901.

NOTE ON THE SUMS OF TWO SERIES.

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I CONSIDER the two series

$$S = \frac{1}{1+e^{\pi\alpha}} + \frac{1}{3(1+e^{3\pi\alpha})} + \frac{1}{5(1+e^{5\pi\alpha})} + \dots,$$

and

$$S_1 = \frac{1}{2+\pi\alpha} + \frac{1}{3(2+3\pi\alpha)} + \frac{1}{5(2+5\pi\alpha)} + \dots,$$

where α is real, positive, and indefinitely small; these would at first sight appear to be equal to each other, but this is not in fact the case.

Taking first the series S_1, putting therein $\pi\alpha = 2x$, this is

$$2S_1 = \frac{1}{1+x} + \frac{1}{3(1+3x)} + \frac{1}{5(1+5x)} + \dots.$$

Now we have, (Legendre, *Théorie des Fonctions Elliptiques*, t. II. p. 438),

$$\frac{y}{1+y} + \frac{y}{2(2+y)} + \frac{y}{3(3+y)} + \dots = C + \frac{d}{dy}\log\Gamma(y+1),$$

where C is Euler's constant, $= \cdot 577\dots$; and if y be real, positive, and very large, then

$$\Gamma(y+1) = \sqrt{(2\pi)}\, y^{y+\frac{1}{2}} e^{-y+\frac{1}{12y}+\dots};$$

whence, differentiating the logarithm and neglecting the terms which contain negative powers of y, then the value is $= C + \log y$; hence, writing $y = \frac{1}{x}$, we obtain

$$\frac{1}{1+x} + \frac{1}{2(1+2x)} + \frac{1}{3(1+3x)} + \dots = C - \log x.$$

Writing herein $2x$ for x, and dividing by 2, we have

$$\frac{1}{2(1+2x)}+\frac{1}{4(1+4x)}+\ldots=\tfrac{1}{2}C-\tfrac{1}{2}\log 2x,\ =\tfrac{1}{2}(C-\log 2)-\tfrac{1}{2}\log x;$$

or, subtracting,

$$\frac{1}{1+x}+\frac{1}{3(1+3x)}+\frac{1}{5(1+5x)}+\ldots=\tfrac{1}{2}(C+\log 2)-\tfrac{1}{2}\log x.$$

Hence, writing for x its value, $=\tfrac{1}{2}\pi\alpha$, we have

$$S_1=\tfrac{1}{4}(C+\log 2)-\tfrac{1}{4}\log\tfrac{1}{2}\pi\alpha,\quad =\tfrac{1}{4}(C+2\log 2-\log\pi)-\tfrac{1}{4}\log\alpha.$$

For the series S, we have (*Fundamenta Nova,* p. 103*) the formula

$$\tfrac{1}{4}\log\frac{2K}{\pi}=\frac{1}{1+q^{-1}}+\frac{1}{3(1+q^{-3})}+\frac{1}{5(1+q^{-5})}+\ldots;$$

or, putting herein $\alpha=\dfrac{K'}{K}$, then $q=e^{-\frac{\pi K'}{K}}=e^{-\pi\alpha}$, and thence

$$S=\frac{1}{1+e^{\pi\alpha}}+\frac{1}{2(1+e^{2\pi\alpha})}+\frac{1}{3(1+e^{3\pi\alpha})}+\ldots,\quad =\tfrac{1}{4}\log\frac{2K}{\pi}.$$

We have $q=e^{-\pi\alpha}$, which is real, positive, and less than but indefinitely near to 1; hence also k is real, positive, and less than but indefinitely near to 1, say the value is $=1-\beta$; thence $k'=\sqrt{(2\beta)}$, and $K=\log\dfrac{4}{k'}$, $=\log\dfrac{2\sqrt{2}}{\sqrt{\beta}}$; also $K'=\tfrac{1}{2}\pi$, and therefore $\alpha=\dfrac{K'}{K}=\tfrac{1}{2}\pi\div\log\dfrac{2\sqrt{2}}{\sqrt{\beta}}$, whence $\log\dfrac{2\sqrt{2}}{\sqrt{\beta}}=\dfrac{\pi}{2\alpha}$, which is the relation between α and β; and we thus have $\dfrac{2K}{\pi}=\dfrac{2}{\pi}\log\dfrac{2\sqrt{2}}{\sqrt{\beta}}$, $=\dfrac{1}{\alpha}$; and consequently $S=\tfrac{1}{4}\log\dfrac{2K}{\pi}$, $=-\tfrac{1}{4}\log\alpha$. The two values thus are

$$S=-\tfrac{1}{4}\log\alpha,\quad S_1=\tfrac{1}{4}(C+2\log 2-\log\pi)-\tfrac{1}{4}\log\alpha,$$

each depending on $\log\alpha$, and having for this term the same coefficient $-\tfrac{1}{4}$; but there is in S_1 a constant term $\tfrac{1}{4}(C+2\log 2-\log\pi)$, where C is the constant ·577....

It is easy to see why the series S is not reducible to S_1; however small α may be in the general term $\dfrac{1}{n(1+e^{n\pi\alpha})}$, then taking n sufficiently large, not only $n\pi\alpha$ is not indefinitely small, but it in fact becomes indefinitely large; the general term of the first series thus approximates to $\dfrac{1}{ne^{n\pi\alpha}}$, or the terms diminish somewhat more rapidly than in a geometric series with the ratio $e^{-\pi\alpha}$ (a small positive value less than but very near to 1), whereas in the second series the general term approximates to $\dfrac{1}{n^2\pi\alpha}$, or the convergence is ultimately that of the series $\dfrac{1}{n^2}+\dfrac{1}{(n+1)^2}+\ldots$.

[* Jacobi's *Gesammelte Werke*, t. I, p. 159.]