907.

NOTE ON THE NINTH ROOTS OF UNITY.

[From the *Messenger of Mathematics*, vol. XX. (1891), p. 63.]

LET θ be a prime ninth root of unity, so that $\theta^6 + \theta^3 + 1 = 0$; and write

$$a = \theta + \theta^8,$$
$$b = \theta^2 + \theta^7,$$
$$c = \theta^4 + \theta^5;$$

then

$$a + b + c + \theta^3 + \theta^6 = \frac{1 - \theta^9}{1 - \theta} - 1, = -1,$$

that is,

$$a + b + c = 0.$$

Also

$$a^2 = b + 2, \quad bc = b - 1,$$
$$b^2 = c + 2, \quad ca = c - 1,$$
$$c^2 = a + 2, \quad ab = a - 1,$$

whence

$$ab + ac + bc = -3,$$
$$abc = -1;$$

and a, b, c are thus the roots of the equation $x^3 - 3x + 1 = 0$. We have

$$a^2b + b^2c + c^2a = a^2 + b^2 + c^2 = 6,$$
$$ab^2 + bc^2 + ca^2 = bc + ca + ab = -3,$$

and thence

$$-(a^2b + b^2c + c^2a) + (ab^2 + bc^2 + ca^2) = (b - c)(c - a)(a - b) = -6 - 3, = -9.$$

The equation $x^3 - 3x + 1 = 0$ is thus such that $a^2b + b^2c + c^2a$, and consequently any rational function whatever of a, b, c, invariable by the cyclical interchange (abc) of the roots, has a rational value.