890.

NOTE ON THE HYDRODYNAMICAL EQUATIONS.

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WRITING for shortness $D = \frac{d}{dt} + u\frac{d}{dx} + v\frac{d}{dy} + w\frac{d}{dz}$, then if from the hydrodynamical equations

$$Du = \frac{d}{dx}\left(V - \frac{p}{\rho}\right),\ Dv = \frac{d}{dy}\left(V - \frac{p}{\rho}\right),\ Dw = \frac{d}{dz}\left(V - \frac{p}{\rho}\right),$$

without the aid of the equation

$$\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz} = 0,$$

we eliminate $V - \frac{p}{\rho}$, we obtain equations not equivalent to those of Helmholtz,

$$D\xi = \left(\xi\frac{d}{dx} + \eta\frac{d}{dy} + \zeta\frac{d}{dz}\right)u, \ = \xi\frac{du}{dx} + \eta\frac{dv}{dx} + \zeta\frac{dw}{dx},\ \text{\&c.},$$

$\left(2\xi,\ 2\eta,\ 2\zeta = \frac{dv}{dz} - \frac{dw}{dy},\ \frac{dw}{dx} - \frac{du}{dz},\ \frac{du}{dy} - \frac{dv}{dx}\right.$, as usual), but which, transforming them by means of the omitted equation, agree as they should do with his equations. But the form of the equations obtained directly by elimination as above, is an interesting one, which it is worth while to give.

We have

$$\begin{aligned} D\left(\frac{dv}{dz} - \frac{dw}{dy}\right) &= D\left(\frac{dv}{dz} - \frac{dw}{dy}\right) - \frac{d}{dz}Dv + \frac{d}{dy}Dw, \\ &= \left(\frac{d}{dt} + u\frac{d}{dx} + v\frac{d}{dy} + w\frac{d}{dz}\right)\left(\frac{dv}{dz} - \frac{dw}{dy}\right) \\ &\quad - \frac{d}{dz}\left(\frac{dv}{dt} + u\frac{dv}{dx} + v\frac{dv}{dy} + w\frac{dv}{dz}\right) \\ &\quad + \frac{d}{dy}\left(\frac{dw}{dt} + u\frac{dw}{dx} + v\frac{dw}{dy} + w\frac{dw}{dz}\right), \end{aligned}$$

where the terms containing second derived functions disappear of themselves, and the expression on the right-hand is thus

$$= -\frac{du}{dz}\frac{dv}{dx} - \frac{dv}{dz}\frac{dv}{dy} - \frac{dw}{dz}\frac{dv}{dz}$$
$$+\frac{du}{dy}\frac{dw}{dx} + \frac{dv}{dy}\frac{dw}{dy} + \frac{dw}{dy}\frac{dw}{dz}.$$

Representing for shortness the Matrix

$$\left|\begin{matrix} \frac{du}{dx}, & \frac{du}{dy}, & \frac{du}{dz} \\ \frac{dv}{dx}, & \frac{dv}{dy}, & \frac{dv}{dz} \\ \frac{dw}{dx}, & \frac{dw}{dy}, & \frac{dw}{dz} \end{matrix}\right| \text{ by } \left|\begin{matrix} a, & b, & c \\ a', & b', & c' \\ a'', & b'', & c'' \end{matrix}\right|, \text{ and its square by } \left|\begin{matrix} A, & B, & C \\ A', & B', & C' \\ A'', & B'', & C'' \end{matrix}\right|,$$

we have

$$\left|\begin{matrix} A, & B, & C \\ A', & B', & C' \\ A'', & B'', & C'' \end{matrix}\right| = \begin{array}{ccc|ccc} & & & \left(\frac{du}{dx}, \frac{dv}{dx}, \frac{dw}{dx}\right), & \left(\frac{du}{dy}, \frac{dv}{dy}, \frac{dw}{dy}\right), & \left(\frac{du}{dz}, \frac{dv}{dz}, \frac{dw}{dz}\right) \\ \hline \frac{du}{dx}, & \frac{du}{dy}, & \frac{du}{dz} & ,, & ,, & ,, \\ \frac{dv}{dx}, & \frac{dv}{dy}, & \frac{dv}{dz} & ,, & ,, & ,, \\ \frac{dw}{dx}, & \frac{dw}{dy}, & \frac{dw}{dz} & ,, & ,, & ,, \end{array}$$

viz. the combinations which enter into the foregoing formula are

$$C' = \frac{dv}{dx}\frac{du}{dz} + \frac{dv}{dy}\frac{dv}{dz} + \frac{dv}{dz}\frac{dw}{dz},$$

and

$$B'' = \frac{dw}{dx}\frac{du}{dy} + \frac{dw}{dy}\frac{dv}{dy} + \frac{dw}{dz}\frac{dw}{dy},$$

and the equation thus is $D(c' - b'') + C' - B'' = 0$; viz. the three equations are

$$D(c' - b'') + C' - B'' = 0,$$
$$D(a'' - c) + A'' - C = 0,$$
$$D(b - a') + B - A' = 0,$$

which are the equations in question.

Observe that we have

$$C' - B'' = (a', b', c'')(c, c', c'') - (a'', b'', c'')(b, b', b'')$$
$$= a'c' + b'c' + c'c'' \qquad - a''b - b'b'' - b''c'',$$

and thence, writing

$$\rho = a(c' - b'') + b(a'' - c) + c(b - a'),$$
$$= ac' - ab'' + a''b - a'c,$$

we have

$$C' - B'' + \rho = (a + b' + c'')(c' - b'') = 0,$$

if $a+b'+c''=0$; viz. this being so, $C'-B''=-\rho$, or the first equation is

$$D(c'-b'')=\rho, \ = a(c'-b'')+b(a''-c)+c(b-a'),$$

that is, $D\xi = \xi\frac{du}{dx}+\eta\frac{du}{dy}+\zeta\frac{du}{dz}$, the first equation of Helmholtz, and we thus have the equations of Helmholtz, if $a-b'+c''=0$, that is, if $\frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}=0$.

The foregoing three equations $D(c'-b'')+C'-B''=0$, &c., are the quaternion equation ($\sigma=iu+jv+kw$, $\nabla = i\frac{d}{dx}+j\frac{d}{dy}+k\frac{d}{dz}$, $\frac{d}{dt}=D$, denotes a complete differentiation),

$$\frac{d}{dt}V\nabla\sigma = V\nabla_1\sigma_2 S\sigma_1\nabla_2$$

of Mr McAulay's paper "Some General Theorems in Quaternion Integration," *Messenger of Mathematics*, vol. XIV. (1884), pp. 26—37; see p. 34.