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ON THE THEORY OF THE SINGULAR SOLUTIONS OF DIFFERENTIAL EQUATIONS OF THE FIRST ORDER.

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IN continuation of the former paper with this title (*Messenger*, vol. II., 1873, pp. 6—12, [545]), I propose to discuss various particular examples, chiefly of cases in which the differential equation is of the form $(L, M, N \between p, 1)^2 = 0$, where L, M, N are rational and integral functions of (x, y), and whether it admits or does not admit of an integral equation $(P, Q, R \between c, 1)^2 = 0$, where P, Q, R are rational and integral functions of (x, y).

The singular solution of the differential equation

$$(L, M, N \between p, 1)^2 = 0,$$

if there be a singular solution, is $S = 0$, where S is either $= LN - M^2$, or a factor of $LN - M^2$. But in general $LN - M^2$ is an indecomposable function, such that $LN - M^2 = 0$ is not a solution of the differential equation, and this being so, there is no singular solution; viz. a differential equation $(L, M, N \between p, 1)^2 = 0$, where L, M, N are rational and integral functions of (x, y), has not in general any singular solution.

Consider now a system of algebraical curves $U = 0$, where U is as regards (x, y) a rational and integral function of the order m, and depends in any manner on an arbitrary parameter C *. *I say that there is always a proper envelope,* which envelope is the singular solution of the differential equation obtained by the elimination of C from the equation $U = 0$, and the derived equation in regard to (x, y). *It follows that the differential equation $(L, M, N \between p, 1)^2 = 0$, which has no singular solution, does not admit of an integral of the form in question $U = 0$, viz. an integral representing a system of algebraic curves.*

* The expressions in the text may be understood as extending to the case where U is a function of any number (α) of constants $c_1, c_2, \ldots, c_\alpha$, connected by an $(\alpha - 1)$fold relation, U thus virtually depending on a single arbitrary parameter.

The theorem just referred to, that the system of algebraic curves $U=0$ has always an envelope, is an interesting theorem, which I proceed to prove. Assume that in general, that is, for an arbitrary value of the parameter, the equation $U=0$ represents a curve of the order m, with δ nodes and κ cusps (and therefore of the class n, with i inflexions and τ double tangents, the numbers m, δ, κ, n, τ, i being connected by Plücker's equations); for particular values of the parameter, the values of δ and κ may be increased, or the curve may break up, but this is immaterial.

The consecutive curve $U+\delta c d_c U=0$ is a curve of the same order m, with δ nodes and κ cusps, consecutive to the nodes and cusps of the original curve U, and the two curves intersect in m^2 points; but of these, there are 2 coinciding with each node, and 3 coinciding with each cusp of the curve $U=0$, as at once appears by drawing a curve with a node or a cusp, and the consecutive curve with a consecutive node or cusp; the number of the remaining intersections is $=m^2-2\delta-3\kappa$, and the envelope is the locus of these $m^2-2\delta-3\kappa$ points. Observe that the two curves have in common n^2 tangents; but of these, 2 coincide with each double tangent and 3 coincide with each stationary tangent of the curve $U=0$, viz. the number of the remaining common tangents is $=n^2-2\tau-3i$ (which is $=m^2-2\delta-3\kappa$): and that these $n^2-2\tau-3i$ common tangents are indefinitely near to the $m^2-2\delta-3\kappa$ common points respectively, and are in fact the tangents of the envelope at the $m^2-2\delta-3\kappa$ points respectively. Now in an algebraic curve we have $m+n=m^2-2\delta-3\kappa$, viz. the number $m^2-2\delta-3\kappa$ cannot be $=0$, and we have therefore always an envelope the locus of the system of the $m^2-2\delta-3\kappa$ points. It might be thought that the conclusion extends to transcendental curves; if this were so, the result would prove too much, viz. it would follow that a differential equation $(L, M, N \between p, 1)^2=0$ without a singular solution had no general integral; but it will appear by an example that the theorem as to the envelope does not extend to transcendental curves.

Ex. 1.

$$p^2-(1-y^2)=0, \text{ that is, } dy^2-(1-y^2)\,dx^2=0.$$

Here there is no algebraical integral, but there is a quasi-algebraical integral of the form $(P, Q, R \between c, 1)^2=0$; viz. starting with the form $y=\sin(x+C)$ and expressing $\sin C$ and $\cos C$ rationally in terms of a new parameter, this is

$$c^2(y+\cos x)-2c\sin x+(y-\cos x)=0,$$

where the coefficients are one-valued functions of (x, y). The discriminant of the differential equation in regard to p and that of the integral equation in regard to c are each $=y^2-1$, and we have a true singular solution $y^2-1=0$.

Ex. 2.

$$(1-x^2)p^2-(1-y^2)=0,$$

that is,

$$(1-x^2)\,dy^2-(1-y^2)\,dx^2=0.$$

We have here an algebraic integral of the proper form, which is at once derived from the circular form

$$C=\cos^{-1}x+\cos^{-1}y$$

by changing the constant, viz. this is

$$c^2 - 2cxy - (1 - x^2 - y^2) = 0.$$

The two discriminants are here each $= (x^2 - 1)(y^2 - 1)$, and we have

$$(x^2 - 1)(y^2 - 1) = 0$$

as a true singular solution. The curves are in fact the system of conics (ellipses and hyperbolas) each touching the four lines $x = 1$, $x = -1$, $y = 1$, $y = -1$.

Ex. 3.

$$(1 - y^2)p^2 - 1 = 0, \text{ that is, } (1 - y^2)\,dy^2 - dx^2 = 0.$$

This is an extremely interesting example: the curve is the orthogonal trajectory of the system of sinusoids $y = \sin(x + c)$, which is the integral of Example 1; and we thus at once see that the real portion of the curve is wholly included between the lines $y = -1$, $y = +1$, being an infinite continuous curve, having a series of equidistant cusps alternately at the one and the other line, and obtained by the continued repetition of the finite portion included between two consecutive cusps on the same line. The discriminant of the differential equation equated to zero gives $y^2 - 1 = 0$, the equation of the two lines in question; but this does not satisfy the differential equation, and it is consequently not a singular solution; by what precedes, it appears that it is, in fact, a cusp-locus.

We thus see that the curves which represent the integral equation have no *real* envelope; but it is to be further shown that there is no imaginary envelope, and that the curve obtained by the elimination of the parameter is, in fact, made up of a (imaginary) node-locus and of the foregoing cusp-locus.

The curve is properly represented by taking x, y each of them a one-valued function of the parameter θ, viz. we may write

$$y = \cos\theta,$$

$$x = c + \tfrac{1}{2}\theta - \tfrac{1}{4}\sin 2\theta.$$

In fact, these values give

$$\frac{dy}{d\theta} = -\sin\theta,\ \frac{dx}{d\theta} = \tfrac{1}{2}(1 - \cos 2\theta) = \sin^2\theta,$$

and therefore

$$p = -\frac{1}{\sin\theta} = \frac{-1}{\sqrt{(1 - y^2)}},$$

that is, $(1 - y^2)p^2 - 1 = 0$, the differential equation.

It is obvious that to a given value of the parameter there corresponds a single point of the curve; and it is to be shown that, conversely, to a given point of the curve corresponds in general a single value of the parameter.

Suppose the coordinates of the given point are $y = \cos\alpha$, $x = c + \frac{1}{2}\alpha - \frac{1}{4}\sin 2\alpha$, where α is a determinate quantity; then, to find θ, we have

$$\cos\theta = \cos\alpha, \quad 2\theta - \sin 2\theta = 2\alpha - \sin 2\alpha.$$

The first equation gives $\theta = 2m\pi \pm \alpha$, and the second equation then is

$$4m\pi \pm 2\alpha \mp \sin 2\alpha = 2\alpha - \sin 2\alpha;$$

viz. taking the upper signs, this is $4m\pi = 0$, giving $m = 0$ and $\theta = \alpha$; and, taking the lower signs, it is $m\pi = \alpha - \sin\alpha$, which, α being given, is not in general satisfied; hence to the given point there corresponds *only* the value α of the parameter θ. If, however, α is such that $\alpha - \sin\alpha$ is equal to a multiple of π, say $r\pi$, then the last-mentioned equation is satisfied by the value $m = r$, so that to the given point of the curve correspond the two values α and $2r\pi - \alpha$ of the parameter; these values are in general unequal, and the point is then a node; but they may be equal, viz. this is so if $\alpha = r\pi$ (the point on the curve being then $y = \cos r\pi, = \pm 1$, $x = c + \frac{1}{2}r\pi$), and the point is then a cusp; showing what was known, that there are on each of the lines $y = -1$, $y = +1$, an infinite series of equidistant cusps.

More definitely, suppose $\alpha = r\pi \pm \beta$, where β is a root of the equation $2\beta - \sin 2\beta = 0$, then

$$\sin 2\alpha = \pm \sin 2\beta, \quad 2\alpha - \sin 2\alpha = 2r\pi \pm (2\beta - \sin 2\beta) = 2r\pi,$$

and to the given point on the curve correspond the two values α and $2r\pi - \alpha$ of the parameter. If $\beta = 0$, we have, as above, the cusps on the two lines $y = +1$, $y = -1$ respectively; but if β be an imaginary root of the equation $2\beta - \sin 2\beta = 0$, then we have an infinite series of nodes on the imaginary line $y = \cos r\pi \cos\beta$; and there are an infinite number of such lines corresponding to the different imaginary roots of the equation $2\beta - \sin 2\beta = 0$.

From the form in which the equation of the curve is given, we cannot directly form the equation of the envelope by equating to zero the discriminant in regard to the constant c; but we may determine the intersections of the curve by the consecutive curve (corresponding to a value $c + \delta c$ of the constant), and thus determine the locus of these intersections.

Consider for a moment the curves belonging to the constants c, c_1, and let θ, θ_1 be the values of the parameter θ belonging to the points of intersection; we have $\cos\theta = \cos\theta_1$, $4c + 2\theta - \sin 2\theta = 4c_1 + 2\theta_1 - \sin 2\theta_1$; we have $\theta_1 = 2r\pi + \theta$, but we cannot thereby satisfy the second equation; or else $\theta_1 = 2r\pi - \theta$, giving

$$4c + 2\theta - \sin 2\theta = 4c_1 + 4r\pi - 2\theta + \sin 2\theta,$$

that is, $2\theta - \sin 2\theta = 2c_1 - 2c + 2r\pi$; and we have thus corresponding to any given value of r a series of values of θ, viz. these are $\theta = r\pi + \beta$, where β is any root of the equation

$$2\beta - \sin 2\beta = 2c_1 - 2c.$$

In particular, taking $c_1 = c$, the intersections are given by $\theta = r\pi + \beta$, where β is any root of the equation $2\beta - \sin 2\beta = 0$; viz. we have thus an infinite number of intersections lying on each of the lines $y = \cos r\pi \cos\beta$. If $\beta = 0$, the intersections lie on the two lines $y = 1$, $y = -1$ respectively; if β be an imaginary root of the equation $2\beta - \sin 2\beta = 0$, then they lie on the imaginary lines $y = \cos r\pi \cos\beta$. But by what precedes, it is clear that in the former case the intersections are nothing else than the cusps on the lines $y = 1$, $y = -1$; and in the latter case nothing else than the nodes on the lines $y = \cos r\pi \cos\beta$; viz. there is no proper envelope, but instead thereof we have lines of cusps and of nodes.

Ex. 4.

$$(1 - y^2)\, p^2 - (1 - x^2) = 0,$$

that is,

$$(1 - y^2)\, dy^2 - (1 - x^2)\, dx^2 = 0.$$

I have not examined this; the curve is the series of orthogonal trajectories of the conics of Example 2, and the integral equation may be represented by $y = \cos\theta$, $x = \cos\phi$, where $c = (2\theta - \sin 2\theta) - (2\phi - \sin 2\phi)$.

Equating to zero the discriminant of the differential equation, we have $(1-y^2)(1-x^2)=0$, viz. the four lines $x = 1$, $x = -1$, $y = 1$, $y = -1$; this is not an envelope, but a locus of cusps.