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A TRANSFORMATION IN ELLIPTIC FUNCTIONS.

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THE formula in question is given in Klein's Memoir "Ueber hyperelliptische Sigmafunctionen," *Math. Ann.* t. XXVII. (1886), pp. 431—464, see p. 454, in the form

$$u=\int_y^x \frac{(zdz)}{\sqrt{\{f(z)\}}},\quad \wp(u)=\frac{\sqrt{\{f(x)\}}\sqrt{\{f(y)\}}+F(x,\ y)}{2\,(xy)^2},$$

(the discovery of it being ascribed to Weierstrass); and it is also given in Halphen's *Traité des Fonctions Elliptiques*, t. II. (1888), p. 357.

The algebraic foundation of the theorem is as follows: writing

$$z=\tfrac{1}{2}\,\{\sqrt{(ae)}+c\},$$

and as usual I, J for the two invariants of the quartic function $(a,\ b,\ c,\ d,\ e \between x,\ y)^4$, we have identically, as is easily verified,

$$4z^3-Iz-J=\{d\sqrt{(a)}+b\sqrt{(e)}\}^2,$$

or say

$$\sqrt{(4z^3-Iz-J)}=d\sqrt{(a)}+b\sqrt{(e)}.$$

Hence putting

$$A=(a,\ b,\ c,\ d,\ e \between x_1,\ x_2)^4,$$

$$B=(a, \quad \between x_1,\ x_2)^3\,(y_1,\ y_2),$$

$$\vdots$$

$$E=(a, \quad \between y_1,\ y_2)^4;$$

and

$$\lambda=x_1y_2-x_2y_1,$$

so that

$$\lambda^4 I = AE - 4BD + 3C^2,$$

$$\lambda^6 J = ACE - AD^2 - B^2E + 2BCD - C^3,$$

then writing

$$z = \frac{1}{2\lambda^2}\{\sqrt{(AE)} + C\},$$

so that $\lambda^2 z$ is a bipartite irrational quadric function of the variables (x_1, x_2) and (y_1, y_2), we have

$$\sqrt{(4z^3 - Iz - J)} = \frac{1}{\lambda^3}\{D\sqrt{(A)} + B\sqrt{(E)}\},$$

and we thence infer the existence of a transformation

$$\frac{dz}{\sqrt{(4z^3 - Iz - J)}} = M\left\{\frac{x_1dx_2 - x_2dx_1}{\sqrt{(A)}} + \frac{y_1dy_2 - y_2dy_1}{\sqrt{(E)}}\right\}.$$

But for the verification hereof and a direct determination of the value of M, observe that we have

$$dz = \frac{1}{2\lambda^2}\left\{\sqrt{(E)}\frac{dA}{2\sqrt{(A)}} + dC - \{\sqrt{(AE)} + C\}\frac{2}{\lambda}d\lambda\right\},$$

where, attending only to the terms in dx_1, dx_2,

$$dA = 4\{(a, b, c, d \between x_1, x_2)^3 dx_1 + (b, c, d, e \between x_1, x_2)^3 dx_2\},$$

$$= 4(A_1dx_1 + A_2dx_2), \text{ suppose};$$

$$C = (a, b, c \between y_1, y_2)^2 x_1^2 + 2(b, c, d \between y_1, y_2)^2 x_1x_2 + (c, d, e \between y_1, y_2)^2 x_2^2,$$

$$= E_1x_1^2 + 2E_2x_1x_2 + E_3x_2^2, \text{ suppose};$$

and therefore

$$dC = 2\{(E_1x_1 + E_2x_2)dx_1 + (E_2x_1 + E_3x_2)dx_2\}.$$

Hence

$$dz = \frac{1}{2\lambda^2}\left\{\tfrac{1}{2}\frac{\sqrt{(E)}}{\sqrt{(A)}}4(A_1dx_1 + A_2dx_2) + 2(E_1x_1 + E_2x_2)dx_1\right.$$

$$\left. + 2(E_2x_1 + E_3x_2)dx_2 - \{\sqrt{(AE)} + C\}\frac{2}{\lambda}(y_2dx_1 - y_1dx_2)\right\}.$$

The whole coefficient herein of dx_1 is

$$= \frac{1}{\lambda^2}\left\{\frac{\sqrt{(E)}}{\sqrt{(A)}}A_1 + (E_1x_1 + E_2x_2) - \frac{\sqrt{(AE)} + C}{\lambda}y_2\right\},$$

which is

$$= \frac{1}{\lambda^2\sqrt{(A)}}\left\{A_1\sqrt{(E)} + (E_1x_1 + E_2x_2)\sqrt{(A)} - \frac{1}{\lambda}\{A\sqrt{(E)} + C\sqrt{(A)}\}y_2\right\}$$

$$= \frac{1}{\lambda^3\sqrt{(A)}}\{[-Cy_2 + \lambda(E_1x_1 + E_2x_2)]\sqrt{(A)} + [A_1\lambda - Ay_2]\sqrt{(E)}\},$$

which is

$$= \frac{1}{\lambda^3 \sqrt{(A)}} (-x_2) \{D \sqrt{(A)} + B \sqrt{(E)}\};$$

and similarly the whole coefficient of dx_2 is

$$= \frac{1}{\lambda^3 \sqrt{(A)}} \quad (x_1) \{D \sqrt{(A)} + B \sqrt{(E)}\}.$$

Hence the terms in dx_1, dx_2 are together

$$= \frac{1}{\sqrt{(A)}} (x_1 dx_2 - x_2 dx_1) \frac{D \sqrt{(A)} + B \sqrt{(E)}}{\lambda^3};$$

and since the terms in dy_1, dy_2 are of the like form, we have

$$dz = \left\{ \frac{1}{\sqrt{(A)}} (x_1 dx_2 - x_2 dx_1) + \frac{1}{\sqrt{(E)}} (y_1 dy_2 - y_2 dy_1) \right\} \times \frac{D \sqrt{(A)} + E \sqrt{(B)}}{\lambda^3},$$

and combining herewith the foregoing value

$$\sqrt{(4z^3 - Iz - J)} = \frac{D \sqrt{(A)} + E \sqrt{(B)}}{\lambda^3},$$

we have the required formula

$$\frac{dz}{\sqrt{(4z^3 - Iz - J)}} = \left\{ \frac{x_1 dx_2 - x_2 dx_1}{\sqrt{(A)}} + \frac{y_1 dy_2 - y_2 dy_1}{\sqrt{(B)}} \right\}.$$

As a very simple verification, suppose $a = 1$, $b = c = d = e = 0$; then

$$(A, B, C, D, E) = (x_1^4, x_1^3 y_1, x_1^2 y_1^2, x_1 y_1^3, y_1^4),$$

and if $\lambda = x_1 y_2 - x_2 y_1$ as before, then

$$z = \frac{x_1^2 y_1^2}{(x_1 y_2 - x_2 y_1)^2} = \theta^2,$$

where

$$\theta = \frac{-x_1 y_1}{x_1 y_2 - x_2 y_1}, \quad \text{or} \quad \frac{1}{\theta} = \frac{x_2}{x_1} - \frac{y_2}{y_1}.$$

Also $I = 0$, $J = 0$, and consequently

$$\frac{dz}{\sqrt{(4z^3 - Iz - J)}} = \frac{dz}{2 \sqrt{(z^3)}} = \frac{d\theta}{\theta^2} = \frac{x_1 dx_2 - x_2 dx_1}{x_1^2} - \frac{y_1 dy_2 - y_2 dy_1}{y_1^2},$$

which, in virtue of the foregoing values of A, E, is

$$= \frac{x_1 dx_2 - x_2 dx_1}{\sqrt{(A)}} - \frac{y_1 dy_2 - y_2 dy_1}{\sqrt{(E)}}.$$

I remark that an even more simple transformation from the general quartic radical to the Weierstrassian cubic radical was obtained by Hermite; this is alluded to in my "Note sur les Covariants, &c.," *Crelle*, t. L. (1855), pp. 285—287, [135], and is given with a demonstration in my paper "Sur quelques formules pour la transformation des intégrales elliptiques," *Crelle*, t. LV. (1858), pp. 15—24, [235], see No. IV.; viz. from the identical relation $JU^3 - IU^2H + 4H^3 = -\Phi^2$, which connects the covariants of a quartic function, it at once follows that if

$$U = (a,\ b,\ c,\ d,\ e \between x,\ 1)^4,$$

and H is the Hessian hereof,

$$H = (ac - b^2,\quad (ad - bc),\quad ae + 2bd - 3c^2,\quad 2(be - cd),\quad ce - d^2 \between x,\ 1)^4;$$

then writing

$$z = \frac{H}{U},$$

we have

$$\frac{dz}{\sqrt{(-4z^3 + zI - J)}} = \frac{2dx}{\sqrt{\{(a,\ b,\ c,\ d,\ e \between x,\ 1)^4\}}},$$

which is the transformation in question.