909.

ON A PARTICULAR CASE OF KUMMER'S DIFFERENTIAL EQUATION OF THE THIRD ORDER.

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THE general form of equation in question is

$$\frac{x'''}{x'}-\tfrac{3}{2}\left(\frac{x''}{x'}\right)^2+x'^2\left\{\frac{A}{(x-1)^2}+\frac{B}{x(x-1)}+\frac{C}{x^2}\right\}-\left\{\frac{A'}{(t-1)^2}+\frac{B'}{t(t-1)}+\frac{C'}{t^2}\right\}=0,$$

here x is a function of t; and A, B, C, A', B', C' are numerical constants. For various given values of A, B, C, and values determined thereby of A', B', C', the equation admits of a solution in the form $x=$ rational function of t; the theory in reference to the cases considered by Schwarz is considered in my paper "On the Schwarzian Derivative and the Polyhedral Functions," *Camb. Phil. Trans.*, t. XIII. (1883), pp. 5—68, [744]. But the theory is considered in a more general and exhaustive manner in Goursat's memoir, "Recherches sur l'équation de Kummer," *Acta Soc. Sci. Fennicæ*, t. XV. (1888), pp. 47—127. I consider here one of the solutions given by him, viz. writing

$$\begin{aligned} P&=4t-5, & X&=t^2P^3,\\ Q&=5t-4, & Y&=Q^3,\\ R&=8t^2-11t+8, & Z&=-(t-1)^2R^2,\end{aligned}$$

so that, identically, $X+Y+Z=0$; then the solution is expressed by either of the equivalent equations

$$x=-\frac{X}{Z}=\frac{t^2P^3}{(t-1)^2R^2},$$

$$x-1=\frac{Y}{Z}=-\frac{Q^3}{(t-1)^2R^2}.$$

The values of the constants to which this solution belongs are

$$A=\tfrac{4}{9},\ B=-\tfrac{37}{32},\ C=\tfrac{4}{9};\ A'=\tfrac{3}{8},\ B'=\tfrac{131}{144},\ C'=\tfrac{5}{18}.$$

But instead of assuming these values in the first instance, I leave the values indeterminate; and starting from the foregoing expression for x, I substitute this in

$$\Omega=\frac{x'''}{x'}-\tfrac{3}{2}\left(\frac{x''}{x'}\right)^2+x'^2\left\{\frac{A}{(x-1)^2}+\frac{B}{x(x-1)}+\frac{C}{x^2}\right\}-\left\{\frac{A'}{(t-1)^2}+\frac{B'}{t(t-1)}+\frac{C'}{t^2}\right\},$$

thus obtaining Ω as a function of t which, as will appear, vanishes identically when A, B, C, A', B', C' have the foregoing values.

I remark that this is, in effect, doing in a somewhat different form for the particular case what Goursat does for the general case, viz. starting from

$$\Omega_1=\frac{x'''}{x'}-\tfrac{3}{2}\left(\frac{x''}{x'}\right)^2+x'^2\left\{\frac{A}{(x-1)^2}+\frac{B}{x(x-1)}+\frac{C}{x^2}\right\},$$

with values of A, B, C which belong to the solution considered, he shows that this is a function of t having no infinities other than $(0, 1, \infty)$; that ∞ is not an infinity of the function or of the function multiplied into t, and that 0 and 1 are each of them a twofold infinity; that is, that the function is of the form

$$\frac{Lt^2+Mt+N}{t^2(t-1)^2}\ \text{ or }\ \frac{A'}{(t-1)^2}+\frac{B'}{t(t-1)}+\frac{C'}{t^2}.$$

Proceeding to carry out the process, we have

$$\frac{x'}{x-1}=\ -\frac{1}{t-1}+\frac{3Q'}{Q}-\frac{2R'}{R},$$

$$\frac{x'}{x}\ =\frac{2}{t}-\frac{1}{t-1}+\frac{3P'}{P}-\frac{2R'}{R},$$

and from either of these equations, collecting and reducing,

$$x'\ =\frac{5tP^2Q^2}{(t-1)^2R^3},$$

where observe that, from the values of x and $x-1$ respectively, it appears *à priori* that tP^2 and Q^2 must be factors in the numerator of x'. From this value of x', we have

$$\frac{x''}{x'}\ =\frac{1}{t}-\frac{2}{t-1}+\frac{2P'}{P}+\frac{2Q'}{Q}-\frac{3R'}{R};$$

and hence, P' and Q' being mere constants,

$$\frac{x'''}{x'}-\left(\frac{x''}{x'}\right)^2=-\frac{1}{t^2}+\frac{2}{(t-1)^2}-\frac{3R''}{R}-2\left(\frac{P'}{P}\right)^2-2\left(\frac{Q'}{Q}\right)^2+3\left(\frac{R'}{R}\right)^2,$$

and consequently

$$\begin{aligned}\Omega = &-\frac{1}{t^2}+\frac{2}{(t-1)^2}-\frac{3R''}{R}+2\left(\frac{P'}{P}\right)^2-2\left(\frac{Q'}{Q}\right)^2+3\left(\frac{R'}{R}\right)^2\\ &-\tfrac{1}{2}\left(\frac{1}{t}-\frac{2}{t-1}+\frac{2P'}{P}+\frac{2Q'}{Q}-\frac{3R'}{R}\right)^2\\ &+A\left(-\frac{1}{t-1}+\frac{3Q'}{Q}-2\frac{R'}{R}\right)^2\\ &+B\left(-\frac{1}{t-1}+\frac{3Q'}{Q}-2\frac{R'}{R}\right)\left(\frac{2}{t}-\frac{1}{t-1}+\frac{3P'}{P}-\frac{2R'}{R}\right)\\ &+C\left(\frac{2}{t}-\frac{1}{t-1}+\frac{3P'}{P}-\frac{2R'}{R}\right)^2\\ &-\frac{C'}{(t-1)^2}-\frac{B'}{t(t-1)}-\frac{A'}{t^2}.\end{aligned}$$

Putting for shortness

$$\frac{1}{t}=\alpha,\quad \frac{1}{t-1}=\beta,\quad \frac{P'}{P}=p,\quad \frac{Q'}{Q}=q,\quad \frac{R'}{R}=r,$$

this equation gives

$$\begin{aligned}\Omega = &-\alpha^2+2\beta^2-\frac{3R''}{R}-2p^2-2q^2+3r^2\\ &-\tfrac{1}{2}(\alpha-2\beta+2p+2q-3r)^2\\ &+A(-\beta+3q-2r)^2\\ &+B(-\beta+3q-2r)(2\alpha-\beta+3p-2r)\\ &+C(2\alpha-\beta+3p-2r)^2\\ &-C'\alpha^2-B'\alpha\beta-A'\beta^2,\end{aligned}$$

which is

$$\begin{aligned}= \quad &\alpha^2(-\tfrac{3}{2}+4C-C')-\frac{3R''}{R}\text{: say it is} = && L\alpha^2-\frac{48}{R}\\ &+\alpha\beta(2-2B-4C-B') && +M\alpha\beta\\ &+\beta^2(A+B+C-A') && +N\beta^2\\ &+\alpha p(-2+12C) && +F\alpha p\\ &+\alpha q(-2+6B) && +G\alpha q\\ &+\alpha r(3-4B-8C) && +H\alpha r\\ &+\beta p(4-3B-6C) && +F'\beta p\\ &+\beta q(4-6A-3B) && +G'\beta q\\ &+\beta r(-6+4A+4B+4C) && +H'\beta r\\ &+p^2(-4+9C) && +A''p^2\\ &+q^2(-4+9A) && +B''q^2\\ &+r^2(-\tfrac{3}{2}+4A+4B+4C) && +C''r^2\\ &+qr(6-12A-6B) && +F''qr\\ &+rp(6-6B-12C) && +G''rp\\ &+pq(-4+9B) && +H''pq.\end{aligned}$$

By decomposing $\alpha\beta$, αp, &c., into simple fractions, this becomes

$$\begin{aligned}\Omega = \quad & L\alpha^2 - \frac{48}{R} \\ & + M(-\alpha+\beta) \\ & + N\beta^2 \\ & + F(-\tfrac{4}{5}\alpha \ \ + \tfrac{4}{5}p) \\ & + G(-\tfrac{5}{4}\alpha \ \ + \tfrac{5}{4}q) \\ & + H\left(-\tfrac{11}{8}\alpha + \frac{11t+\frac{7}{8}}{R}\right) \\ & + F' \ (-4\beta + 4p) \\ & + G' \ (\ \ 5\beta - 5q) \\ & + H'\left(\quad \beta + \frac{-8t+19}{R}\right) \\ & + A''p^2 \\ & + B''q^2 \\ & + C''r^2 \\ & + F'' \tfrac{5}{12}\left(q - \frac{8t-43}{R}\right) \\ & + G'' \tfrac{4}{3} \left(p - \frac{8t-13}{R}\right) \\ & + H'' \tfrac{20}{9}(p-q).\end{aligned}$$

This is

$$\begin{aligned} = \quad & \alpha^2 L \\ & + \alpha(-M - \tfrac{4}{5}F - \tfrac{5}{4}G - \tfrac{11}{8}H) \\ & + \beta^2 N \\ & + \beta(M - 4F' + 5G' + H') \\ & + p^2 A'' \\ & + p(4F' + \tfrac{4}{3}G'' + \tfrac{20}{9}H'' + \tfrac{4}{5}F) \\ & + q^2 B'' \\ & + q(-5G' - \tfrac{5}{12}F'' - \tfrac{20}{9}H'' + \tfrac{5}{4}G) \\ & + r^2 C'' \\ & + \frac{1}{R}\{-48 + H(11t + \tfrac{7}{8}) + H'(-8t+19) - \tfrac{5}{12}F''(8t-43) - \tfrac{4}{3}G''(8t-13)\}.\end{aligned}$$

This should be identically $=0$; making $A''=0$, $B''=0$, $C''=0$, we find

$$A = \tfrac{4}{9},\ B = -\tfrac{37}{32},\ C = \tfrac{4}{9};\ (A+B+C = \tfrac{3}{8}).$$

and thence

$$F=\tfrac{10}{3},\ G=-\tfrac{61}{12},\ H=\tfrac{3}{2};\ F'=\tfrac{23}{8},\ G'=\tfrac{23}{8},\ H'=-\tfrac{9}{2};$$

$$F''=\tfrac{15}{4},\ G''=\tfrac{15}{4},\ H''=-\tfrac{68}{9}.$$

These values make the coefficients of p and q to be each $=0$; and they make the coefficient of R to be identically $=0$, viz. we have

$$0=-48+\tfrac{7}{8}H+19H'+\tfrac{215}{12}F''+\tfrac{52}{3}G'',$$

and

$$0=\quad 11H-8H'-\tfrac{10}{3}F''-\tfrac{32}{3}G''.$$

We have, moreover,

$$L=\tfrac{5}{18}-C',\ M=\tfrac{365}{144}-B',\ N=\tfrac{3}{8}-A';$$

and the coefficients of α and β are $=-M+\tfrac{13}{8}$ and $M-\tfrac{13}{8}$ respectively; hence the coefficients of α^2, β^2, α and β will all vanish if only $L=0$, $M=\tfrac{13}{8}$, $N=0$, that is,

$$A'=\tfrac{3}{8},\ B'=\tfrac{131}{144},\ C'=\tfrac{5}{8};$$

and we have thus identically $\Omega=0$, if only A, B, C, A', B', C' have the above-mentioned values.