919.

ON THE PROBLEM OF TACTIONS.

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1. I REMARK that the problem "to draw a circle touching each of three given circles" is not properly a problem with eight solutions, but it is a set of four problems each with two solutions: viz. if a, b, c are the radii of the given circles, ϑ the radius of the tangent circle, and r, s, t the distances of its centre from the centres of the given circles respectively, then in the four problems respectively we have

$$\begin{array}{llll} r=a+\vartheta, & r=\ \ a+\vartheta, & r=\ \ a+\vartheta, & r=\ \ a+\vartheta, \\ s=b+\vartheta, & s=-b+\vartheta, & s=\ \ b+\vartheta, & s=-b+\vartheta, \\ t=c+\vartheta, & t=\ \ c+\vartheta, & t=-c+\vartheta, & t=-c+\vartheta; \end{array}$$

and thence also

$$\begin{array}{llll} s-t=b-c, & s-t=-b-c, & s-t=\ \ b+c, & s-t=-b+c, \\ t-r=c-a, & t-r=\ \ c-a, & t-r=-c-a, & t-r=-c-a, \\ r-s=a-b, & r-s=\ \ a+b, & r-s=\ \ a-b, & r-s=\ \ a+b, \end{array}$$

where a, b, c may be regarded each of them as positive; but the sign of each distance r, s, t, and of the radius ϑ is not assumable at pleasure, but analytically it comes out as a result in the solution, or it may be found by geometrical considerations. Thus, if the given circles are external to each other, then in the first problem we have two solutions, a first tangent circle touched externally by each of the given circles, and a second tangent circle touched internally by each of the given circles; and taking r, s, t, ϑ_1, ϑ_2, each of them as positive, the signs in the two solutions respectively are

$$\begin{array}{lll} r=a+\vartheta_1, & r=-a+\vartheta_2, \text{ or say} & -r=a-\vartheta_2, \\ s=b+\vartheta_1, & s=-b+\vartheta_2, & -s=b-\vartheta_2, \\ t=c+\vartheta_1, & t=-c+\vartheta_2, & -t=c-\vartheta_2, \end{array}$$

and so in other cases. The second, third, and fourth problems are, it is clear, derived from the first problem by the change of (b, c) into $(-b, c)$, $(b, -c)$, $(-b, -c)$ respectively: so that only the first problem need be considered, viz. this is as above

$$r = a + \vartheta, \quad s = b + \vartheta, \quad t = c + \vartheta,$$

whence also

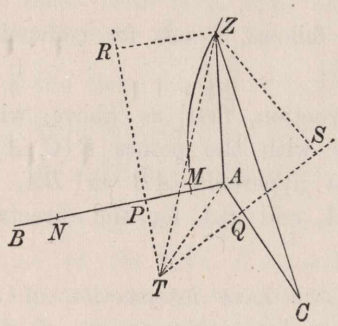
$$s - t = b - c, \quad t - r = c - a, \quad r - s = a - b,$$

where $b - c$, $c - a$, $a - b$ are given magnitudes the algebraical sum of which is $= 0$, viz. they are one of them positive, and the other two each negative, or else two of them each positive, and the remaining one negative.

2. The most simple and straightforward geometrical solution is that given in the *Principia*, Book I. Lemma XVI.; I reproduce this as given in Motte's translation (*The Mathematical Principles of Natural Philosophy*, by Sir Isaac Newton, translated into English by Andrew Motte, 8°, 2 vols., London, 1729).

"Lemma XVI. *From three given points to draw to a fourth point which is not given three right lines whose differences shall be either given or none at all.*

Case 1. Let the given points be A, B, C (see figure) and Z the fourth point which we are to find: because of the given difference of the line AZ, BZ, the locus



of the point Z will be a hyperbola whose foci are A and B and whose principal axe is the given difference. Let that axe be MN. Taking PM to MA as MN is to AB, erect PR perpendicular to AB, and let fall ZR perpendicular to PR; then from the nature of the hyperbola ZR will be to AZ as MN is to AB. And by the like argument the locus of the point Z will be another hyperbola whose foci are A, C, and whose principal axe is the difference between AZ and CZ; and QS a perpendicular on AC may be drawn to which (QS), if from any point Z of this hyperbola a perpendicular ZS is let fall, this (ZS) shall be to AZ as the difference between AZ and CZ is to AC. Wherefore the ratios of ZR and ZS to AZ are given and consequently the ratio of ZR to ZS one to the other: and therefore if the right lines RP, QS meet in T, and TZ and TA are drawn, the figure $TRZS$ will be given in specie, and the right line TZ, in which the point Z is somewhere placed, will be given in position. There will be given also the right line TA and the angle ATZ; and because the ratios of AZ and TZ to ZS are given, their ratio to each other is given also; and thence will be given also the triangle ATZ whose vertex is the point Z. Q. E. I.

Case 2. If two of the three lines, for instance AZ and BZ, are equal, &c.

Case 3. If all the three are equal, &c.

This problematic lemma is likewise resolved in Apollonius's Book of Tactions restored by Vieta."

3. Newton, in fact, considers the hyperbolas AB and AC, each of given axis, having the foci (A, B) and (A, C) respectively, and having PR, QS for the directrices which in the two hyperbolas respectively belong to the common focus A. The required point Z thus lies on a given line through the intersection T of these two directrices; and its position on this line is determined by the condition that the distances AZ, TZ shall be in a given ratio: the locus of the points which satisfy this last condition is of course a circle; and the position of Z is thus determined as the intersection of the given line by a given circle (which I will call a Newton-circle); there are two intersections giving points Z_1, Z_2, which are the centres of the two tangent circles respectively: and the line as a *locus in quo* of these two points is of course a determinate line, but Newton's circle is only one of a singly infinite series of circles through the two points: any other solution of the problem gives therefore the same line, but not in general the same circle.

4. In what immediately follows, I use for convenience the letter F in place of the foregoing letter T.

Effecting Newton's construction, first as above, with the points $A(B, C)$; and then in like manner, secondly with the points $B(C, A)$, and thirdly with the points $C(A, B)$; and in regard to a hyperbola AB or BA, writing AB for the directrix which belongs to the focus A, and BA for the directrix which belongs to the focus B; then

For hyperbolas AB, AC, we have intersection of directrices AB, AC is a point F; for hyperbolas BC, BA, we have intersection of directrices BC, BA is a point G; for hyperbolas CA, CB, we have intersection of directrices CA, CB is a point H.

Hence these three points F, G, H lie in a line, which is the line containing the required points Z_1, Z_2; or say it is the line Z_1Z_2.

The points Z_1, Z_2 are determined as the intersections of this line by a circle which is the locus of the points whose distances A, F are in a given ratio; similarly they are determined as the intersections by a circle which is the locus of the points whose distances from B, G are in a given ratio; and they are determined as the intersections by a circle which is the locus of the points whose distances from C, H are in a given ratio. We have thus three Newton-circles: if the centres of these are F', G', H' respectively, then clearly these points lie on a line $F'G'H'$, which bisects at right angles the line (or chord) Z_1Z_2; the points A, F, F' are obviously in a line, as are also the points B, G, G', and the points C, H, H'; or (what is the same thing) considering for a moment the line $F'G'H'$ as a given line bisecting Z_1Z_2 at right angles, then the centres F', G', H' would be found as the intersections of this line $F'G'H'$ with the lines AF, BG, CH respectively.

The points Z_1, Z_2 being determined as above, then the points of contact α_1, β_1, γ_1 of the circle Z_1 with the circles A, B, C respectively are points of intersection of the lines Z_1A, Z_1B, Z_1C with these circles respectively; and similarly the points of contact α_2, β_2, γ_2 of the circle Z_2 with the same circles respectively are points of intersection of the lines Z_2A, Z_2B, Z_2C with these circles respectively.

5. I compare with Newton's the construction in which the centres Z_1, Z_2 are determined by means of the points of contact with the given circles: I may refer to Prop. 10, pp. 118—120 of Casey's *Sequel to Euclid* (12°, Dublin, 1881). We have here the line Z_1Z_2 determined as the line through the radical centre Ω of the three circles A, B, C, perpendicular to an axis of symmetry (say the axis containing the three centres of direct symmetry) of the same circles: this point Ω is the centre of the orthotomic circle. And if the common chords of the orthotomic circle and the circles A, B, C respectively meet the axis of symmetry in the points a, b, c; then we have α_1, α_2 as the points of contact of the tangents from a to the circle A; β_1, β_2 as the points of contact of the tangents from b to the circle B; and γ_1, γ_2 as the points of contact of the tangents from c to the circle C: the suffixes 1 and 2 can and must be so applied that the three lines $A\alpha_1$, $B\beta_1$, $C\gamma_1$ meet in a point Z_1 of the line Z_1Z_2, and the three lines $A\alpha_2$, $B\beta_2$, $C\gamma_2$ in a point Z_2 of the same line. We thus obtain the required points Z_1 and Z_2.

6. Taking the equations of the three circles to be

$$(x-\alpha)^2+(y-\alpha_1)^2=a^2,$$
$$(x-\beta)^2+(y-\beta_1)^2=b^2,$$
$$(x-\gamma)^2+(y-\gamma_1)^2=c^2,$$

I wish to obtain the equations of the line Z_1Z_2 and of the three Newton-circles; but I will first find, by a separate analytical investigation, an expression for the length of the chord Z_1Z_2.

Writing f, g, h for the distances BC, CA, AB of the points A, B, C from each other; r_1, s_1, t_1 for the distances of Z_1 from these points respectively, and r_2, s_2, t_2 for the distances of Z_2 from these points respectively; we have a triangle whose sides are f, g, h, and two points Z_1, Z_2 whose distances from the vertices are r_1, s_1, t_1 and r_2, s_2, t_2 respectively; and we can in terms of these data find an expression for the distance x of the points Z_1, Z_2 from each other. In fact, considering any four points 1, 2, 3, 4 and any other four points 1′, 2′, 3′, 4′, then if 11′, 12′, &c., denote the squared distances of the points 1 and 1′ from each other, of the points 1 and 2′ from each other, &c., we have between the several distances the relation

$$\begin{vmatrix} 0, & 1\ , & 1\ , & 1\ , & 1 \\ 1, & 11', & 12', & 13', & 14' \\ 1, & 21', & 22', & 23', & 24' \\ 1, & 31', & 32', & 33', & 34' \\ 1, & 41', & 42', & 43', & 44' \end{vmatrix} = 0,$$

and thence, taking the points 1, 2, 3 and also the points 1′, 2′, 3′ to be the points A, B, C respectively, and the points 4 and 4′ to be the points Z_1 and Z_2 respectively, we have the required relation

$$\begin{vmatrix} 0, & 1, & 1, & 1, & 1 \\ 1, & 0, & h^2, & g^2, & r_1^2 \\ 1, & h^2, & 0, & f^2, & s_1^2 \\ 1, & g^2, & f^2, & 0, & t_1^2 \\ 1, & r^2, & s^2, & t^2, & x^2 \end{vmatrix} = 0,$$

viz. putting for shortness

$$\Delta = f^4 + g^4 + h^4 - 2g^2h^2 - 2h^2f^2 - 2f^2g^2,$$

this equation is

$$\Delta x^2 + \begin{vmatrix} 0, & 1, & 1, & 1, & 1 \\ 1, & 0, & h^2, & g^2, & r_1^2 \\ 1, & h^2, & 0, & f^2, & s_1^2 \\ 1, & g^2, & f^2, & 0, & t_1^2 \\ 1, & r^2, & s^2, & t^2, & 0 \end{vmatrix} = 0,$$

where the determinant has the value

$$\begin{aligned} &(r^2 + r_1^2) f^2 (-f^2 + g^2 + h^2) + f^2 \{(t_1^2 - r_1^2)(r_2^2 - s_2^2) + (t_2^2 - r_2^2)(r_1^2 - s_1^2)\} \\ &+ (s^2 + s_1^2)\, g^2 (\ \ f^2 - g^2 + h^2) + g^2 \{(r_1^2 - s_1^2)(s_2^2 - t_2^2) + (r_2^2 - s_2^2)(s_1^2 - t_1^2)\} \\ &+ (t_1^2 + t_2^2)\, h^2 (\ \ f^2 + g^2 - h^2) + h^2 \{(s_1^2 - t_1^2)(t_2^2 - r_2^2) + (s_2^2 - t_2^2)(t_1^2 - r_1^2)\} \\ &- 2f^2g^2h^2. \end{aligned}$$

7. For the points Z_1, Z_2, the distances r_1, s_1, t_1 and r_2, s_2, t_2 have the values $a + \vartheta_1$, $b + \vartheta_1$, $c + \vartheta_1$ and $a + \vartheta_2$, $b + \vartheta_2$, $c + \vartheta_2$ respectively, where ϑ_1, ϑ_2 are the radii of the tangent circles; substituting these values, we find

$$\Delta x^2 + 2\mathfrak{A} + 2\mathfrak{B}(\vartheta_1 + \vartheta_2) + \mathfrak{D}(\vartheta_1^2 + \vartheta_2^2) + 2\mathfrak{E}\vartheta_1\vartheta_2 = 0,$$

where

$$\begin{aligned} \mathfrak{A} = \ & a^2f^2(-f^2 + g^2 + h^2) + f^2(c^2 - a^2)(a^2 - b^2) - f^2g^2h^2 \\ & + b^2g^2 (\ \ f^2 - g^2 + h^2) + g^2(a^2 - b^2)(b^2 - c^2) \\ & + c^2h^2 (\ \ f^2 + g^2 - h^2) + h^2(b^2 - c^2)(c^2 - a^2), \\ \mathfrak{B} = \ & af^2 (-f^2 + g^2 + h^2) + f^2(c - a)(a - b)(2a + b + c) \\ & + bg^2 (\ \ f^2 - g^2 + h^2) + g^2(a - b)(b - c)(a + 2b + c) \\ & + ch^2 (\ \ f^2 + g^2 - h^2) + h^2(b - c)(c - a)(a + b + 2c), \\ \mathfrak{D} = \ & f^2(-f^2 + g^2 + h^2) \\ & + g^2(\ \ f^2 - g^2 + h^2) \\ & + h^2(\ \ f^2 + g^2 - h^2), \ = -\Delta, \\ \mathfrak{E} = \ & 4f^2(c - a)(a - b) \\ & + 4g^2(a - b)(b - c) \\ & + 4h^2(b - c)(c - a). \end{aligned}$$

But ϑ_1, ϑ_2 are the roots of an equation in ϑ, which is at once obtained from the foregoing equation by putting therein $x=0$, $\vartheta_1=\vartheta_2$, $=\vartheta$: viz. if, for shortness,

$$\mathfrak{C}=\mathfrak{D}+\mathfrak{E},\ =-\Delta+4\{f^2(c-a)(a-b)+g^2(a-b)(b-c)+h^2(b-c)(c-a)\},$$

then the equation in ϑ is

$$\mathfrak{A}+2\mathfrak{B}\vartheta+\mathfrak{C}\vartheta^2=0\,;$$

we have therefore

$$\vartheta_1+\vartheta_2=-\frac{2\mathfrak{B}}{\mathfrak{C}},\quad \vartheta_1\vartheta_2=\frac{\mathfrak{A}}{\mathfrak{C}};$$

and consequently

$$\Delta x^2+2\mathfrak{A}-\frac{4\mathfrak{B}^2}{\mathfrak{C}}+\mathfrak{D}\,\frac{4\mathfrak{B}^2-2\mathfrak{A}\mathfrak{C}}{\mathfrak{C}^2}+\frac{2\mathfrak{A}\mathfrak{E}}{\mathfrak{C}}=0,$$

that is,

$$\Delta\mathfrak{C}^2x^2+2\mathfrak{A}\mathfrak{C}^2-4\mathfrak{B}^2\mathfrak{C}+\mathfrak{D}(4\mathfrak{B}^2-2\mathfrak{A}\mathfrak{C})+2\mathfrak{A}\mathfrak{C}(\mathfrak{C}-\mathfrak{D})=0,$$

or, reducing,

$$\Delta\mathfrak{C}^2x^2+4(\mathfrak{B}^2-\mathfrak{A}\mathfrak{C})(\mathfrak{D}-\mathfrak{C})=0,$$

that is,

$$\Delta\mathfrak{C}^2x^2=4(\mathfrak{B}^2-\mathfrak{A}\mathfrak{C})\,\mathfrak{E},$$

where Δ, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{E}$ have the values given above; we hence find

$$\mathfrak{B}^2-\mathfrak{A}\mathfrak{C}=-\{f^2-(b-c)^2\}\{g^2-(c-a)^2\}\{h^2-(a-b)^2\}\times(f^4+g^4+h^4-2g^2h^2-2h^2f^2-2f^2g^2)$$
$$=-\{f^2-(b-c)^2\}\{g^2-(c-a)^2\}\{h^2-(a-b)^2\}\,\Delta\,;$$

and, putting for x its value ZZ_1, the equation thus reduces itself to

$$\mathfrak{C}^2(ZZ_1)^2=-4\{f^2-(b-c)^2\}\{g^2-(c-a)^2\}\{h^2-(a-b)^2\}\,\mathfrak{E}.$$

8. The denominator factor $\mathfrak{C}^2$ and the several numerator factors of $(ZZ_1)^2$ may be accounted for. It is to be observed that $(ZZ_1)^2$ does not contain the factor Δ of $\mathfrak{B}^2-\mathfrak{A}\mathfrak{C}$. If $\Delta=0$, the centres of the circles A, B, C are in a line; the two tangent circles are circles situate symmetrically in regard to the line ABC, that is, their radii are equal, and the line through their centres is bisected at right angles by the line ABC; the radii are equal, and thus $\mathfrak{B}^2-\mathfrak{A}\mathfrak{C}=0$, but the centres are not coincident, and thus ZZ_1 is not $=0$. The expression for $(ZZ_1)^2$ assumes a more simple form, for we have $\mathfrak{C}=-\Delta+\mathfrak{E}=\mathfrak{E}$, and the formula thus becomes

$$\mathfrak{E}(ZZ_1)^2=-4\{f^2-(b-c)^2\}\{g^2-(c-a)^2\}\{h^2-(a-b)^2\}.$$

If in this formula $\mathfrak{E}=0$, then we have $ZZ_1=\infty$; in fact, the circles A, B, C have here a pair of common tangents, and the tangent circles are these common tangents: each of the centres is thus a point at infinity, and the distance of the two centres is to be regarded as $=$ infinity. In verification observe that, if the circles have a pair of common tangents, then, taking the intersection of these for the origin, if P, Q, R be the distances of the centres from this origin, we have

$$P:Q:R=a:b:c\,;$$

and therefore

$$f:g:h=b-c:c-a:a-b\,;$$

whence

$$\mathfrak{E},\ =(c-a)(a-b)f^2+(a-b)(b-c)g^2+(b-c)(c-a)h^2,$$

contains a factor $(b-c)+(c-a)+(a-b)$, and is thus $=0$.

Supposing Δ not $=0$, then $\mathfrak{B}^2-\mathfrak{A}\mathfrak{C}$ is $=0$, that is, the two radii are equal only if one of the factors $f^2-(b-c)^2$, $g^2-(c-a)^2$, $h^2-(a-b)^2$ is $=0$, and in this case we have also $ZZ_1=0$; the two tangent circles are here coincident. The equation $f^2-(b-c)^2=0$ signifies that the circles B, C touch each other internally, or, what is the same thing (if, as was initially assumed, the radii b, c be taken to be each of them positive), the point of contact is a direct centre of symmetry; hence, when $f^2-(b-c)^2=0$, the two tangent circles are coincident. And similarly if $g^2-(c-a)^2=0$, or if $h^2-(a-b)^2=0$.

9. If in the general formula we have $\mathfrak{C}=0$, then $ZZ_1=\infty$; the circles A, B, C have here a common tangent and one of the tangent circles becomes this common tangent; we have thus a centre at infinity, and the distance of the two centres is thus also infinite.

To show that $\mathfrak{C}=0$ is the condition of a common tangent, suppose that the three circles have a common tangent, and let P, Q, R denote the distances of the points of contact from any fixed point on this tangent: we have

$$f^2=(Q-R)^2+(b-c)^2,$$
$$g^2=(R-P)^2+(c-a)^2,$$
$$h^2=(P-Q)^2+(a-b)^2,$$

equations which I represent by $f^2,\ g^2,\ h^2=\alpha^2+l^2,\ \beta^2+m^2,\ \gamma^2+n^2$, where $\alpha+\beta+\gamma=0$, $l+m+n=0$. We have

$$-f^2+g^2+h^2=-\alpha^2+\beta^2+\gamma^2-l^2+m^2+n^2,\ =-2\beta\gamma-2mn,$$

and forming the corresponding values of $f^2-g^2+h^2$, $f^2+g^2-h^2$, and multiplying by $f^2,\ g^2,\ h^2=\alpha^2+l^2,\ \beta^2+m^2,\ \gamma^2+n^2$ respectively, we find

$$-\Delta=-2\alpha^2mn-2\beta^2nl-2\gamma^2lm-2l^2\beta\gamma-2m^2\gamma\alpha-2n^2\alpha\beta.$$

But the last three terms hereof are

$$=l^2(-\alpha^2+\beta^2+\gamma^2)+m^2(-\beta^2+\gamma^2+\alpha^2)+n^2(-\gamma^2+\alpha^2+\beta^2),$$

which is

$$=\alpha^2(m^2+n^2-l^2)+\beta^2(n^2+l^2-m^2)+\gamma^2(l^2+m^2-n^2),$$

and this is

$$=-2\alpha^2mn-2\beta^2nl-2\gamma^2lm.$$

Hence we have

$$-\Delta=-4\alpha^2mn-4\beta^2nl-4\gamma^2lm,$$

that is,

$$=-4f^2mn-4g^2nl-4h^2lm,$$

or replacing l, m, n by their values, we find

$$-\Delta=-4\{f^2(c-a)(a-b)+g^2(a-b)(b-c)+h^2(b-c)(c-a)\},$$

which is the required condition $\mathfrak{C}=0$.

10. It would at first sight appear that the distance ZZ_1 of the two centres

would always vanish if $\mathfrak{C}=0$. But if A, B, C are real circles, this condition $\mathfrak{C}=0$, implies

$$\frac{f^2}{(b-c)^2}=\frac{g^2}{(c-a)^2}=\frac{h^2}{(a-b)^2},$$

whence $\Delta=0$, and this being so we have $\mathfrak{C}=-\Delta+\mathfrak{C}, =0$, and the value of ZZ_1 instead of being $=0$, is or appears to be infinite. In proof, take for a moment the origin at A and the line AB for the axis of x; we have thus $(0, h)$ for the coordinates of B, and taking (x, y) for the coordinates of $\mathfrak{C}$, we have $g^2=x^2+y^2$; $f^2=(h-x)^2+y^2$. Writing as before l, m, n, to denote $b-c$, $c-a$, $a-b$ respectively, we have

$$\begin{aligned}
\tfrac{1}{4}\mathfrak{C} &= mn\{(h-x)^2+y^2\}+nl(x^2+y^2)+lmh^2,\\
&= m(n+l)h^2+n(l+m)(x^2+y^2)-2mnhx,\\
&=-m^2h^2-n^2(x^2+y^2)-2mnhx,\\
&=-(mh+nx)^2-n^2y^2,
\end{aligned}$$

and thus, for real values, $\mathfrak{C}$ can only vanish for $y=0$, $x=-\dfrac{mh}{n}$; these values of x, y give $f^2=\dfrac{l^2h^2}{n^2}$, $g^2=\dfrac{m^2h^2}{n^2}$, that is, $\dfrac{f^2}{l^2}=\dfrac{g^2}{m^2}=\dfrac{h^2}{n^2}$, or writing for l, m, n their values, they give

$$\frac{f^2}{(b-c)^2}=\frac{g^2}{(c-a)^2}=\frac{h^2}{(a-b)^2}.$$

11. But for imaginary circles the condition $\mathfrak{C}=0$ does not imply $\Delta=0$, and supposing $\mathfrak{C}=0$, the distance Z_1Z_2 is $=0$; the equation $\mathfrak{B}^2-\mathfrak{A}\mathfrak{C}=0$, is not satisfied, and thus the two radii are unequal; it would seem that we have *concentric* circles Z_1, Z_2 each touching the three given circles A, B, C, and this would imply that the radii a, b, c were equal to each other: this cannot be the case, for the only relation is that given by the foregoing condition $\mathfrak{C}=0$. The explanation of this paradox is that the two circles Z_1, Z_2 are not really concentric, but it is only the distance Z_1Z_2 of the centres which is $=0$, viz. the centres are points on an imaginary line $x-\alpha\pm i(y-\beta)=0$.

In verification hereof, I start from two circles Z_1, Z_2,

$$\begin{aligned}
(x+1)^2+(y+i)^2&=m^2,\\
(x-1)^2+(y-i)^2&=n^2,
\end{aligned}$$

having for centres the two points $(-1, -i)$, $(1, i)$ the distance of which two points from each other is $=0$. Consider for a moment a conic having these two imaginary points for its foci; viz. writing ξ, η for the coordinates of a point of the conic, the equation is

$$\sqrt{\{(\xi+1)^2+(\eta+i)^2\}}-\sqrt{\{(\xi-1)^2+(\eta-i)^2\}}=m-n;$$

we thence obtain

$$(\xi+1)^2+(\eta+i)^2=(m-n)^2+2(m-n)\sqrt{\{(\xi-1)^2+(\eta-i)^2\}}+(\xi-1)^2+(\eta-i)^2,$$

that is,

$$4(\xi+i\eta)-(m-n)^2=2(m-n)\sqrt{\{(\xi-1)^2+(\eta-i)^2\}},$$

or putting $m-n=2k$, we have

$$\xi+i\eta-k^2=k\sqrt{\{(\xi-1)^2+(\eta-i)^2\}},$$

and thence

$$k^2(\xi^2+\eta^2)-(\xi+i\eta)^2=k^4,$$

for the equation of the conic. The last preceding equation gives

$$\sqrt{\{(\xi-1)^2+(\eta-i)^2\}}=-k+\frac{\xi+i\eta}{k},\ =-\tfrac{1}{2}(m-n)+\frac{\xi+i\eta}{k},$$

or say

$$\sqrt{\{(\xi-1)^2+(\eta-i)^2\}}-n=-\tfrac{1}{2}(m+n)+\frac{\xi+i\eta}{k};$$

and we have similarly

$$\sqrt{\{(\xi+1)^2+(\eta+i)^2\}}-m=-\tfrac{1}{2}(m+n)+\frac{\xi+i\eta}{k}.$$

This being so, it at once appears that, if (ξ, η) are coordinates of a point on the conic, then the circle

$$(x-\xi)^2+(y-\eta)^2=a^2,$$

where

$$a=-\tfrac{1}{2}(m+n)+\frac{\xi+i\eta}{k},$$

is a circle touching each of the given circles Z_1, Z_2. In fact, the distance of the centre from the point Z_1 is $\sqrt{\{(\xi+1)^2+(\eta+i)^2\}}$, which is $=a+m$, the sum of the two radii; and similarly the distance of the centre from the point Z_2 is $\sqrt{\{(\xi-1)^2+(\eta-i)^2\}}$, which is $=a+n$, the sum of the two radii.

Hence if (ξ', η'), (ξ'', η'') belong to any other two points on the conic, and we write

$$a=-\tfrac{1}{2}(m+n)+\frac{\xi+i\eta}{k},$$

$$b=-\tfrac{1}{2}(m+n)+\frac{\xi'+i\eta'}{k},$$

$$c=-\tfrac{1}{2}(m+n)+\frac{\xi''+i\eta''}{k},$$

we have

$$(x-\xi\ )^2+(y-\eta\ )^2=a^2,$$

$$(x-\xi')^2+(y-\eta')^2=b^2,$$

$$(x-\xi'')^2+(y-\eta'')^2=c^2,$$

for the equations of three circles A, B, C each touching the two circles Z_1, Z_2. Writing as before f, g, h for the mutual distances BC, CA, AB of the centres of these circles, then

$$f^2=(\xi'-\xi'')^2+(\eta'-\eta'')^2,$$

and similarly for g^2 and h^2. But we have

$$b-c=\frac{1}{k}\{(\xi'-\xi'')+i(\eta'-\eta'')\},$$

and therefore

$$\frac{f^2}{b-c} = k\{\xi' - \xi'' - i(\eta' - \eta'')\};$$

and similarly

$$\frac{g^2}{c-a} = k\{\xi'' - \xi - i(\eta'' - \eta)\},$$

$$\frac{h^2}{a-b} = k\{\xi - \xi' - i(\eta - \eta')\},$$

and hence

$$\frac{f^2}{b-c} + \frac{g^2}{c-a} + \frac{h^2}{a-b} = 0, \text{ that is, } \mathfrak{E} = 0,$$

viz. it thus appears that the condition $\mathfrak{E} = 0$ applies to a pair of circles Z_1, Z_2 which are not concentric, but which have for their centres two imaginary points the distance of which from each other is $= 0$.

This completes the explanation of the denominator and numerator factors in the expression for the distance Z_1Z_2 between the centres of the two tangent circles.

12. I consider now the analytical solution: the equations of the given circles A, B, C are

$$(X-\alpha)^2 + (Y-\alpha_1)^2 - a^2 = 0,$$
$$(X-\beta)^2 + (Y-\beta_1)^2 - b^2 = 0,$$
$$(X-\gamma)^2 + (Y-\gamma_1)^2 - c^2 = 0,$$

viz. (α, α_1), (β, β_1), and (γ, γ_1) are the coordinates of the centres and (a, b, c) are the radii. Taking (x, y) for the coordinates of the centre of the tangent circle and ϑ for its radius, the equation of the tangent circle is

$$(X-x)^2 + (Y-y)^2 - \vartheta^2 = 0;$$

and if we write r, s, t for the distances of this centre from the points A, B, C respectively, that is,

$$r = \sqrt{\{(x-\alpha)^2 + (y-\alpha_1)^2\}},$$
$$s = \sqrt{\{(x-\beta)^2 + (y-\beta_1)^2\}},$$
$$t = \sqrt{\{(x-\gamma)^2 + (y-\gamma_1)^2\}};$$

then for the determination of the unknown quantities x, y, ϑ we have the three equations

$$r = a + \vartheta, \quad s = b + \vartheta, \quad t = c + \vartheta,$$

or eliminating ϑ, the centre is determined by means of the hyperbolas

$$s - t = b - c, \quad t - r = c - a, \quad r - s = a - b;$$

these three hyperbolas have, in fact, two common intersections which are the two centres Z_1, Z_2.

In all that follows, I write, as before, $b-c,\ c-a,\ a-b=l,\ m,\ n$; the last-mentioned equations are therefore

$$s-t=l,\quad t-r=m,\quad r-s=n,$$

and we deduce

$$r=\tfrac{1}{2}n+\frac{r^2-s^2}{2n}=-\tfrac{1}{2}m+\frac{t^2-r^2}{2m},$$

$$s=\tfrac{1}{2}l+\frac{s^2-t^2}{2l}=-\tfrac{1}{2}n+\frac{r^2-s^2}{2n},$$

$$t=\tfrac{1}{2}m+\frac{t^2-r^2}{2m}=-\tfrac{1}{2}l+\frac{s^2-t^2}{2l},$$

viz. writing

$$R=\frac{s^2-t^2}{2l},\quad S=\frac{t^2-r^2}{2m},\quad T=\frac{r^2-s^2}{2n};$$

and therefore

$$lR+mS+nT=0,$$

these equations are

$$r=S-\tfrac{1}{2}m,\quad s=T-\tfrac{1}{2}n,\quad t=R-\tfrac{1}{2}l,$$

$$=T+\tfrac{1}{2}n,\quad =R+\tfrac{1}{2}l,\quad =S+\tfrac{1}{2}m;$$

$R,\ S,\ T$ are each of them a linear function of the coordinates $(x,\ y)$, say we have

$$R=\lambda x+\lambda_1 y+\lambda_2,$$

$$S=\mu x+\mu_1 y+\mu_2,$$

$$T=\nu x+\nu_1 y+\nu_2,$$

where

$$\lambda,\ \lambda_1,\ \lambda_2=-\frac{\beta-\gamma}{l},\quad -\frac{\beta_1-\gamma_1}{l},\quad \frac{\beta^2+\beta_1^2-\gamma^2-\gamma_1^2}{2l},$$

$$\mu,\ \mu_1,\ \mu_2=-\frac{\gamma-\alpha}{m},\quad -\frac{\gamma_1-\alpha_1}{m},\quad \frac{\gamma^2+\gamma_1^2-\alpha^2-\alpha_1^2}{2m},$$

$$\nu,\ \nu_1,\ \nu_2=-\frac{\alpha-\beta}{n},\quad -\frac{\alpha_1-\beta_1}{n},\quad \frac{\alpha^2+\alpha_1^2-\beta^2-\beta_1^2}{2n}.$$

13. From the two equations $r=S-\tfrac{1}{2}m=T+\tfrac{1}{2}n$, we deduce the equations of a line and a circle.

The line is $S-T-\tfrac{1}{2}(m+n)=0$, viz. substituting for S and T their values, this is

$$\frac{t^2-r^2}{m}-\frac{r^2-s^2}{n}-(m+n)=0,$$

that is,

$$n(t^2-r^2)-m(r^2-s^2)-mn(m+n)=0,$$

or, since $l+m+n=0$, the equation is

$$lr^2+ms^2+nt^2+lmn=0,$$

which is symmetrical in regard to the three circles. The equation may be written

$$l(r^2 - a^2) + m(s^2 - b^2) + n(t^2 - c^2) = 0;$$

and it thus appears that the line passes through the radical centre of the three circles.

We have

$$(\nu - \mu) r = \nu (S - \tfrac{1}{2}m) - \mu (T + \tfrac{1}{2}n) = -(\mu\nu_1 - \mu_1\nu) y + \nu\mu_2 - \mu\nu_2 - \tfrac{1}{2}(m\nu + n\mu),$$

$$(\nu_1 - \mu_1) r = \nu_1 (S - \tfrac{1}{2}m) - \mu_1 (T + \tfrac{1}{2}n) = (\mu\nu_1 - \mu_1\nu) x + \nu\mu_2 - \mu\nu_2 - \tfrac{1}{2}(m\nu_1 + n\mu_1),$$

and thence

$$\{(\nu - \mu)^2 + (\nu_1 - \mu_1)^2\} r^2 = \{\nu(S - \tfrac{1}{2}m) - \mu(T + \tfrac{1}{2}n)\}^2 + \{\nu_1(S - \tfrac{1}{2}m) - \mu_1(T + \tfrac{1}{2}n)\}^2,$$

which is the equation of a circle; in fact, on the left-hand side and right-hand side the only terms of the second order in (x, y) are $\{(\nu - \mu)^2 + (\nu_1 - \mu_1)^2\}(x^2 + y^2)$ and $(\mu\nu_1 - \mu_1\nu)^2(x^2 + y^2)$ respectively. We have thus the equation of the Newton-circle F; but I reduce the form by substituting for $\mu, \mu_1, \mu_2, \nu, \nu_1, \nu_2$ their values. Writing for shortness

$$l\alpha + m\beta + n\gamma = K,$$

$$l\alpha_1 + m\beta_1 + n\gamma_1 = K_1,$$

$$\beta\gamma_1 - \beta_1\gamma + \gamma\alpha_1 - \gamma_1\alpha + \alpha\beta_1 - \alpha_1\beta = \Omega,$$

$$(\beta - \gamma)(\alpha^2 + \alpha_1^2) + (\gamma - \alpha)(\beta^2 + \beta_1^2) + (\alpha - \beta)(\gamma^2 + \gamma_1^2) = \Pi,$$

$$(\beta_1 - \gamma_1)(\alpha^2 + \alpha_1^2) + (\gamma_1 - \alpha_1)(\beta^2 + \beta_1^2) + (\alpha_1 - \beta_1)(\gamma^2 + \gamma_1^2) = \Pi_1,$$

after some easy reductions the equation is found to be

$$\begin{aligned}4(K^2 + K_1^2)(x^2 + y^2 - 2\alpha x - 2\alpha_1 y + \alpha^2 + \alpha_1^2) &\\ = \{\ 2\Omega y + \Pi + (\beta - \gamma) mn + (m - n) K\}^2 &\\ + \{-2\Omega x + \Pi_1 + (\beta_1 - \gamma_1) mn + (m - n) K_1\}^2. &\end{aligned}$$

14. To further abbreviate, I write

$$(\beta - \gamma) mn + (m - n) K = F, \quad (\beta_1 - \gamma_1) mn + (m - n) K_1 = F_1,$$

$$(\gamma - \alpha) nl + (n - l) K = G, \quad (\gamma_1 - \alpha_1) nl + (n - l) K_1 = G_1,$$

$$(\alpha - \beta) lm + (l - m) K = H, \quad (\alpha_1 - \beta_1) lm + (l - m) K_1 = H_1;$$

also

$$l(\alpha^2 + \alpha_1^2) + m(\beta^2 + \beta_1^2) + n(\gamma^2 + \gamma_1^2) = \Theta;$$

and then writing down the three equations, we have

$$4(K^2 + K_1^2) r^2 = (-2\Omega x + \Pi_1 + F_1)^2 + (2\Omega y + \Pi + F)^2,$$

$$4(K^2 + K_1^2) s^2 = (-2\Omega x + \Pi_1 + G_1)^2 + (2\Omega y + \Pi + G)^2,$$

$$4(K^2 + K_1^2) t^2 = (-2\Omega x + \Pi_1 + H_1)^2 + (2\Omega y + \Pi + H)^2,$$

which are the equations of the three Newton-circles, each meeting the chord

$$lr^2 + ms^2 + nt^2 + lmn = 0,$$

or, say

$$-2Kx - 2K_1 y + \Theta + lmn = 0,$$

in the points Z_1, Z_2.

15. The first of these equations is

$$\begin{aligned}&4(K^2+K_1^2-\Omega^2)(x^2+y^2)\\&\qquad -2\{4(K^2+K_1^2)\alpha-2\Omega(\Pi_1+F_1)\}x\\&\qquad -2\{4(K^2+K_1^2)\alpha_1+2\Omega(\Pi+F)\}y\\&\qquad +4(K^2+K_1^2)(\alpha^2+\alpha_1^2)-(\Pi_1+F_1)^2-(\Pi+F)^2=0,\end{aligned}$$

that is,

$$\begin{aligned}&\{4(K^2+K_1^2-\Omega^2)x-4(K^2+K_1^2)\alpha+2\Omega(\Pi_1+F_1)\}^2\\&+\{4(K^2+K_1^2-\Omega^2)y-4(K^2+K_1^2)\alpha_1-2\Omega(\Pi+F)\}^2\\&+4(K^2+K_1^2-\Omega^2)\{4(K^2+K_1^2)(\alpha^2+\alpha_1^2)-(\Pi_1+F_1)^2-(\Pi+F)^2\}\\&-\{4(K^2+K_1^2)\alpha-2\Omega(\Pi_1+F_1)\}^2\\&-\{4(K^2+K_1^2)\alpha_1+2\Omega(\Pi+F)\}^2=0,\end{aligned}$$

where the last term is

$$\begin{aligned}=\;&16(K^2+K_1^2)^2(\alpha^2+\alpha_1^2)\\&-4(K^2+K_1^2)(\Pi_1+F_1)^2\\&-4(K^2+K_1^2)(\Pi+F)^2\\&-16(K^2+K_1^2)(\alpha^2+\alpha_1^2)\Omega^2\\&-16(K^2+K_1^2)^2(\alpha^2+\alpha_1^2)\\&+16(K^2+K_1^2)\alpha\Omega(\Pi_1+F_1)\\&-16(K^2+K_1^2)\alpha_1\Omega(\Pi+F)\\=\;&(K^2+K_1^2)\{-4(\Pi_1+F_1)^2-4(\Pi+F)^2\\&\qquad -16(\alpha^2+\alpha_1^2)\Omega^2+16\alpha\Omega(\Pi_1+F_1)-16\alpha_1\Omega(\Pi+F)\}.\end{aligned}$$

It thus appears that the equation of the Newton-circle F is

$$\begin{aligned}&4(K^2+K_1^2-\Omega^2)^2\{(x-\mathrm{f})^2+(y-\mathrm{f}_1)^2\}\\&=(K^2+K_1^2)\{(\Pi_1+F_1)^2+(\Pi+F)^2\\&\qquad -4\alpha\Omega(\Pi_1+F_1)+4\alpha_1\Omega(\Pi+F)+4(\alpha^2+\alpha_1^2)\Omega^2\}\\&=(K^2+K_1^2)\{(\Pi_1+F_1-2\alpha\Omega)^2+(\Pi+F+2\alpha_1\Omega)^2\},\end{aligned}$$

where the coordinates of the centre are

$$\mathrm{f}=\frac{2(K^2+K_1^2)\alpha-\Omega(\Pi_1+F_1)}{2(K^2+K_1^2-\Omega^2)},$$

$$\mathrm{f}_1=\frac{2(K^2+K_1^2)\alpha_1+\Omega(\Pi+F)}{2(K^2+K_1^2-\Omega^2)},$$

and

$$\mathrm{rad}^2=\frac{K^2+K_1^2}{4(K^2+K_1^2-\Omega^2)^2}\{(\Pi_1+F_1-2\alpha\Omega)^2+(\Pi+F+2\alpha_1\Omega)^2\};$$

and similarly for the Newton-circles G and H.

16. The centres are in a line at right angles to

$$-2Kx-2K_1y+\Theta+lmn=0,$$

say the equation of this line is

$$K_1x-\ Ky\ +\Psi=0\,;$$

then we ought to have

$$\frac{K_1\{2(K^2+K_1^2)\alpha-\Omega(\Pi_1+F_1)\}-K\{2(K^2+K_1^2)\alpha_1+\Omega(\Pi+F)\}}{2(K^2+K_1^2-\Omega^2)}+\Psi=0,$$

that is,

$$2(K^2+K_1^2)(K_1\alpha-K\alpha_1)-(K_1\Pi_1+K\Pi)\Omega-(K_1F_1+KF)\Omega+2(K^2+K_1^2-\Omega^2)\Psi=0.$$

This should agree with

$$2(K^2+K_1^2)(K_1\beta-K\beta_1)-(K_1\Pi_1+K\Pi)\Omega-(K_1G_1+KG)\Omega+2(K^2+K_1^2-\Omega^2)\Psi=0,$$

viz. we ought to have

$$2(K^2+K_1^2)\{K_1(\alpha-\beta)-K(\alpha_1-\beta_1)\}-\{K_1(F_1-G_1)+K(F-G)\}\Omega=0.$$

This can be true only if $K_1(\alpha-\beta)-K(\alpha_1-\beta_1)$ is a multiple of Ω; and, in fact,

$$\begin{aligned}K_1(\alpha-\beta)&-K(\alpha_1-\beta_1)\\&=(\alpha-\beta)(l\alpha_1+m\beta_1+n\gamma_1)-(\alpha_1-\beta_1)(l\alpha+m\beta+n\gamma)\\&=l(\alpha\beta_1-\alpha_1\beta)+m(\alpha\beta_1-\alpha_1\beta)+n\{-(\beta\gamma_1-\beta_1\gamma)-(\gamma\alpha_1-\gamma_1\alpha)\}\\&=-n(\beta\gamma_1-\beta_1\gamma+\gamma\alpha_1-\gamma_1\alpha+\alpha\beta_1-\alpha_1\beta),\ =-n\Omega.\end{aligned}$$

The equation to be verified thus is

$$-2n(K^2+K_1^2)-K_1(F_1-G_1)-K(F-G)=0,$$

and here

$$\begin{aligned}F-G&=(\beta-\gamma)mn-(\gamma-\alpha)ln+(l+m-2n)K\\&=\alpha ln+\beta mn-\gamma(l+m)n-3nK=(n-3n)K=-2nK.\end{aligned}$$

Hence

$$-K\ (F\ -G\ )=2nK^2,$$

and similarly

$$-K_1(F_1-G_1)=2nK_1^2;$$

and thus the equation is verified.

17. Writing for shortness

$$\beta\gamma_1-\beta_1\gamma,\quad \gamma\alpha_1-\gamma_1\alpha,\quad \alpha\beta_1-\alpha_1\beta=X,\ Y,\ Z,$$

and therefore $\Omega=X+Y+Z$; we have

$$\begin{aligned}2(K^2+K_1^2)(K_1\alpha-K\alpha_1)-(KF+K_1F_1)\Omega\\=\ K\ \{mn(\beta-\gamma)(\ \ X-Y-Z)\\+\ nl\ (\gamma-\alpha)(-X+Y-Z)\\+\ lm\ (\alpha-\beta)(-X-Y+Z)\}\\+K_1\{mn(\beta_1-\gamma_1)(\ \ X-Y-Z)\\+\ nl\ (\gamma_1-\alpha_1)(-X+Y-Z)\\+\ lm\ (\alpha_1-\beta_1)(-X-Y+Z)\}.\end{aligned}$$

To verify this, observe that the left-hand side is

$$2(K^2+K_1^2)(mZ-nY)-\{(\beta-\gamma)mnK+(\beta_1-\gamma_1)mnK_1+(m-n)(K^2+K_1^2)\Omega\},$$

or putting herein $X+Y+Z$ for Ω, this is

$$(K^2+K_1^2)\{(n-m)X+lY-lZ\}-mn\{(\beta-\gamma)K+(\beta_1-\gamma_1)K_1\}(X+Y+Z),$$

which is thus

$$\begin{aligned} &= mn\{(\beta-\gamma)K+(\beta_1-\gamma_1)K_1\}(X-Y-Z)\\ &+ nl\ \{(\gamma-\alpha)K+(\gamma_1-\alpha_1)K_1\}(-X+Y-Z)\\ &+ lm\ \{(\alpha-\beta)K+(\alpha_1-\beta_1)K_1\}(-X-Y+Z). \end{aligned}$$

The equation to be verified thus becomes

$$\begin{aligned} &(K^2+K_1^2)\{(n-m)X+lY-lZ\}\\ &\qquad = mn\{(\beta-\gamma)K+(\beta_1-\gamma_1)K_1\}2X\\ &\qquad + nl\ \{(\gamma-\alpha)K+(\gamma_1-\alpha_1)K_1\}(-X+Y-Z)\\ &\qquad + lm\ \{(\alpha-\beta)K+(\alpha_1-\beta_1)K_1\}(-X-Y+Z). \end{aligned}$$

This breaks up into two equations,

$$\begin{aligned} K\{(n-m)X+lY-lZ\} = {} & mn(\beta-\gamma)2X\\ & + nl\ (\gamma-\alpha)(-X+Y-Z)\\ & + lm(\alpha-\beta)(-X-Y+Z); \end{aligned}$$

and a like equation with the suffixed letters. And the equation just written down, observing that each side is a linear function of X and $Y-Z$, again breaks up into the two equations

$$\begin{aligned} (n-m)K &= 2mn(\beta-\gamma)-nl(\gamma-\alpha)-lm(\alpha-\beta),\\ lK &= -nl(\gamma-\alpha)-lm(\alpha-\beta), \end{aligned}$$

which are at once verified: in fact, for the first equation the right-hand side is

$$\begin{aligned} &=(nl-lm)\ \alpha+(2mn+lm)\beta+(-2mn-nl)\gamma,\\ &=(n-m)\ l\alpha+\ (2n+l)m\beta+(-2m\ -l)n\gamma,\\ &=(n-m)(l\alpha+\qquad m\beta+\qquad n\gamma),\ =(n-m)K; \end{aligned}$$

and similarly in the second equation the right-hand side is

$$(-nl-lm)\alpha+lm\beta+ln\gamma,\quad =l(l\alpha+m\beta+n\gamma),\quad =lK.$$

Writing then

$$\begin{aligned} \Phi = {} & K\ \{mn(\beta-\gamma)(X-Y-Z)\\ & +nl(\gamma-\alpha)(-X+Y-Z)+lm(\alpha-\beta)(-X-Y+Z)\}\\ & +K_1\{mn(\beta_1-\gamma_1)(X-Y-Z)\\ & +nl(\gamma_1-\alpha_1)(-X+Y-Z)+lm(\alpha_1-\beta_1)(-X-Y+Z)\}, \end{aligned}$$

we have

$$\Phi=(K_1\Pi_1+K\Pi)\Omega-2(K^2+K_1^2-\Omega^2)\Psi.$$

This equation determines Φ, and thus the equation of the line of centres is

$$2(K^2+K_1^2-\Omega^2)(K_1x-Ky)+(K\Pi+K_1\Pi_1)\,\Omega-\Phi=0.$$

18. This line meets

$$-2Kx-2K_1y+\Theta+lmn=0,$$

in the mid-point of the chord Z_1Z_2. We thus have for the coordinates x, y of this mid-point

$$2(K^2+K_1^2-\Omega^2)(K^2+K_1^2)\,x+K_1\{(K\Pi+K_1\Pi_1)\,\Omega-\Phi\}-K\,(K^2+K_1^2-\Omega^2)(\Theta+lmn)=0,$$

$$2(K^2+K_1^2-\Omega^2)(K^2+K_1^2)\,y-K\,\{(K\Pi+K_1\Pi_1)\,\Omega-\Phi\}-K_1(K^2+K_1^2-\Omega^2)(\Theta+lmn)=0.$$

The perpendicular distance of the centre of the circle F from the chord is

$$=\frac{-2Kf-2K_1f_1+\Theta+lmn}{2\sqrt{(K^2+K_1^2)}};$$

here

$$2(Kf+K_1f_1)=\frac{1}{K^2+K_1^2-\Omega^2}[2(K^2+K_1^2)(K\alpha+K_1\alpha_1)-\Omega\,\{K(\Pi_1+F_1)-K_1(\Pi+F)\}],$$

$$\begin{aligned}K\Pi_1-K_1\Pi=\ &(\alpha^2+\alpha_1^2)\,\{K(\beta_1-\gamma_1)-K_1(\beta-\gamma)\}\\&+(\beta^2+\beta_1^2)\,\{K(\gamma_1-\alpha_1)-K_1(\gamma-\alpha)\}\\&+(\gamma^2+\gamma_1^2)\,\{K(\alpha_1-\beta_1)-K_1(\alpha-\beta)\}\\=\ &\Omega\,\{l(\alpha^2+\alpha_1^2)+m(\beta^2+\beta_1^2)+n(\gamma^2+\gamma_1^2)\}=\Omega\Theta,\\KF_1-K_1F=\ &mn\,\{K(\beta_1-\gamma_1)-K_1(\beta-\gamma)\}=lmn\Omega.\end{aligned}$$

Thus

$$2(Kf+K_1f_1)=\frac{1}{K^2+K_1^2-\Omega^2}\{2(K^2+K_1^2)(K\alpha+K_1\alpha_1)-\Omega^2(\Theta+lmn)\},$$

and hence the numerator of the fraction is

$$\frac{1}{K^2+K_1^2-\Omega^2}\{-2(K^2+K_1^2)(K\alpha+K_1\alpha_1)+\Omega^2(\Theta+lmn)+(K^2+K_1^2-\Omega^2)(\Theta+lmn)\}$$

$$=\frac{1}{K^2+K_1^2-\Omega^2}(K^2+K_1^2)\{-2(K\alpha+K_1\alpha_1)+\Theta+lmn\}.$$

Thus the perpendicular distance of the centre of the circle F from the chord Z_1Z_2 is

$$=\frac{\sqrt{(K^2+K_1^2)}}{2(K^2+K_1^2-\Omega^2)}\{-2(K\alpha+K_1\alpha_1)+\Theta+lmn\};$$

moreover, by what precedes, we have

$$\text{Radius}=\frac{\sqrt{(K^2+K_1^2)}}{2(K^2+K_1^2-\Omega^2)}\sqrt{\{(\Pi+F+2\alpha_1\Omega)^2+(\Pi_1+F_1-2\alpha\Omega)^2\}}.$$

19. Hence also

$$(Z_1Z_2)^2=\frac{K^2+K_1^2}{(K^2+K_1^2-\Omega^2)^2}[(\Pi+F+2\alpha_1\Omega)^2+(\Pi_1+F_1-2\alpha\Omega)^2-\{\Theta+lmn-2(K\alpha+K_1\alpha_1)\}^2].$$

We have

$$K^2 + K_1^2 - \Omega^2 = (l\alpha + m\beta + n\gamma)^2 + (l\alpha_1 + m\beta_1 + n\gamma_1)^2 - (\beta\gamma_1 - \beta_1\gamma + \gamma\alpha_1 - \gamma_1\alpha + \alpha\beta_1 - \alpha_1\beta)^2;$$

but

$$f^2 = (\beta - \gamma)^2 + (\beta_1 - \gamma_1)^2,$$

$$g^2 = (\gamma - \alpha)^2 + (\gamma_1 - \alpha_1)^2,$$

$$h^2 = (\alpha - \beta)^2 + (\alpha_1 - \beta_1)^2,$$

and hence

$$-f^2 + g^2 + h^2 = -2(\gamma - \alpha)(\alpha - \beta) - 2(\gamma_1 - \alpha_1)(\alpha_1 - \beta_1),$$

whence

$$-f^2(-f^2 + g^2 + h^2) = \{2(\gamma - \alpha)(\alpha - \beta) + 2(\gamma_1 - \alpha_1)(\alpha_1 - \beta_1)\}\{(\beta - \gamma)^2 + (\beta_1 - \gamma_1)^2\};$$

or forming the like values of $-g^2(f^2 - g^2 + h^2)$ and $-h^2(f^2 + g^2 - h^2)$ and adding, we have

$$\begin{aligned}\Delta &= f^4 + g^4 + h^4 - 2g^2h^2 - 2h^2f^2 - 2f^2g^2 \\ &= \ \ 2(\gamma - \alpha)(\alpha - \beta)(\beta_1 - \gamma_1)^2 + 2(\gamma_1 - \alpha_1)(\alpha_1 - \beta_1)(\beta - \gamma)^2 \\ &\quad + 2(\alpha - \beta)(\beta - \gamma)(\gamma_1 - \alpha_1)^2 + 2(\alpha_1 - \beta_1)(\beta_1 - \gamma_1)(\gamma - \alpha)^2 \\ &\quad + 2(\beta - \gamma)(\gamma - \alpha)(\alpha_1 - \beta_1)^2 + 2(\beta_1 - \gamma_1)(\gamma_1 - \alpha_1)(\alpha - \beta)^2 \\ &= -4\{\alpha_1(\beta - \gamma) + \beta_1(\gamma - \alpha) + \gamma_1(\alpha - \beta)\}^2 = -4\Omega^2.\end{aligned}$$

But

$$\begin{aligned}mnf^2 + nlg^2 + lmh^2 = &\quad mn\{\beta^2 + \beta_1^2 + \gamma^2 + \gamma_1^2 - 2(\beta\gamma + \beta_1\gamma_1)\} \\ &+ nl\ \{\gamma^2 + \gamma_1^2 + \alpha^2 + \alpha_1^2 - 2(\gamma\alpha + \gamma_1\alpha_1)\} \\ &+ lm\{\alpha^2 + \alpha_1^2 + \beta^2 + \beta_1^2 - 2(\alpha\beta + \alpha_1\beta_1)\}, \\ = &- l^2(\alpha^2 + \alpha_1^2) - m^2(\beta^2 + \beta_1^2) - n^2(\gamma^2 + \gamma_1^2) \\ &- 2mn(\beta\gamma_1 + \beta_1\gamma) - 2nl(\gamma\alpha_1 + \gamma_1\alpha) - 2lm(\alpha\beta_1 + \alpha_1\beta), \\ = &-(l\alpha + m\beta + n\gamma)^2 - (l\alpha_1 + m\beta_1 + n\gamma_1)^2, \\ = &- K^2 - K_1^2.\end{aligned}$$

Thus

$$4(K^2 + K_1^2) = -4(mnf^2 + nlg^2 + lmh^2), \quad -4\Omega^2 = \Delta,$$

and therefore

$$4(K^2 + K_1^2 - \Omega^2) = \Delta - 4(mnf^2 + nlg^2 + lmh^2).$$

But

$$\mathfrak{C} = \qquad 4(mnf^2 + nlg^2 + lmh^2),$$

$$\mathfrak{C} = -\Delta + 4(mnf^2 + nlg^2 + lmh^2),$$

and thus

$$4(K^2 + K_1^2) \qquad = -\mathfrak{C},$$

$$4(K^2 + K_1^2 - \Omega^2) = -\mathfrak{C},$$

hence

$$\frac{K^2 + K_1^2}{(K^2 + K_1^2 - \Omega^2)^2} = \frac{-4\mathfrak{C}}{\mathfrak{C}^2},$$

and we have

$$(Z_1Z_2)^2 = \frac{-4\mathfrak{C}}{\mathfrak{C}^2}\left[(\Pi + F + 2\alpha_1\Omega)^2 + (\Pi_1 + F_1 - 2\alpha\Omega)^2 - \{\Theta + lmn - 2(K\alpha + K_1\alpha_1)\}^2\right].$$

20. This should agree with the expression in No. 7, that is, we ought to have

$$(\Pi + F + 2\alpha_1\Omega)^2 + (\Pi_1 + F_1 - 2\alpha\Omega)^2 - \{\Theta + lmn - 2(K\alpha + K_1\alpha_1)\}^2 = (f^2 - l^2)(g^2 - m^2)(h^2 - n^2),$$

and this breaks up into the equations

$$\begin{aligned}
(\Pi + 2\alpha_1\Omega) + (\Pi_1 - 2\alpha\Omega)^2 &= f^2g^2h^2\\
2F(\Pi + 2\alpha_1\Omega) + 2F_1(\Pi_1 - 2\alpha\Omega) - \{\Theta - 2(K\alpha + K_1\alpha_1)\}^2 &= -g^2h^2l^2 - h^2f^2m^2 - f^2g^2n^2\\
F^2 + F_1^2 - 2lmn\{\Theta - 2(K\alpha + K_1\alpha_1)\} &= f^2m^2n^2 + g^2n^2l^2 + h^2l^2m^2\\
-l^2m^2n^2 &= -l^2m^2n^2,
\end{aligned}$$

which may be separately verified.

21. In fact, we have

$$\begin{aligned}
\Pi + 2\alpha_1\Omega = {} & (\beta - \gamma)(\alpha^2 + \alpha_1^2) + (\gamma - \alpha)(\beta^2 + \beta_1^2) + (\alpha - \beta)(\gamma^2 + \gamma_1^2)\\
& \qquad + 2\alpha_1\{-(\beta - \gamma)\alpha_1 - (\gamma - \alpha)\beta_1 - (\alpha - \beta)\gamma_1\},\\
= {} & (\beta - \gamma)(\alpha^2 - \alpha_1^2) + (\gamma - \alpha)\{\beta^2 - \alpha_1^2 + (\alpha_1 - \beta_1)^2\} + (\alpha - \beta)\{\gamma^2 - \alpha_1^2 + (\gamma_1 - \alpha_1)^2\},\\
= {} & \alpha^2(\beta - \gamma) + \beta^2(\gamma - \alpha) + \gamma^2(\alpha - \beta) + (\gamma - \alpha)(\alpha_1 - \beta_1)^2 + (\alpha - \beta)(\gamma_1 - \alpha_1)^2,\\
= {} & -(\beta - \gamma)(\gamma - \alpha)(\alpha - \beta) + (\gamma - \alpha)(\alpha_1 - \beta_1)^2 + (\alpha - \beta)(\gamma_1 - \alpha_1)^2,
\end{aligned}$$

or, putting for shortness

$$\beta - \gamma,\ \gamma - \alpha,\ \alpha - \beta = \lambda,\ \mu,\ \nu;\quad \beta_1 - \gamma_1,\ \gamma_1 - \alpha_1,\ \alpha_1 - \beta_1 = \lambda_1,\ \mu_1,\ \nu_1,$$

(where the letters λ, μ, ν, λ_1, μ_1, ν_1 have a meaning different from that assigned to them in No. 7), this is

$$\Pi + 2\alpha_1\Omega = -\lambda\mu\nu + \mu\nu_1^2 + \nu\mu_1^2,$$

and similarly

$$\Pi_1 - 2\alpha\Omega = -\lambda_1\mu_1\nu_1 + \mu_1\nu^2 + \nu_1\mu^2.$$

Also

$$f^2,\ g^2,\ h^2 = \lambda^2 + \lambda_1^2,\ \mu^2 + \mu_1^2,\ \nu^2 + \nu_1^2,$$

and the first equation thus becomes

$$(-\lambda\mu\nu + \mu\nu_1^2 + \nu\mu_1^2)^2 + (-\lambda_1\mu_1\nu_1 + \mu_1\nu^2 + \nu_1\mu^2)^2 = (\lambda^2 + \lambda_1^2)(\mu^2 + \mu_1^2)(\nu^2 + \nu_1^2),$$

or if on the left-hand we write $\lambda = -\mu - \nu$, $\lambda_1 = -\mu_1 - \nu_1$, this is

$$\{\nu(\mu^2 + \mu_1^2) + \mu(\nu^2 + \nu_1^2)\}^2 + \{\nu_1(\mu^2 + \mu_1^2) + \mu_1(\nu^2 + \nu_1^2)\}^2,$$

which is

$$\begin{aligned}
&= (\mu^2 + \mu_1^2)(\nu^2 + \nu_1^2)(\mu^2 + 2\mu\nu + \nu^2 + \mu_1^2 + 2\mu_1\nu_1 + \nu_1^2)\\
&= (\mu^2 + \mu_1^2)(\nu^2 + \nu_1^2)(\lambda^2 + \lambda_1^2),
\end{aligned}$$

which is right.

22. For the second equation, we use the values

$$\Pi + 2\alpha_1\Omega = \nu\ (\mu^2 + \mu_1^2) + \mu\ (\nu^2 + \nu_1^2),$$

$$\Pi_1 - 2\alpha\ \Omega = \nu_1(\mu^2 + \mu_1^2) + \mu_1(\nu^2 + \nu_1^2);$$

and the equation to be verified thus is

$$\begin{aligned} &2F\ \{\nu\ (\mu^2 + \mu_1^2) + \mu\ (\nu^2 + \nu_1^2)\} &&= -\ l^2\ (\mu^2 + \mu_1^2)(\nu^2 + \nu_1^2) \\ &+ 2F_1\{\nu_1(\mu^2 + \mu_1^2) + \mu_1(\nu^2 + \nu_1^2)\} &&\quad - m^2(\nu^2 + \nu_1^2)(\lambda^2 + \lambda_1^2) \\ &- \{\Theta - 2(K\alpha + K_1\alpha_1)\}^2 &&\quad - n^2\ (\lambda^2 + \lambda_1^2)(\mu^2 + \mu_1^2); \end{aligned}$$

we have

$$F = mn\lambda + (m - n)(l\alpha + m\beta + n\gamma) = mn\lambda + (m - n)(-m\nu + n\mu),$$

that is,

$$F = -m^2\nu - n^2\mu;$$

and similarly

$$F_1 = -m^2\nu_1 - n^2\mu_1.$$

Also

$$\begin{aligned} &\Theta - 2(K\alpha + K_1\alpha_1) \\ &= \ l(\alpha^2 + \alpha_1^2) + m(\beta^2 + \beta_1^2) + n(\gamma^2 + \gamma_1^2) - 2\{l(\alpha^2 + \alpha_1^2) + m(\alpha\beta + \alpha_1\beta_1) + n(\alpha\gamma + \alpha_1\gamma_1)\}, \\ &= -l(\alpha^2 + \alpha_1^2) + m(\beta^2 + \beta_1^2 - 2\alpha\beta - 2\alpha_1\beta_1) + n(\gamma^2 + \gamma_1^2 - 2\alpha\gamma - 2\alpha_1\gamma_1), \\ &= \ m\{(\alpha - \beta)^2 + (\alpha_1 - \beta_1)^2\} + n\{(\gamma - \alpha)^2 + (\gamma_1 - \alpha_1)^2\}, \\ &= \ m(\nu^2 + \nu_1^2) + n(\mu^2 + \mu_1^2). \end{aligned}$$

The left-hand side is

$$\begin{aligned} &-2(m^2\nu + n^2\mu)\{\nu\ (\mu^2 + \mu_1^2) + \mu\ (\nu^2 + \nu_1^2)\} \\ &-2(m^2\nu_1 + n^2\mu_1)\{\nu_1(\mu^2 + \mu_1^2) + \mu_1(\nu^2 + \nu_1^2)\} \\ &-\{m(\nu^2 + \nu_1^2) + n(\mu^2 + \mu_1^2)\}^2, \end{aligned}$$

which is

$$\begin{aligned} = \ & m^2\{-2(\nu^2 + \nu_1^2)(\mu^2 + \mu_1^2) - 2(\mu\nu + \mu_1\nu_1)(\nu^2 + \nu_1^2) - (\nu^2 + \nu_1^2)^2\} \\ & + n^2\{-2(\mu\nu + \mu_1\nu_1)(\mu^2 + \mu_1^2) - 2(\mu^2 + \mu_1^2)(\nu^2 + \nu_1^2) - (\mu^2 + \mu_1^2)^2\} \\ & + \quad (-l^2 + m^2 + n^2)(\mu^2 + \mu_1^2)(\nu^2 + \nu_1^2), \\ = & -l^2(\mu^2 + \mu_1^2)(\nu^2 + \nu_1^2) \\ & - m^2(\nu^2 + \nu_1^2)(\mu^2 + 2\mu\nu + \nu^2 + \mu_1^2 + 2\mu_1\nu_1 + \nu_1^2) \\ & - n^2(\mu^2 + \mu_1^2)(\mu^2 + 2\mu\nu + \nu^2 + \mu_1^2 + 2\mu_1\nu_1 + \nu_1^2), \end{aligned}$$

which is equal to the right-hand side.

23. For the third equation we have as above

$$F = -m^2\nu - n^2\mu, \quad F_1 = -m^2\nu_1 - n^2\mu_1,$$

$$\Theta - 2(K\alpha + K_1\alpha_1) = m(\nu^2 + \nu_1^2) + n(\mu^2 + \mu_1^2),$$

and the equation thus is

$$(m^2\nu + n^2\mu)^2 + (m^2\nu_1 + n^2\mu_1)^2 - 2lmn\,\{m\,(\nu^2+\nu_1^2) + n\,(\mu^2+\mu_1^2)\}$$
$$= (\lambda^2+\lambda_1^2)\,m^2n^2 + (\mu^2+\mu_1^2)\,n^2l^2 + (\nu^2+\nu_1^2)\,l^2m^2;$$

here the left-hand side is

$$\begin{aligned} = \quad & (m^4 - 2lm^2n)\,(\nu^2+\nu_1^2) \\ & + (n^4 - 2lmn^2)\,(\mu^2+\mu_1^2) \\ & + m^2n^2\,(\lambda^2+\lambda_1^2-\mu^2-\mu_1^2-\nu^2-\nu_1^2), \end{aligned}$$

which is

$$\begin{aligned} = \quad & (\lambda^2+\lambda_1^2)\,m^2n^2 \\ & + (\mu^2+\mu_1^2)\,n^2\,(n^2 - 2lm - m^2) \\ & + (\nu^2+\nu_1^2)\,m^2\,(m^2 - 2ln - n^2), \end{aligned}$$

which is equal to the right-hand side.

24. The fourth equation is the identity $-l^2m^2n^2 = -l^2m^2n^2$, and the whole equation is thus verified: viz. the analytical solution leads to the expression

$$\mathfrak{C}^2\,(Z_1Z_2)^2 = -4\,\{f^2-(b-c)^2\}\,\{g^2-(c-a)^2\}\,\{h^2-(a-b)^2\}\,\mathfrak{C},$$

obtained by an independent process in No. 7 for the squared distance of the two centres.