\frac{dn}{dp}$: using a subscript $({}_1)$ to denote derivation in regard to p, we have $\Delta = E\lambda - \omega^2$, and thence

$$\begin{aligned}
\Delta_1 &= E_1\lambda + E\lambda_1 - 2\omega\omega_1 \\
&= 2\omega\lambda + E \,.\, 2(\alpha\alpha_1 + \beta\beta_1 + \gamma\gamma_1) - 2\omega(\lambda + a\alpha_1 + b\beta_1 + c\gamma_1), \\
&= 2\{E(\alpha\alpha_1 + \beta\beta_1 + \gamma\gamma_1) - \omega(a\alpha_1 + b\beta_1 + c\gamma_1)\}, \\
&= 2(\alpha_1 X + \beta_1 Y + \gamma_1 Z),
\end{aligned}$$

if, for a moment,

$$X,\ Y,\ Z = E\alpha - a\omega,\ E\beta - b\omega,\ E\gamma - c\omega\,;$$

these values give identically

$$\alpha X + \beta Y + \gamma Z = \Delta, \quad \text{and} \quad aX + bY + cZ = 0.$$

Hence we have

$$\frac{d}{dp}\frac{b\gamma - c\beta}{\sqrt{\Delta}} = \frac{b\gamma_1 - c\beta_1}{\sqrt{\Delta}} - \frac{(b\gamma - c\beta)(\alpha_1 X + \beta_1 Y + \gamma_1 Z)}{\Delta\sqrt{\Delta}},$$

which must be

$$= \frac{X}{\sqrt{E\Delta}}\frac{dn}{dp};$$

we thus have

$$(b\gamma_1 - c\beta_1)\Delta - (b\gamma - c\beta)(\alpha_1 X + \beta_1 Y + \gamma_1 Z) = \frac{\Delta X}{\sqrt{E}}\frac{dn}{dp};$$

or, putting the left-hand side in the form

$$\begin{aligned}&(b\gamma_1 - c\beta_1)(\alpha X + \beta Y + \gamma Z)\\ &-(b\gamma - c\beta)(\alpha_1 X + \beta_1 Y + \gamma_1 Z)\\ &-(\beta\gamma_1 - \beta_1\gamma)(a X + b Y + c Z),\end{aligned}$$

this is

$$= -X\begin{vmatrix} a, & b, & c \\ \alpha, & \beta, & \gamma \\ \alpha_1, & \beta_1, & \gamma_1 \end{vmatrix},$$

and we thus find

$$\frac{dn}{dp} = -\frac{\sqrt{E}}{\Delta}\begin{vmatrix} a, & b, & c \\ \alpha, & \beta, & \gamma \\ \alpha_1, & \beta_1, & \gamma_1 \end{vmatrix}.$$

For the verification of the equation

$$\frac{d}{dp}\frac{E\alpha - a\omega}{\sqrt{E\Delta}} = -\frac{a\sqrt{\Delta}}{E\sqrt{E}} + \frac{(b\gamma - c\beta)\sqrt{E}}{\Delta\sqrt{\Delta}}\begin{vmatrix} a, & b, & c \\ \alpha, & \beta, & \gamma \\ \alpha_1, & \beta_1, & \gamma_1 \end{vmatrix},$$

we have

$$(E\alpha - a\omega)_1 = E\alpha_1 - a(a\alpha_1 + b\beta_1 + c\gamma_1) + \alpha\omega - a\lambda,$$

$$(E\Delta)_1 = 2E^2(\alpha\alpha_1 + \beta\beta_1 + \gamma\gamma_1) - 2E\omega(a\alpha_1 + b\beta_1 + c\gamma_1) + 2\Delta\omega,$$

and hence the equation is

$$-\frac{\{E^2(\alpha\alpha_1 + \beta\beta_1 + \gamma\gamma_1) - E\omega(a\alpha_1 + b\beta_1 + c\gamma_1) + \omega\}(E\alpha - a\omega)}{E\Delta\sqrt{E\Delta}} + \frac{E\alpha_1 - a(a\alpha_1 + b\beta_1 + c\gamma_1) + \alpha\omega - a\lambda}{\sqrt{E\Delta}}$$

$$= -\frac{a\sqrt{\Delta}}{E\sqrt{E}} + \frac{(b\gamma - c\beta)\sqrt{E}}{\Delta\sqrt{\Delta}}\begin{vmatrix} a, & b, & c \\ \alpha, & \beta, & \gamma \\ \alpha_1, & \beta_1, & \gamma_1 \end{vmatrix}.$$

Considering first the terms without α_1, β_1, γ_1, these give

$$-\omega(E\alpha - a\omega) + E(\alpha\omega - a\lambda) = -a(E\lambda - \omega^2),$$

which is identically true; and then the remaining terms with $\alpha_1, \beta_1, \gamma_1$ give

$$-\{E(\alpha\alpha_1+\beta\beta_1+\gamma\gamma_1)-\omega(a\alpha_1+b\beta_1+c\gamma_1)\}(E\alpha-a\omega)+\{E\alpha_1-a(a\alpha_1+b\beta_1+c\gamma_1)\}(E\lambda-\omega^2)$$

$$=(b\gamma-c\beta)E\begin{vmatrix} a, & b, & c \\ \alpha, & \beta, & \gamma \\ \alpha_1, & \beta_1, & \gamma_1 \end{vmatrix}.$$

On the left-hand side, the whole coefficient of α_1 is

$$=-(E\alpha-a\omega)^2+(b^2+c^2)(E\lambda-\omega^2),$$

which is

$$=-E^2\alpha^2+2a\alpha E\omega-a^2\omega^2+(b^2+c^2)E\lambda-(b^2+c^2)\omega^2,$$

$$=E[-E\alpha^2+2a\alpha\omega+(b^2+c^2)\lambda-\omega^2];$$

and, substituting for E, ω, and λ their values, this is found to be

$$=E\{(b^2+c^2)(\beta^2+\gamma^2)-(b\beta+c\gamma)^2\},\quad =E(b\gamma-c\beta)^2.$$

Similarly the whole coefficients of β_1 and γ_1 are found to be

$$=E(b\gamma-c\beta)(c\alpha-a\gamma),\quad \text{and}\quad E(b\gamma-c\beta)(a\beta-b\alpha),$$

respectively; and thus the left-hand side becomes

$$=(b\gamma-c\beta)E\begin{vmatrix} a, & b, & c \\ \alpha, & \beta, & \gamma \\ \alpha_1, & \beta_1, & \gamma_1 \end{vmatrix},$$

as it should do.

We have

$$\tan l=\frac{P}{Q},$$

where

$$P=E'\sqrt{E}=\sqrt{E}(A\alpha+B\beta+C\gamma),\qquad =\mathrm{a}\alpha+\mathrm{b}\beta+\mathrm{c}\gamma,\ \text{suppose},$$

$$Q=E\varpi-F\omega=(cB-bC)\alpha+(aC-cA)\beta+(bA-aB)\gamma,\quad =\mathrm{a}'\alpha+\mathrm{b}'\beta+\mathrm{c}'\gamma,\ \text{suppose},$$

$$P^2+Q^2=V^2\Delta,$$

and we have hence to find $\dfrac{dl}{dp}$. Using, as before, a subscript ($_1$) to denote derivation in regard to p, we have

$$\frac{dl}{dp}=\frac{QP_1-PQ_1}{P^2+Q^2};$$

the numerator is

$$=\ (\mathrm{a}'\alpha+\mathrm{b}'\beta+\mathrm{c}'\gamma)(\mathrm{a}\,\alpha_1+\mathrm{b}\,\beta_1+\mathrm{c}\,\gamma_1+\mathrm{a}_1\alpha+\mathrm{b}_1\beta+\mathrm{c}_1\gamma)$$
$$-(\mathrm{a}\,\alpha+\mathrm{b}\,\beta+\mathrm{c}\,\gamma)(\mathrm{a}'\alpha_1+\mathrm{b}'\beta_1+\mathrm{c}'\gamma_1+\mathrm{a}_1'\alpha+\mathrm{b}_1'\beta+\mathrm{c}_1'\gamma)$$
$$=-[(\mathrm{bc}'-\mathrm{b}'\mathrm{c})(\beta\gamma_1-\beta_1\gamma)+(\mathrm{ca}'-\mathrm{c}'\mathrm{a})(\gamma\alpha_1-\gamma_1\alpha)+(\mathrm{ab}'-\mathrm{a}'\mathrm{b})(\alpha\beta_1-\alpha_1\beta)]$$
$$+Q(\mathrm{a}_1\alpha+\mathrm{b}_1\beta+\mathrm{c}_1\gamma)-P(\mathrm{a}_1'\alpha+\mathrm{b}_1'\beta+\mathrm{c}_1'\gamma).$$

For the first part hereof,

$$\begin{aligned} \mathrm{bc}' - \mathrm{b}'\mathrm{c} &= \sqrt{E}\,\{B\,(bA - aB) - C\,(aC - cA)\} \\ &= \sqrt{E}\,\{A\,(aA + bB + cC) - a\,(A^2 + B^2 + C^2)\}, \quad = -\,aV^2\sqrt{E}, \end{aligned}$$

since $aA + bB + cC = 0$; and similarly

$$\mathrm{ca}' - \mathrm{c}'\mathrm{a}, \quad \mathrm{ab}' - \mathrm{a}'\mathrm{b} = -\,bV^2\sqrt{E}, \quad -\,cV^2\sqrt{E},$$

respectively; and thus the first part is

$$= V^2\sqrt{E}\,\{a\,(\beta\gamma_1 - \beta_1\gamma) + b\,(\gamma\alpha_1 - \gamma_1\alpha) + c\,(\alpha\beta_1 - \alpha_1\beta)\},$$

$$= V^2\sqrt{E}\begin{vmatrix} a, & b, & c \\ \alpha, & \beta, & \gamma \\ \alpha_1, & \beta_1, & \gamma_1 \end{vmatrix}.$$

Hence, dividing by

$$P^2 + Q^2, \quad = V^2\Delta,$$

the first part of $\dfrac{dl}{dp}$ is

$$= \frac{\sqrt{E}}{\Delta}\begin{vmatrix} a, & b, & c \\ \alpha, & \beta, & \gamma \\ \alpha_1, & \beta_1, & \gamma_1 \end{vmatrix}.$$

For the second part of the numerator, we require

$$\alpha\mathrm{a}_1 + \beta\mathrm{b}_1 + \gamma\mathrm{c}_1 \quad \text{and} \quad \alpha\mathrm{a}_1' + \beta\mathrm{b}_1' + \gamma\mathrm{c}_1';$$

the values of a, b, c are $A\sqrt{E}$, $B\sqrt{E}$, $C\sqrt{E}$, where $E_1 = 2\omega$, and hence

$$\begin{aligned} \mathrm{a}_1\alpha + \mathrm{b}_1\beta + \mathrm{c}_1\gamma &= \left(A_1\sqrt{E} + \frac{A\omega}{\sqrt{E}}\right)\alpha + \left(B_1\sqrt{E} + \frac{B\omega}{\sqrt{E}}\right)\beta + \left(C_1\sqrt{E} + \frac{C\omega}{\sqrt{E}}\right)\gamma, \\ &= \sqrt{E}\,(A_1\alpha + B_1\beta + C_1\gamma) + \frac{\omega}{\sqrt{E}}(A\alpha + B\beta + C\gamma). \end{aligned}$$

From the values

$$A,\ B,\ C = bc' - b'c,\ ca' - c'a,\ ab' - a'b,$$

we have

$$A_1,\ B_1,\ C_1 = \beta c' - \gamma b' + b\gamma' - c\beta',\ \gamma a' - \alpha c' + c\alpha' - a\gamma',\ \alpha b' - \beta a' + a\beta' - b\alpha',$$

and thence

$$A_1\alpha + B_1\beta + C_1\gamma = -\,[a\,(\beta\gamma' - \beta'\gamma) + b\,(\gamma\alpha' - \gamma'\alpha) + c\,(\alpha\beta' - \alpha'\beta)],$$

$$= -\begin{vmatrix} a, & b, & c \\ \alpha, & \beta, & \gamma \\ \alpha', & \beta', & \gamma' \end{vmatrix}.$$

Hence

$$\mathrm{a}_1\alpha + \mathrm{b}_1\beta + \mathrm{c}_1\gamma = -\sqrt{E}\begin{vmatrix} a, & b, & c \\ \alpha, & \beta, & \gamma \\ \alpha', & \beta', & \gamma' \end{vmatrix} + \frac{\omega E'}{\sqrt{E}}.$$

Next, we have

$$\mathrm{a}', \mathrm{b}', \mathrm{c}' = cB - bC,\ aC - cA,\ bA - aB,$$

and thence

$$\begin{aligned}
&\alpha \mathrm{a}_1' + \beta \mathrm{b}_1' + \gamma \mathrm{c}_1' \\
&\quad = \alpha (\gamma B - \beta C + cB_1 - bC_1) + \beta (\alpha C - \gamma A + aC_1 - cA_1) + \gamma (\beta A - \alpha B + bA_1 - aB_1) \\
&\quad = (b\gamma - c\beta) A_1 + (c\alpha - a\gamma) B_1 + (a\beta - b\alpha) C_1 \\
&\quad = \quad (b\gamma - c\beta)(c'\beta - b'\gamma + b\gamma' - c\beta') \\
&\qquad + (c\alpha - a\gamma)(a'\gamma - c'\alpha + c\alpha' - a\gamma') \\
&\qquad + (a\beta - b\alpha)(b'\alpha - a'\beta + a\beta' - b\alpha');
\end{aligned}$$

the portion hereof, which is quadric in α, β, γ, is

$$\begin{aligned}
&= -(\alpha^2 + \beta^2 + \gamma^2)(aa' + bb' + cc') + (a\alpha + b\beta + c\gamma)(a'\alpha + b'\beta + c'\gamma), \\
&= -\lambda F + \omega\varpi,
\end{aligned}$$

and the remaining portion, which is lineo-linear in α, β, γ and α', β', γ', is

$$\begin{aligned}
&= (\alpha\alpha' + \beta\beta' + \gamma\gamma')(a^2 + b^2 + c^2) - (a\alpha + b\beta + c\gamma)(a\alpha' + b\beta' + c\gamma'), \\
&= E\mu'' - \omega\omega';
\end{aligned}$$

we thus have

$$\alpha \mathrm{a}_1' + \beta \mathrm{b}_1' + \gamma \mathrm{c}_1' = -\lambda F + E\mu'' + \omega(\varpi - \omega').$$

Hence the second portion of the numerator, or

$$Q(\alpha \mathrm{a}_1 + \beta \mathrm{b}_1 + \gamma \mathrm{c}_1) - P(\alpha \mathrm{a}_1' + \beta \mathrm{b}_1' + \gamma \mathrm{c}_1'),$$

is

$$= (E\varpi - F\omega)\left\{-\sqrt{E}\begin{vmatrix} a, & b, & c \\ \alpha, & \beta, & \gamma \\ \alpha', & \beta', & \gamma' \end{vmatrix} + \frac{\omega E'}{\sqrt{E}}\right\} - E'\sqrt{E}\{-\lambda F + \mu'' E + \omega(\varpi - \omega')\}.$$

There are two terms, $+E'\sqrt{E}\omega\varpi$ and $-E'\sqrt{E}\omega\varpi$, which destroy each other, and thus the whole second part of the numerator is

$$= -(E\varpi - F\omega)\begin{vmatrix} a, & b, & c \\ \alpha, & \beta, & \gamma \\ \alpha', & \beta', & \gamma' \end{vmatrix} - E'\sqrt{E}(\mu'' E - \lambda F) + \frac{E'\omega}{\sqrt{E}}(\omega' E - \omega F),$$

and, for the corresponding part of $\frac{dl}{dp}$, we must divide this by $P^2 + Q^2, = V^2\Delta$.

Hence, finally, we have

$$\begin{aligned}
\frac{dl}{dp} &= \frac{\sqrt{E}}{\Delta}\begin{vmatrix} a, & b, & c \\ \alpha, & \beta, & \gamma \\ \alpha_1, & \beta_1, & \gamma_1 \end{vmatrix} \\
&\quad + \frac{1}{V^2\Delta}\left\{-(\varpi E - \omega F)\begin{vmatrix} a, & b, & c \\ \alpha, & \beta, & \gamma \\ \alpha', & \beta', & \gamma' \end{vmatrix} - E'\sqrt{E}(\mu'' E - \lambda F) + \frac{E'\omega}{\sqrt{E}}(\omega' E - \omega F)\right\},
\end{aligned}$$

where I recall that the values of ω, ϖ, ω', λ, and μ'' are $a\alpha+b\beta+c\gamma$, $a'\alpha+b'\beta+c'\gamma$, $a\alpha'+b\beta'+c\gamma'$, $\alpha^2+\beta^2+\gamma^2$, and $\alpha\alpha'+\beta\beta'+\gamma\gamma'$, respectively. Observe that the first term in this expression for $\frac{dl}{dp}$ is

$$=-\frac{dn}{dp}.$$

We thus have

$$u=\frac{dm}{dp}\cos l=\frac{E\varpi-F\omega}{VE},$$

$$v=\frac{dm}{dp}\sin l=\frac{E'}{V\sqrt{E}},$$

$$w=\frac{dn}{dp}-\frac{dl}{dp}=-\frac{2\sqrt{E}}{\sqrt{\Delta}}\begin{vmatrix} a, & b, & c \\ \alpha, & \beta, & \gamma \\ \alpha_1, & \beta_1, & \gamma_1 \end{vmatrix}$$

$$+\frac{1}{V^2\Delta}\left\{(\varpi E-\omega F)\begin{vmatrix} a, & b, & c \\ \alpha, & \beta, & \gamma \\ \alpha', & \beta', & \gamma' \end{vmatrix}+E'\sqrt{E}\,(\mu''E-\lambda F)-\frac{E'\omega}{\sqrt{E}}(\omega'E-\omega F)\right\},$$

and we thence obtain at once the values of U, V, W; viz. these are

$$U=\frac{dM}{dq}\cos L=\frac{G\omega''-F\varpi''}{VG},$$

$$V=\frac{dM}{dq}\sin L=\frac{G'}{V\sqrt{G}},$$

$$W=\frac{dN}{dq}-\frac{dL}{dq}=-\frac{2\sqrt{G}}{\sqrt{\Delta''}}$$

$$+\frac{1}{V^2\Delta''}\left\{(\omega''G-\varpi''F)\begin{vmatrix} a', & b', & c' \\ \alpha'', & \beta'', & \gamma'' \\ \alpha_2'', & \beta_2'', & \gamma_2'' \end{vmatrix}+G'\sqrt{G}\,(\mu G-\lambda''F)-\frac{G'\varpi''}{\sqrt{G}}(\varpi'G-\varpi''F)\right\},$$

where α_2'', β_2'', γ_2'' denote the derived functions of α'', β'', γ'' in regard to q, viz. these are the third derived functions of x, y, z in regard to q.

We have, moreover,

$$dx^2+dy^2+dz^2=E\,dp^2+2F\,dp\,dq+G\,dq^2,\ =r^2\,dt^2+2rR\cos\theta\,dt\,dT+R^2\,dT^2;$$

that is,

$$r=\sqrt{E},\quad R=\sqrt{G},\quad \cos\theta=\frac{F}{\sqrt{EG}},$$

and therefore also

$$\sin\theta=\frac{\sqrt{EG-F^2}}{\sqrt{EG}},\quad =\frac{V}{\sqrt{EG}}.$$

Writing p, q in place of t, T, and for r, R substituting their values, then, with the foregoing values of u, v, w, U, V, W, Codazzi's six equations are

$$\frac{du}{dq} = \frac{dU}{dp}\cos\theta + \frac{dW}{dp}\sin\theta + w\left(V - \frac{d\theta}{dq}\right) + (U\sin\theta - W\cos\theta)\left(v - \frac{d\theta}{dp}\right),$$

$$\frac{dU}{dp} = \frac{du}{dq}\cos\theta + \frac{dW}{dq}\sin\theta + W\left(v - \frac{d\theta}{dp}\right) + (u\sin\theta - w\cos\theta)\left(V - \frac{d\theta}{dq}\right),$$

$$\left(\frac{dv}{dq} + \frac{dV}{dp}\right)\frac{d^2\theta}{dp\,dq} + \sin\theta\,(uU - wW) - \cos\theta\,(uW + wU) = 0,$$

$$\sqrt{G}\,(u\cos\theta + w\sin\theta) = \sqrt{E}\,(U\cos\theta + W\sin\theta),$$

$$\sqrt{G}\sin\theta\left(v - \frac{d\theta}{dp}\right) + \frac{d\sqrt{G}}{dp}\cos\theta = \frac{d\sqrt{E}}{dq},$$

$$\sqrt{E}\sin\theta\left(V - \frac{d\theta}{dq}\right) + \frac{d\sqrt{E}}{dq}\cos\theta = \frac{d\sqrt{G}}{dp}.$$

I take the opportunity of remarking that Gauss, in § 11 of his memoir, gives the first and second of the formulæ

$$\alpha\, V^2 + a(\varpi F - \omega G) + a'(\omega F - \varpi E) - AE' = 0,$$

$$\alpha'\, V^2 + a(\varpi' F - \omega' G) + a'(\omega' F - \varpi' E) - AF' = 0,$$

$$\alpha'' V^2 + a(\varpi'' F - \omega'' G) + a'(\omega'' F - \varpi'' E) - AG' = 0,$$

(each of them one out of a system of three like equations), where, as before,

$$V^2 = EG - F^2,$$

$$E',\ F',\ G' = A\alpha + B\beta + C\gamma,\ A\alpha' + B\beta' + C\gamma',\ A\alpha'' + B\beta'' + C\gamma'',$$

$$\omega,\ \omega',\ \omega'' = a\alpha + b\beta + c\gamma,\ a\alpha' + b\beta' + c\gamma',\ a\alpha'' + b\beta'' + c\gamma'',$$

$$= \tfrac{1}{2}E_1,\ \tfrac{1}{2}E_2,\ F_2 - \tfrac{1}{2}G_1,$$

$$\varpi,\ \varpi',\ \varpi'' = a'\alpha + b'\beta + c'\gamma,\ a'\alpha' + b'\beta' + c'\gamma',\ a'\alpha'' + b'\beta'' + c'\gamma'',$$

$$= F_1 - \tfrac{1}{2}E_2,\ \tfrac{1}{2}G_1,\ \tfrac{1}{2}G_2.$$

These are, in fact, the formulæ (IV.),

$$\frac{d^2x}{dp^2} - \begin{Bmatrix}11\\1\end{Bmatrix}\frac{dx}{dp} - \begin{Bmatrix}11\\2\end{Bmatrix}\frac{dx}{dq} + c_{11}X = 0,$$

$$\frac{d^2x}{dp\,dq} - \begin{Bmatrix}12\\1\end{Bmatrix}\frac{dx}{dp} - \begin{Bmatrix}12\\2\end{Bmatrix}\frac{dx}{dq} + c_{12}X = 0,$$

$$\frac{d^2x}{dq^2} - \begin{Bmatrix}22\\1\end{Bmatrix}\frac{dx}{dp} - \begin{Bmatrix}22\\2\end{Bmatrix}\frac{dx}{dq} + c_{22}X = 0,$$

of the memoir, Weingarten, "Ueber die Deformation einer biegsamen unausdehnbaren

Fläche," *Crelle,* t. C. (1887), pp. 296—310; viz. the symbols correspond to those of Weingarten, as follows:—

$$E,\ F,\ G = a_{11},\ a_{12},\ a_{22},$$

$$V^2 = EG - F^2 = a = \rho^2,$$

$$E',\ F',\ G' = -c_{11}\sqrt{a},\ -c_{12}\sqrt{a},\ -c_{22}\sqrt{a},$$

$$A,\ B,\ C = \rho X,\ \rho Y,\ \rho Z,$$

$$\omega,\ \omega',\ \omega'' = \tfrac{1}{2}\frac{da_{11}}{dp},\qquad \tfrac{1}{2}\frac{da_{11}}{dq},\qquad \frac{da_{12}}{dq} - \tfrac{1}{2}\frac{da_{22}}{dp},$$

$$\varpi,\ \varpi',\ \varpi'' = \frac{da_{12}}{dp} - \tfrac{1}{2}\frac{da_{11}}{dq},\ \tfrac{1}{2}\frac{da_{22}}{dp},\ \tfrac{1}{2}\frac{da_{22}}{dq},$$

and thus Weingarten's symbols, $\left\{ {11 \atop 1} \right\}$, &c., have the values

$$\left\{ {11 \atop 1} \right\} = \frac{1}{a}\left[a_{12}\left(-\frac{da_{12}}{dp} + \tfrac{1}{2}\frac{da_{11}}{dq}\right) + a_{22}\left(\tfrac{1}{2}\frac{da_{11}}{dp}\right)\right],$$

$$\left\{ {12 \atop 1} \right\} = \frac{1}{a}\left[a_{12}\left(-\tfrac{1}{2}\frac{da_{22}}{dp}\right) + a_{22}\left(\tfrac{1}{2}\frac{da_{11}}{dq}\right)\right],$$

$$\left\{ {22 \atop 1} \right\} = \frac{1}{a}\left[a_{12}\left(-\tfrac{1}{2}\frac{da_{22}}{dq}\right) + a_{22}\left(\frac{da_{12}}{dq} - \tfrac{1}{2}\frac{da_{22}}{dp}\right)\right],$$

$$\left\{ {11 \atop 2} \right\} = \frac{1}{a}\left[a_{12}\left(-\tfrac{1}{2}\frac{da_{11}}{dp}\right) + a_{11}\left(\frac{da_{12}}{dp} - \tfrac{1}{2}\frac{da_{11}}{dq}\right)\right],$$

$$\left\{ {12 \atop 2} \right\} = \frac{1}{a}\left[a_{12}\left(-\tfrac{1}{2}\frac{da_{11}}{dq}\right) + a_{11}\left(\tfrac{1}{2}\frac{da_{22}}{dp}\right)\right],$$

$$\left\{ {22 \atop 2} \right\} = \frac{1}{a}\left[a_{12}\left(-\frac{da_{12}}{dq} + \tfrac{1}{2}\frac{da_{22}}{dp}\right) + a_{11}\left(\tfrac{1}{2}\frac{da_{22}}{dq}\right)\right],$$

values which give, as they should do,

$$\left\{ {11 \atop 1} \right\} + \left\{ {12 \atop 2} \right\} = \frac{1}{\sqrt{a}}\frac{d\sqrt{a}}{dp},\ \text{and}\ \left\{ {12 \atop 1} \right\} + \left\{ {22 \atop 2} \right\} = \frac{1}{\sqrt{a}}\frac{d\sqrt{a}}{dq}.$$

The foregoing comparison serves to explain the notation of Weingarten's valuable memoir.