906.

NOTE ON SCHLAEFLI'S MODULAR EQUATION FOR THE CUBIC TRANSFORMATION; WITH A CORRECTION.

[From the *Messenger of Mathematics*, vol. XX. (1891), pp. 59, 60; 120.]

THE equation in question, *Crelle*, t. LXXII. (1870), p. 369, is

$$S^4 + T^4 + 8ST - S^3T^3 = 0,$$

where

$$S = \frac{\sqrt{2}}{\sqrt[4]{(kk')}}, \quad T = \frac{\sqrt{2}}{\sqrt[4]{(\lambda\lambda')}};$$

which must be of course equivalent to the ordinary modular equation

$$u^4 - v^4 + 2uv - 2u^3v^3 = 0,$$

where

$$u = \sqrt[4]{k}, \quad v = \sqrt[4]{\lambda};$$

the resemblance in form between the two equations, with such different meanings of the S and T in the one case, and the u and v in the other, is very noticeable.

Schlaefli's equation is

$$\frac{4}{kk'} + \frac{4}{\lambda\lambda'} + \frac{16}{(kk'\lambda\lambda')^{\frac{1}{4}}} - \frac{8}{(kk'\lambda\lambda')^{\frac{3}{4}}} = 0,$$

that is,

$$kk' + \lambda\lambda' = 2\,(kk'\lambda\lambda')^{\frac{1}{4}} - 4\,(kk'\lambda\lambda')^{\frac{3}{4}},$$

or say

$$k^2k'^2 + \lambda^2\lambda'^2 = 4\,(kk'\lambda\lambda')^{\frac{1}{2}} - 18\,(kk'\lambda\lambda') + 16\,(kk'\lambda\lambda')^{\frac{3}{2}}.$$

To deduce this from the uv-modular equation, we have (Jacobi's *Fund. Nova*, p. 68, *Ges. Werke*, t. I., p. 124),

$$(1 - u^8)(1 - v^8) = (1 - u^2v^2)^4,$$

or, multiplying each side by u^8v^8, and extracting the fourth root, we have

$$\sqrt{(kk'\lambda\lambda')} = u^2v^2(1-u^2v^2) = x - x^2,$$

if for shortness we write $x = u^2v^2$.

The equation to be proved thus is

$$u^8(1-u^8) + v^8(1-v^8) = 4(x-x^2) - 18(x-x^2)^2 + 16(x-x^2)^3.$$

But from the foregoing equation

$$(1-u^8)(1-v^8) = (1-u^2v^2)^4,$$

we have

$$u^8 + v^8 = 4u^2v^2 - 6u^4v^4 + 4u^6v^6, \quad = 4x - 6x^2 + 4x^3,$$

and thence

$$u^{16} + v^{16} = (4x - 6x^2 + 4x^3)^2 - 2x^4;$$

and the equation to be proved thus becomes

$$(4x - 6x^2 + 4x^3) - (4x - 6x^2 + 4x^3)^2 + 2x^4 = 4(x-x^2) - 18(x-x^2)^2 + 16(x-x^2)^3,$$

which is in fact an identity, each side being

$$= 4x - 22x^2 + 52x^3 - 66x^4 + 48x^5 - 16x^6.$$

CORRECTION, p. 120.

I find that I misquoted Schlaefli's equation, viz. in effect, I took it to be

$$S_1^4 + T_1^4 + 8S_1T_1 - S_1^3T_1^3 = 0, \text{ where } S_1 = \frac{\sqrt{2}}{\sqrt[4]{(kk')}}, \; T_1 = \frac{\sqrt{2}}{\sqrt[4]{(\lambda\lambda')}}:$$

whereas his equation really is

$$S^4 + T^4 - 8ST + S^3T^3 = 0, \text{ where } S = 2\sqrt[4]{(kk')}, \; T = 2\sqrt[4]{(\lambda\lambda')}.$$

The change is only a change of form, for writing $S_1 = \frac{2\sqrt{2}}{S}$ and $T_1 = \frac{2\sqrt{2}}{T}$, the equation in (S_1, T_1) is converted into that in (S, T); but it was quite an unnecessary one, and I cannot account for having made it, as the paper in *Crelle* must have been before me.