925.

ON WARING'S FORMULA FOR THE SUM OF THE mth POWERS OF THE ROOTS OF AN EQUATION.

[From the *Messenger of Mathematics*, vol. XXI. (1892), pp. 133—137.]

THE formula in question, Prob. I. of Waring's *Meditationes Algebraicæ*, Cambridge, 1782, making therein a slight change of notation, is as follows: viz. the equation being

$$x^n + bx^{n-1} + cx^{n-2} + dx^{n-3} + \ldots = 0,$$

then we have

$$(-)^m S_m =$$

$$\begin{array}{rl}
 & b^m \\
-mc & b^{m-2} \\
+md & b^{m-3} \\
\left.\begin{array}{r} -me \\ +\frac{1}{2}m\,.\,m-3\,.\,c^2 \end{array}\right\} & b^{m-4} \\
\left.\begin{array}{r} +mf \\ -m\,.\,m-4\,.\,cd \end{array}\right\} & b^{m-5} \\
\left.\begin{array}{r} -m\,.\,g \\ +m\,.\,m-5\,.\,ce \\ +\frac{1}{2}m\,.\,m-5\,.\,d^2 \\ -\frac{1}{6}m\,.\,m-4\,.\,m-5\,.\,c^3 \end{array}\right\} & b^{m-6}
\end{array}
\qquad
\begin{array}{rl}
\left.\begin{array}{r} +m\,.\,h \\ -m\,.\,m-6\,.\,cf \\ -m\,.\,m-6\,.\,de \\ +\frac{1}{2}m\,.\,m-5\,.\,m-6\,.\,c^2d \end{array}\right\} & b^{m-7} \\
\left.\begin{array}{r} +m\,.\,i \\ +m\,.\,m-7\,.\,cg \\ +m\,.\,m-7\,.\,df \\ +\frac{1}{2}m\,.\,m-7\,.\,e^2 \\ -\frac{1}{2}m\,.\,m-6\,.\,m-7\,.\,c^2e \\ -\frac{1}{2}m\,.\,m-6\,.\,m-7\,.\,cd^2 \\ +\frac{1}{24}m\,.\,m-5\,.\,m-6\,.\,m-7\,.\,c^4 \end{array}\right\} & b^{m-8} \\
+\,\&c., &
\end{array}$$

where, reckoning the weights of b, c, d, e, ... as 1, 2, 3, 4 ..., respectively, the several terms are all the terms of the weight m, or (what is the same thing) in the coefficient of $b^{m-\theta}$ we have all the combinations of $c, d, e, \ldots$, (or say all the non-unitary combinations) of the weight θ, and where the numerical coefficient of

$$b^{m-\theta} c^{\mathrm{c}} d^{\mathrm{d}} e^{\mathrm{e}} \ldots \;(\mathrm{c}+\mathrm{d}+\mathrm{e}+\ldots=\theta),$$

is

$$=(-)^{\mathrm{c}+\mathrm{e}+\mathrm{g}+\ldots} \frac{m \,.\, m-(\theta-\delta+1) \,.\, m-(\theta-\delta+2) \ldots m-(\theta-1)}{\Pi \mathrm{c} \,.\, \Pi \mathrm{d} \,.\, \Pi \mathrm{e} \ldots}.$$

Thus for the term $b^{m-8} c^2 e^1$, $\theta=8$; c, d, e $=2, 4, 1$ respectively (the other exponents each vanishing), and the coefficient is

$$(-)^3 \frac{m \,.\, m-6 \,.\, m-7}{1 \,.\, 2 \,.\, 1}, =-\tfrac{1}{2} m \,.\, m-6 \,.\, m-7,$$

as above; and so in other cases.

For the MacMahon form

$$1+b x+\frac{c x^2}{1 \,.\, 2}+\ldots=(1-\alpha x)(1-\beta x) \ldots,$$

or say

$$y^n+\frac{b}{1} y^{n-1}+\frac{c}{1 \,.\, 2} y^{n-2}+\ldots=(y-\alpha)(y-\beta) \ldots,$$

we must for $b, c, d, \ldots$, write $b, \dfrac{c}{1 \,.\, 2}, \dfrac{d}{1 \,.\, 2 \,.\, 3}, \ldots$ respectively: we thus have

$$\begin{array}{rll}
(-)^m \,.\, S_m = & b^m & \\
& -m \dfrac{c}{1 \,.\, 2} & b^{m-2} \\
& +m \dfrac{d}{1 \,.\, 2 \,.\, 3} & b^{m-3} \\
& \left.\begin{array}{l} -m \dfrac{e}{1 \,.\, 2 \,.\, 3 \,.\, 4} \\ +\tfrac{1}{2} m \,.\, m-3 \left(\dfrac{c}{1 \,.\, 2}\right)^2 \end{array}\right\} & b^{m-4} \\
& +\&c., &
\end{array}$$

or say

$$\begin{array}{rll}
(-)^m \,\Pi\,(m-1)\, S_m = & \Pi\,(m-1) & b^4 \\
& -\Pi m \dfrac{c}{1 \,.\, 2} & b^{m-2} \\
& +\Pi m \dfrac{d}{1 \,.\, 2 \,.\, 3} & b^{m-3} \\
& \left.\begin{array}{l} -\Pi m \dfrac{e}{1 \,.\, 2 \,.\, 3 \,.\, 4} \\ +\Pi m \dfrac{m-3}{2} \left(\dfrac{c}{1 \,.\, 2}\right)^2 \end{array}\right\} & b^{m-4} \\
& +\&c., &
\end{array}$$

the numerical coefficient of

$$b^{m-\theta}c^{\mathrm{c}}d^{\mathrm{d}}e^{\mathrm{e}}\ldots\ (\mathrm{c}+\mathrm{d}+\mathrm{e}+\ldots=\theta)$$

being

$$(-)^{\mathrm{c}+\mathrm{e}+\mathrm{g}+\ldots}\frac{\Pi m\,.\,m-(\theta-\delta+1)\,.\,m-(\theta-\delta+2)\ldots m-(\theta-1)}{\Pi\mathrm{c}\,.\,\Pi\mathrm{d}\,.\,\Pi\mathrm{e}\ldots(\Pi 2)^{\mathrm{c}}\,(\Pi 3)^{\mathrm{d}}\,(\Pi 4)^{\mathrm{e}}\ldots}.$$

It is convenient to write down the literal terms in alphabetical order (AO), calculating and affixing to each term the proper numerical coefficient; thus taking

$$1+bx+c\frac{x^2}{1.2}+\ldots=(1-\alpha x)(1-\beta x)(1-\gamma x)\ldots,$$

we find

$-120S_6=$	g	.	1
	bf	. −	6
	ce	. −	15
	d^2	. −	10
	b^2e	. +	30
	bcd	. +	120
	c^3	. +	30
	b^3d	. −	120
	b^2c^2	. −	270
	b^4c	. +	360
	b^6	. −	120
			± 541

this expression, as representing the value of the non-unitary function S_6, being in fact a seminvariant.

It is to be remarked that the foregoing expression for the sum of the mth powers of the roots of the equation

$$x^n+bx^{n-1}+cx^{n-2}+\ldots=0$$

is, in fact, the series for x^m continued so far only as the exponent of b is not negative: see as to this Note XI. of Lagrange's *Équations Numériques*. For the *à posteriori* verification, observe that we have

$$x+b+\frac{c}{x}+\frac{d}{x^2}+\ldots=0,$$

or writing for a moment $u=-b$, say this is

$$x=u+fx,$$

where

$$fx = -\frac{c}{x} - \frac{d}{x^2} - \frac{e}{x^3} - \&c.$$

Hence, by Lagrange's theorem,

$$\begin{aligned} x^m = \ & u^m \\ & - mu^{m-1}\left(\frac{c}{u} + \frac{d}{u^2} + \frac{e}{u^3} + \dots\right) \\ & + \left\{mu^{m-1}\left(\frac{c}{u} + \frac{d}{u^2} + \frac{e}{u^3} + \dots\right)^2\right\}' \frac{1}{1.2} \\ & - \left\{mu^{m-1}\left(\frac{c}{u} + \frac{d}{u^2} + \frac{e}{u^3} + \dots\right)^3\right\}'' \frac{1}{1.2.3} \\ & + \&c., \end{aligned}$$

where the accents denote differentiations in regard to u. This is

$$\begin{aligned} = \ & u^m \\ & - m\{cu^{m-2} + du^{m-3} + eu^{m-4} + fu^{m-5} + gu^{m-6} + \dots\} \\ & + \tfrac{1}{2}m\{(m-3)c^2u^{m-4} + (m-4)2cdu^{m-5} + (m-5)(d^2+2ce)u^{m-6} + \dots\} \\ & - \tfrac{1}{6}m\{(m-4)(m-5)c^3u^{m-3} + \dots\} \\ & + \&c. \\ = \ & u^m \\ & + u^{m-2}. - mc \\ & + u^{m-3}. - md \\ & + u^{m-4}. - me + \tfrac{1}{2}m.m-3.c^2 \\ & + u^{m-5}. - mf + \tfrac{1}{2}m.m-4.2cd \\ & + u^{m-6}. - mg + \tfrac{1}{2}m.m-5.(d^2+2ce) - \tfrac{1}{6}m.m-4.m-5.c^3 \\ & + \&c., \end{aligned}$$

which, putting therein $u = -b$ and multiplying each side by $(-)^m$, is the before-mentioned formula for $(-)^m S\alpha^m$: in that formula the series being continued only so far as the exponent of b is not negative.

I notice also that we cannot easily, by means of the known formula

$$S\alpha^m\beta^p = S\alpha^m . S\alpha^p - S\alpha^{m+p},$$

deduce an expression for $S\alpha^m\beta^p$: in fact, forming the product of the series for $S\alpha^m$, $S\alpha^p$ respectively, this product is identically equal to the series for $S\alpha^{m+p}$, or we seem to obtain $0 = S\alpha^m S\alpha^p - S\alpha^{m+p}$; to obtain the correct formula, we have to take each of the three series only so far as the exponent of b therein respectively is not negative: and it is not easy to see how the resulting formula is to be expressed.