894.

THE INVESTIGATION BY WALLIS OF HIS EXPRESSION FOR π.

[From the *Quarterly Journal of Pure and Applied Mathematics*, vol. XXIII. (1889), pp. 165—169.]

THE following is in effect the investigation by Wallis in the *Arithmetica Infinitorum* (Oxford, 1656) of his well-known expression for π. He obtains the series of equations which in modern notation are

$$\int_0^1 (x - x^2)^0 \, dx = 1,$$

$$\int_0^1 (x - x^2)^1 \, dx = \tfrac{1}{2} \cdot \tfrac{1}{3},$$

$$\int_0^1 (x - x^2)^2 \, dx = \tfrac{1}{6} \cdot \tfrac{1}{5},$$

$$\int_0^1 (x - x^2)^3 \, dx = \tfrac{1}{20} \cdot \tfrac{1}{7},$$

or say, in general,

$$\int_0^1 (x - x^2)^n \, dx = \frac{1}{\phi(n)} \frac{1}{2n+1}.$$

In the case $n = \frac{1}{2}$, the integral gives the area of a semicircle, diameter $= 1$, viz. this is $= \frac{1}{8}\pi$, or we have $\frac{1}{8}\pi = \dfrac{1}{2\phi(\frac{1}{2})}$; or writing with him $\square = \dfrac{4}{\pi}$, this is $\square = \phi(\frac{1}{2})$, where $\phi(0)$, $\phi(1)$, $\phi(2)$, ... are the series of numbers 1, 2, 6, 20, ..., which are in fact the middle binomial coefficients of the several even powers, and are the diagonal numbers in the table which he subsequently considers. As regards the form in which Wallis exhibits these results, his theorem CXXXIII. is:

Si exponatur series Primanorum mulctata serie secundanorum, Residuorum Quadrata, Cubi, Biquadrata, &c. ad seriem Æqualium rationem habebunt cognitam.

And then, after an algebraical calculation,

$$\frac{1}{2}-\frac{1}{3}=\frac{1}{6} \quad \Big| \quad \frac{1}{3}-\frac{2}{4}+\frac{1}{5}=\frac{1}{30} \quad \Big| \quad \frac{1}{4}-\frac{3}{5}+\frac{3}{6}-\frac{1}{7}=\frac{1}{140}$$

$$\frac{1}{2.3}=\frac{1}{6} \quad \Big| \quad \frac{1.2}{3.4.5}=\frac{1}{30} \quad \Big| \quad \frac{1.2.3}{4.5.6.7}=\frac{1}{140}$$

$$\frac{1}{2.3}=\frac{1}{6} \quad \Big| \quad \frac{1}{2.3}\cdot\frac{4}{4.5}=\frac{1}{30} \quad \Big| \quad \frac{1}{2.3}\cdot\frac{4}{4.5}\cdot\frac{9}{6.7}=\frac{1}{140}$$

...et sic deinceps; continue multiplicando Numeratores per numeros quadratos et denominatores per binos continue sequentes arithmetice proportionales—viz. he thus explains the law of his calculated numbers.

He then forms the table

	1		$\frac{1}{2}$		$\frac{3}{8}$		$\frac{15}{48}$		$\frac{105}{384}$	$-\frac{1}{2}$
1	1	1	1	1	1	1	1	1	1	0
	1		$\frac{3}{2}$		$\frac{15}{8}$		$\frac{105}{48}$		$\frac{945}{384}$	$\frac{1}{2}$
$\frac{1}{2}$	1	$\frac{3}{2}$	2	$\frac{5}{2}$	3	$\frac{7}{2}$	4	$\frac{9}{2}$	5	1
	1		$\frac{5}{2}$		$\frac{35}{8}$		$\frac{315}{48}$		$\frac{3465}{384}$	$\frac{3}{2}$
$\frac{3}{8}$	1	$\frac{15}{8}$	3	$\frac{35}{8}$	6	$\frac{63}{8}$	10	$\frac{99}{8}$	15	2
	1		$\frac{7}{2}$		$\frac{63}{8}$		$\frac{693}{48}$		$\frac{9009}{384}$	$\frac{5}{2}$
$\frac{15}{48}$	1	$\frac{105}{48}$	4	$\frac{315}{48}$	10	$\frac{693}{48}$	20	$\frac{1287}{48}$	35	3
	1		$\frac{9}{2}$		$\frac{99}{8}$		$\frac{1287}{48}$		$\frac{19305}{384}$	$\frac{7}{2}$
$\frac{105}{384}$	1	$\frac{945}{384}$	5	$\frac{3465}{384}$	15	$\frac{9009}{384}$	35	$\frac{19305}{384}$	70	4
$-\frac{1}{2}$	0	$\frac{1}{2}$	1	$\frac{3}{2}$	2	$\frac{5}{2}$	3	$\frac{7}{2}$	4	

where the figures in the second column are $=1$, those in the fourth column are $=\frac{n+1}{1}$, those of the sixth column are $=\frac{(n+1)(n+2)}{1.2}$, and so on: n being in each case the rank in the column as shown by the right-hand outside column, viz. these ranks are $-\frac{1}{2}$, 0, $\frac{1}{2}$, 1, ... at intervals of $\frac{1}{2}$. And the several even lines contain the same figures as the several even columns respectively.

Wallis then remarks that, if in any line the first and second terms respectively are A and 1, then the even lines are as follows, viz. the second line is

$$A,\ 1,\ A\frac{1}{1},\ 1.\frac{2}{2},\ A\frac{1.3}{1.3},\ 1.\frac{2.4}{2.4},\ A\frac{1.3.5}{1.3.5},\ 1.\frac{2.4.6}{2.4.6},\ \ldots,$$

the fourth line is

$$A,\ 1,\ A\frac{3}{1},\ 1.\frac{4}{2},\ A\frac{3.5}{1.3},\ 1.\frac{4.6}{2.4},\ A\frac{3.5.7}{1.3.5},\ 1.\frac{4.6.8}{2.4.6},\ \ldots,$$

the sixth line is

$$A,\ 1,\ A\frac{5}{1},\ 1.\frac{6}{2},\ A\frac{5.7}{1.3},\ 1.\frac{6.8}{2.4},\ A\frac{5.7.9}{1.3.5},\ 1.\frac{6.8.10}{2.4.6},\ \ldots;$$

and this being so, he completes the table by inserting a term $\square$ in the diagonal line between the 1 and 2, and calculating the odd lines as follows: viz. if in each case A is the first term and B the second term of the line, then the complete rule is: the first line is

$$A,\ 1,\ A\frac{0}{1},\ 1.\frac{1}{2},\ A\frac{0.2}{1.3},\ 1.\frac{1.3}{2.4},\ A\frac{0.2.4}{1.3.5},\ 1.\frac{1.3.5}{2.4.6},$$

the second line is

$$A,\ 1,\ A\frac{1}{1},\ 1.\frac{2}{2},\ A\frac{1.3}{1.3},\ 1.\frac{2.4}{2.4},\ A\frac{1.3.5}{1.3.5},\ 1.\frac{2.4.6}{2.4.6},$$

the third line is

$$A,\ 1,\ A\frac{2}{1},\ 1.\frac{3}{2},\ A\frac{2.4}{1.3},\ 1.\frac{3.5}{2.4},\ A\frac{2.4.6}{1.3.5},\ 1.\frac{3.5.7}{2.4.6},$$

the fourth line is

$$A,\ 1,\ A\frac{3}{1},\ 1.\frac{4}{2},\ A\frac{3.5}{1.3},\ 1.\frac{4.6}{2.4},\ A\frac{3.5.7}{1.3.5},\ 1.\frac{4.6.8}{2.4.6},$$

the fifth line is

$$A,\ 1,\ A\frac{4}{1},\ 1.\frac{5}{2},\ A\frac{4.6}{1.3},\ 1.\frac{5.7}{2.4},\ A\frac{4.6.8}{1.3.5},\ 1.\frac{5.7.9}{2.4.6};$$

and so on, the rule for the odd lines being thus an interpolation from that for the even lines. Observe that, in the first line A is $=\infty$, and that, in the several odd terms after the first, $A\frac{0}{1}$ is $=1$.

The table thus calculated (I have given here the fractions in their least terms) is

∞	1	$\frac{1}{2}\square$	$\frac{1}{2}$	$\frac{1}{3}\square$	$\frac{3}{8}$	$\frac{4}{15}\square$	$\frac{5}{16}$	$\frac{8}{35}\square$	$\frac{35}{128}$	$-\frac{1}{2}$
1	1	1	1	1	1	1	1	1	1	0
$\frac{1}{2}\square$	1	$\square$	$\frac{3}{2}$	$\frac{4}{3}\square$	$\frac{15}{8}$	$\frac{8}{5}\square$	$\frac{35}{16}$	$\frac{64}{35}\square$	$\frac{315}{128}$	$\frac{1}{2}$
$\frac{1}{2}$	1	$\frac{3}{2}$	2	$\frac{5}{2}$	3	$\frac{7}{2}$	4	$\frac{9}{2}$	5	1
$\frac{1}{3}\square$	1	$\frac{4}{3}\square$	$\frac{5}{2}$	$\frac{8}{3}\square$	$\frac{35}{8}$	$\frac{64}{15}\square$	$\frac{105}{16}$	$\frac{128}{21}\square$	$\frac{1155}{128}$	$\frac{3}{2}$
$\frac{3}{8}$	1	$\frac{15}{8}$	3	$\frac{35}{8}$	6	$\frac{63}{8}$	10	$\frac{99}{8}$	15	2
$\frac{4}{15}\square$	1	$\frac{8}{5}\square$	$\frac{7}{2}$	$\frac{64}{15}\square$	$\frac{63}{8}$	$\frac{128}{15}\square$	$\frac{231}{16}$	$\frac{512}{35}\square$	$\frac{3003}{128}$	$\frac{5}{2}$
$\frac{5}{16}$	1	$\frac{35}{16}$	4	$\frac{105}{16}$	10	$\frac{231}{16}$	20	$\frac{429}{16}$	35	3
$\frac{8}{35}\square$	1	$\frac{64}{35}\square$	$\frac{9}{2}$	$\frac{128}{21}\square$	$\frac{99}{8}$	$\frac{512}{35}\square$	$\frac{429}{16}$	$\frac{1024}{35}\square$	$\frac{6435}{128}$	$\frac{7}{2}$
$\frac{35}{128}$	1	$\frac{315}{128}$	5	$\frac{1155}{128}$	15	$\frac{3003}{128}$	35	$\frac{6435}{128}$	70	4
$-\frac{1}{2}$	0	$\frac{1}{2}$	1	$\frac{3}{2}$	2	$\frac{5}{2}$	3	$\frac{7}{2}$	4	

in which table $\square$, as the term interpolated between the diagonal terms 1, 2, denotes the value $\frac{4}{\pi}$ as before.

The third line of the table is

$$\frac{1}{2}\square,\ 1,\ \square,\ \frac{3}{2},\ \frac{4}{3}\square,\ \frac{3.5}{2.4},\ \frac{4.6}{3.5}\square,\ \frac{3.5.7}{2.4.6},\ \frac{4.6.8}{3.5.7}\square,\ \frac{3.5.7.9}{2.4.6.8},\ \ldots.$$

The successive even terms continually increase but tend to equality, and in like manner the successive odd terms continually increase but tend to equality; it seems to have been assumed that, for any three consecutive terms x, y, z, we have $\frac{y}{x} > \frac{z}{y}$, that is, $y^2 > xz$. Taking this to be so, we have

$$1 > \frac{1}{2}\square^2,\quad \square^2 > \frac{3}{2},\quad \frac{3^2}{2^2} > \frac{4}{3}\square^2,\quad \frac{4^2}{3^2}\square^2 > \frac{3^2.5}{2^2.4},\quad \frac{3^2.5^2}{2^2.4^2} > \frac{4^2.6}{3^2.5}\square^2,\ \ldots,$$

and these equations give $\square$

less than	greater than
$\sqrt{\frac{2}{1}}$,	$\sqrt{\frac{3}{2}}$,
$\frac{3^2}{2.4}\sqrt{\frac{4}{3}}$,	$\frac{3^2}{2.4}\sqrt{\frac{5}{4}}$,
$\frac{3^2.5^2}{2.4^2.6}\sqrt{\frac{6}{5}}$,	$\frac{3^2.5^2}{2.4^2.6}\sqrt{\frac{7}{6}}$,
$\frac{3^2.5^2.7^2}{2.4^2.6^2.8}\sqrt{\frac{8}{7}}$,	$\frac{3^2.5^2.7^2}{2.4^2.6^2.8}\sqrt{\frac{9}{8}}$,

limits which tend continually to equality. We thus have

$$\square = \frac{4}{\pi},\quad = \frac{3.3.5.5.7.7\ldots}{2.4.4.6.6.8\ldots},$$

the number of factors in the numerator being always equal to the number in the denominator, and the accuracy of the approximation increasing with the number of factors.

It is to be remarked that for a square; row m and column n; m or $n = -\frac{1}{2}$, 0, $\frac{1}{2}$, 1, $\frac{3}{2}$, ... as before; the term of the square is in general $\Pi(m+n) \div \Pi(m)\,\Pi(n)$; thus $m = n = \frac{1}{2}$, the term is $\Pi(1) \div \{\Pi(\frac{1}{2})\}^2$, $= 1 \div (\frac{1}{2}\pi)^2 = \frac{4}{\pi}$, $= \square$; $m = 3$, $n = \frac{1}{2}$ it is $\Pi(\frac{7}{2}) \div \Pi(3)\,\Pi(\frac{1}{2})$, $= \frac{7}{2}.\frac{5}{2}.\frac{3}{2} \div 6$, $= \frac{105}{48}$; and so in any other case.