631.

SYNOPSIS OF THE THEORY OF EQUATIONS.

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THE following was proposed as one of the subjects of a Dissertation for the Trinity Fellowships:

Synopsis of the theory of equations; i.e. a statement in a logical order, of the divisions of the subject and the leading questions and theorems, but without demonstrations.

In the subject "Theory of Equations," the term equation is used to denote an equation of the form $x^n - p_1x^{n-1} + \ldots \pm p_n = 0$, where $p_1, p_2, \ldots, p_n$ are regarded as known, and x as a quantity to be determined; for shortness, the equation is written $f(x) = 0$.

The equation may be *numerical;* that is, the coefficients $p_1, p_2, \ldots, p_n$ are then numbers; understanding by number, a quantity of the form $\alpha + \beta i$, where α and β have any positive or negative real values whatever; or say, each of these is regarded as susceptible of continuous variation from an indefinitely large negative to an indefinitely large positive value: and i denotes $\surd(-1)$.

Or the equation may be *algebraic;* viz. the coefficients are then not restricted to denote, or are not explicitly considered as denoting, numbers.

I. We consider first numerical equations.

A number a (real or imaginary), such that substituted for x it makes the function $x^n - p_1x^{n-1} + \ldots \pm p_n$ to be $= 0$, or say, such that it satisfies the equation, is said to be a root of the equation; viz. a being a root, we have

$$a^n - p_1a^{n-1} + \ldots \pm p_n = 0, \text{ or say } f(a) = 0;$$

and it is then shown that $x - a$ is a factor of the function $f(x)$, viz. that we have $f(x) = (x-a)f_1(x)$, where $f_1(x)$ is a function $x^{n-1} - q_1x^{n-2} + \ldots \pm q_{n-1}$, of the order $n-1$, with numerical coefficients $q_1, q_2, \ldots, q_{n-1}$.

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In general, a is not a root of the equation $f_1(x) = 0$; but it may be so, viz. $f_1(x)$ may contain the factor $x - a$; when this is so, $f(x)$ will contain the factor $(x-a)^2$; writing then $f(x) = (x-a)^2 f_2(x)$, and assuming that a is not a root of the equation $f_2(x) = 0$, $x = a$ is then said to be a double root of the equation. Similarly, $f(x)$ may contain the factor $(x-a)^3$ and no higher power, and then $x = a$ is said to be a triple root; and so on.

Supposing, in general, that $f(x) = (x-a)^\alpha F(x)$, where α is a positive integer which may be $=1$, and Fx is of the order $n - \alpha$, then if b is a root different from a, we shall have $x - b$ a factor (in general a simple one, but it may be a multiple one) of $F(x)$, and $f(x)$ will in this case become $= (x-a)^\alpha (x-b)^\beta \Phi(x)$, where β is a positive integer which may be $=1$, and Φx is of the order $n - \alpha - \beta$. The original equation $fx = 0$ is in this case said to have α roots each $= a$, β roots each $= b$, and so on.

We have the *theorem*, a numerical equation of the order n has in every case n roots, viz. there exist n numbers $a, b, \ldots$ (in general, all of them distinct, but they may arrange themselves in groups of equal values) such that

$$f(x) = (x-a)(x-b)(x-c)\ldots \text{ identically.}$$

If an equation has equal roots, these can in general be determined; the case is at any rate a special one, which may be here omitted from consideration. It is therefore, in general, assumed that the equation $f(x) = 0$ under consideration has all its roots unequal. If the coefficients $p_1, p_2, \ldots$ are all or any one or more of them imaginary, then the equation $f(x) = 0$, separating the real and imaginary parts, may be written $F(x) + i\Phi(x) = 0$, where $F(x)$, $\Phi(x)$ are each of them a function with real coefficients; and it thus appears that the equation $f(x) = 0$ with imaginary coefficients has not in general any real root; supposing it to have a real root a, this must be at once a root of each of the equations $F(x) = 0$ and $\Phi(x) = 0$.

But an equation with real coefficients may have as well imaginary as real roots; and we have further the *theorem* that for such an equation the imaginary roots enter in pairs, viz. $\alpha + \beta i$ being a root, then will also $\alpha - \beta i$ be a root.

Considering an equation with real coefficients, the question arises as to the number and situation of its real roots; this is completely resolved by means of *Sturm's theorem*, viz. we form a series of functions $f(x)$, $f'(x)$, $f_2(x), \ldots, f_n(x)$ (a constant) of the degrees $n, n-1, \ldots, 2, 1, 0$ respectively; and substituting therein for x any two real values a and b, we find by means of the resulting signs of these functions how many real roots of $f(x)$ lie between the limits a, b.

The same thing can frequently be effected with greater facility by other means, but the only general method is the one just referred to.

In the general case of an equation with imaginary (it may be real) coefficients, the like question arises as to the situation of the (real or imaginary) roots, viz. if for facility of conception we regard the constituents α, β of a root $\alpha + \beta i$ as the coordinates of a point *in plano*, and accordingly represent the root by such point; then drawing in the plane any closed curve or "contour," the question is how many roots lie within such contour.

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This is solved *theoretically* by means of a theorem of Cauchy's, viz. writing in the original equation $x+iy$ in place of x, the function $f(x+iy)$ becomes $=P+iQ$, where P and Q are each of them a rational and integral function (with real coefficients) of (x, y). Imagining the point (x, y) to travel along the contour, and considering the number of changes of sign from $-$ to $+$ and from $+$ to $-$ of the fraction $\frac{P}{Q}$ corresponding to passages of the fraction through zero (that is, to values for which P becomes $=0$, disregarding those for which Q becomes $=0$), the difference of these numbers determines the number of roots within the contour. The investigation leads to a proof of the before-mentioned theorem, that a numerical equation of the order n has precisely n roots.

But, for the actual determination, it is necessary to consider a rectangular contour, and to apply to each of its sides separately a method such as that of Sturm's theorem; and thus the actual determination ultimately depends on a method such as that of Sturm's theorem.

Recurring to the case of an equation with real coefficients, it is important to *separate* the real roots, viz. to determine limits, such that each real root lies alone by itself between two limits l and m. This can be done (with more or less difficulty according to the nearness of the real roots) by repeated applications of Sturm's theorem, or otherwise.

The same thing would be useful, and can theoretically be effected, in regard to the roots of an equation generally, viz. we may, by lines parallel to the axes of x and y respectively, divide the plane into rectangles such that each (real or imaginary) root lies alone by itself in a given rectangle; but the ulterior theory, even as regards the imaginary roots of an equation with real coefficients, has not been developed, and the remarks which immediately follow have reference only to equations with real coefficients, and to the real roots of such equations.

Supposing the roots separated as above, so that a certain root is known to lie alone by itself between two given limits, then it is possible by various processes (Horner's, or Lagrange's method of continued fractions) to obtain to any degree of approximation the numerical value of the real root in question, and thus to obtain (approximately as above) the values of the several real roots.

The real roots can also frequently be obtained, without the necessity of a previous separation of the roots, by other processes of approximation—Newton's, as completed by Fourier, or by a method given by Encke—and the problem of their determination to any degree of approximation may be regarded as completely solved. But this is far from being practically the case even as regards the imaginary roots of such equations, or as regards the roots of an equation with imaginary coefficients.

A class of numerical equations which need to be considered, are the binomial equations $x^n - a = 0$, where $a, = \alpha + \beta i$, is a complex number. The foregoing conclusions apply, viz. there are always n roots, which it may be shown are all unequal. Supposing

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one of these is θ, so that $\theta^n = a$, then, assuming $x = \theta y$, we have $y^n - 1 = 0$, which equation (like the more general one $x^n - a = 0$) has precisely n roots; it is shown that these are $1, \omega, \omega^2, .., \omega^{n-1}$, where ω is a complex number $\alpha + \beta i$ such that $\alpha^2 + \beta^2 = 1$, or, what is the same thing, a complex number of the form $\cos\theta + i\sin\theta$; and it then at once appears that θ may be taken $= \frac{2\pi}{n}$. We have thus the trigonometrical solution of the equation $x^n - 1 = 0$. We may also obtain a like trigonometrical solution of the first-mentioned equation $x^n - a = 0$. We are thus led to the notion (a numerical) of the radical $a^{\frac{1}{n}}$, regarded as an n-valued function, viz. any one of these being denoted by $\sqrt[n]{(a)}$, then the series of values is

$$\sqrt[n]{(a)},\ \omega\sqrt[n]{(a)}, .., \ \omega^{n-1}\sqrt[n]{(a)}.$$

Or we may, if we please, use $\sqrt[n]{(a)}$, instead of $a^{\frac{1}{n}}$, as a symbol to denote the n-valued function.

It is not necessary, as regards the equation $x^n - 1 = 0$, to refer here to the distinctions between the cases n a prime, and a composite, number.

As the coefficients of an algebraical equation may be numerical, all which follows in regard to algebraical equations, is (with, it may be, some few modifications) applicable to numerical equations; and hence, concluding for the present this subject, it will be convenient to pass on to algebraical equations.

II. We consider, secondly, an algebraical equation

$$x^n - p_1 x^{n-1} + \ldots = 0,$$

and we here *assume* the existence of roots, viz. we assume that there are n quantities $a, b, c, \ldots$ (in general, all of them different, but in particular cases they may become equal in sets in any manner), such that

$$x^n - p_1 x^{n-1} + \ldots = (x-a)(x-b)\ldots.$$

Or, looking at the question in a different point of view, and starting with the roots $a, b, c, \ldots$ as given, we express the product of the n factors $x-a$, $x-b, \ldots$ in the foregoing form, and thus arrive at an equation of the order n having the n roots $a, b, c, \ldots$. In either case, we have

$$p_1 = \Sigma a,\ p_2 = \Sigma ab, .., \ p_n = abc\ldots,$$

viz. regarding the coefficients $p_1, p_2, .., p_n$ as given, then we assume the existence of roots $a, b, c, \ldots$ such that $p_1 = \Sigma a$, &c., or regarding the roots as given, then we write p_1, p_2, &c., to denote the functions Σa, Σab, &c.

It is to be noticed that, in virtue of

$$x^n - p_1 x^{n-1} + \ldots = (x-a)(x-b), \text{ \&c.},$$

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or of the equivalent equations $p_1 = \Sigma a$, &c., then

$$a^n - p_1 a^{n-1} + \ldots = 0,$$
$$b^n - p_1 b^{n-1} + \ldots = 0,$$
$$\&c.,$$

(viz. it is for this reason that $a, b, \ldots$ are said to be roots of $x^n - p_1 x^{n-1} + \ldots = 0$); and, moreover, that conversely from the last-mentioned equations, assuming that $a, b, \ldots$ are all different, we deduce

$$p_1 = \Sigma a,\ p_2 = \Sigma ab,\ \&c.,$$

and

$$x^n - p_1 x^{n-1} + \ldots = (x-a)(x-b)\ldots.$$

Observe that, if for instance $a = b$, then the two equations $a^n - p_1 a^{n-1} + \ldots = 0$, $b^n - p_1 b^{n-1} + \ldots = 0$ would reduce themselves to a single equation, which would not of itself express that a was a double root, that is, that $(x-a)^2$ was a factor of $x^n - p_1 x^{n-1} + \&c.$; but by considering b as the limit of $a + h$, h indefinitely small, we obtain a second equation

$$na^{n-1} - (n-1) p_1 a^{n-2} + \ldots = 0,$$

which, with the first, expresses that a is a double root; and then the whole system of equations leads, as before, to the equations $p_1 = \Sigma a$, &c. But this in passing: the general case is when the roots are all unequal.

We have then the *theorem* that every rational symmetrical function of the roots is a rational function of the coefficients; this is an easy consequence from the less general theorem, every rational and integral symmetrical function of the roots is a rational and integral function of the coefficients.

In particular, the sums of powers Σa^2, Σa^3, &c., are rational and integral functions of the coefficients.

An ordinary process, as regards the expression of other functions $\Sigma a^\alpha b^\beta$, &c., in terms of the coefficients, is to make them depend on the functions Σa^α, &c., but this is *very objectionable;* the true theory consists in showing that we have systems of equations

$$p_1 = \Sigma a,$$

$$\begin{cases} p_2 = \Sigma ab, \\ p_1^2 = \Sigma a^2 + 2\Sigma ab, \end{cases}$$

$$\begin{cases} p_3 = \Sigma abc, \\ p_1 p_2 = \Sigma a^2 b + 3\Sigma abc, \\ p_1^3 = \Sigma a^3 + 3\Sigma a^2 b + 6\Sigma abc, \end{cases}$$

$$\&c.,\ \&c.$$

where, in each system, there are precisely as many equations as there are root-functions on the right-hand side, e.g. 3 equations and 3 functions Σabc, $\Sigma a^2 b$, Σa^3. Hence, in each system, the root-functions can be determined linearly in terms of the powers and products of the coefficients.

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It follows that it is possible to determine an equation (of an assignable order) having for roots any given (unsymmetrical) functions of the roots of a given equation. For example, in the case of a quartic equation, roots (a, b, c, d), it is possible to find an equation having the roots ab, ac, ad, bc, bd, cd, being therefore a sextic equation; viz. in the product $(y-ab)(y-ac)(y-ad)(y-bc)(y-bd)(y-cd)$, the coefficients of the several powers of y will be symmetrical functions of a, b, c, d, and therefore rational and integral functions of the coefficients of the original quartic equation.

In connexion herewith, the question arises as to the number of values (obtained by permutations of the roots) of given unsymmetrical functions of the roots; for instance, with roots (a, b, c, d) as before, how many values are there of the function $ab+cd$; or, better, how many functions are there of this form; the answer is 3, viz. $ab+cd$, $ac+bd$, $ad+bc$; or, again, we may ask whether it is possible to obtain functions of a given number of values, 3-valued, 4-valued functions, &c.

We have, moreover, the very important *theorem* that, given the value of any unsymmetrical function, e.g. $ab+cd$, it is in general possible to determine rationally the value of any similar function, e.g. $(a+b)^3+(c+d)^3$.

The *à priori* ground of this theorem may be illustrated by means of a numerical equation. Suppose, e.g. that the roots of a quartic equation are 1, 2, 3, 4; then if it is given that $ab+cd=14$, this in effect determines a, b to be 1, 2 (viz. $a=1$, $b=2$, or else $a=2$, $b=1$) and c, d to be 3, 4 (viz. $c=3$, $d=4$, or else $c=4$, $d=3$); and it therefore in effect determines $(a+b)^3+(c+d)^3$ to be $=370$, and not any other value. And we can in the same way account for cases of failure as regards particular equations; thus, the roots being 1, 2, 3, 4, as above, $a^2b=2$ determines a to be $=1$ and b to be $=2$; but if the roots had been 1, 2, 4, 16, then $a^2b=16$ does not uniquely determine a and b, but only makes them to be 1 and 16, or else 2 and 4, respectively.

As to the *à posteriori* proof, assume, for instance, $t_1=ab+cd$, $y_1=(a+b)^3+(c+d)^3$, and so $t_2=ac+db$, $y_2=(a+c)^3+(d+b)^3$, &c.—in the present case there are only the functions t_1, t_2, t_3 and y_1, y_2, y_3—then $y_1+y_2+y_3$, $t_1y_1+t_2y_2+t_3y_3$, $t_1^2y_1+t_2^2y_2+t_3^2y_3$ will be respectively symmetrical functions of the roots of the quartic, and therefore rational and integral functions of its coefficients, that is, they will be known.

Imagine, in the first instance, that t_1, t_2, t_3 are all known; then the equations being linear in y_1, y_2, y_3, these can be expressed rationally in terms of known functions of the coefficients and of t_1, t_2, t_3, that is, y_1, y_2, y_3 will be known. But observe further, that y_1 is obtained as a function of t_1, t_2, t_3 symmetrical as regards t_2, t_3; it can consequently be expressed as a rational function of t_1 and of t_2+t_3, t_2t_3, or, what is the same thing, of t_1 and $t_1+t_2+t_3$, $t_1t_2+t_1t_3+t_2t_3$, $t_1t_2t_3$; but these last will be symmetrical functions of the roots, and as such expressible rationally in terms of the coefficients: that is, y_1 will be expressed as a rational function of t_1 and of the coefficients, or, t_1 being known, y_1 will be rationally determined.

We may consider now the question of the algebraical solution of equations, or, more accurately, that of the *solution of equations by radicals*.

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In the case of a quadric equation $x^2+px+q=0$, we can find for x, by the assistance of the sign $\sqrt{(\ )}$ or $(\ )^{\frac{1}{2}}$, an expression for x as a two-valued function of the coefficients p, q, such that, substituting this value in the equation, the equation is thereby identically satisfied, viz. we have

$$x=-\tfrac{1}{2}p\pm\sqrt{(\tfrac{1}{4}p^2-q)},$$

giving

$$\begin{aligned} x^2 &= \tfrac{1}{2}p^2-q\mp p\sqrt{(\tfrac{1}{4}p^2-q)}\\ +px &= -\tfrac{1}{2}p^2 \pm p\sqrt{(\tfrac{1}{4}p^2-q)}\\ +q &= +q \\ \hline x^2+px+q &=0, \end{aligned}$$

and the equation is on this account said to be algebraically solvable, or, more accurately, to be *solvable by radicals*. Or we may, by writing $x=-\frac{1}{2}p+z$, reduce the equation to $z^2=\frac{1}{4}p^2-q$, viz. to an equation of the form $z^2=a$, and, in virtue of its being thus reducible, we may say that the equation is solvable by radicals. And the question for an equation of any higher order is, say of the order n, can we by means of radicals, that is, by aid of the sign $\sqrt[m]{(\ )}$ or $(\ )^{\frac{1}{m}}$, using as many as we please of such signs and with any values of m, find an n-valued function (or any function) of the coefficients, which substituted for x in the equation shall satisfy it identically.

It will be observed that the coefficients $p, q, \ldots$ are not explicitly considered as numbers, but that even if they do denote numbers, the question whether a numerical equation admits of solution by radicals is wholly unconnected with the before-mentioned theorem of the existence of the n roots of such an equation. It does not even follow that, in the case of a numerical equation solvable by radicals, the algebraical expression of x gives the numerical solution; but this requires explanation. Consider, first, a numerical quadric equation with imaginary coefficients; in the formula $x=-\frac{1}{2}p\pm\sqrt{(\frac{1}{4}p^2-q)}$, substituting for p, q their given numerical values we obtain for x an expression of the form $x=\alpha+\beta i\pm\sqrt{(\gamma+\delta i)}$, where α, β, γ, δ are real numbers; this value substituted in the numerical equation would satisfy it identically and it is thus an algebraical solution; but there is no obvious *à priori* reason why the expression $\sqrt{(\gamma+\delta i)}$ should have a value $=c+di$, where c and d are real numbers calculable by the extraction of a root or roots of real numbers; it appears upon investigation that $\sqrt{(\gamma+\delta i)}$ has such a value calculable by means of the radical expression $\sqrt{\{\sqrt{(\gamma^2+\delta^2)}\pm\gamma\}}$; and hence that the algebraical solution of a quadric equation does in every case give the numerical solution of a numerical quadric. The case of a numerical cubic will be considered presently.

A cubic equation can be solved by radicals, viz. taking for greater simplicity the cubic in the reduced form $x^3-qx-r=0$, and writing $x=a+b$, this will be a solution if only $3ab=q$, and $a^3+b^3=r$, or say $\frac{1}{2}(a^3+b^3)=\frac{1}{2}r$; whence

$$\tfrac{1}{2}(a^3-b^3)=\pm\sqrt{(\tfrac{1}{4}r^2-\tfrac{1}{27}q^3)},$$

and therefore

$$a = \sqrt[3]{\{\tfrac{1}{2}r \pm \sqrt{(\tfrac{1}{4}r^2 - \tfrac{1}{27}q^3)}\}},$$

a six-valued function of q, r. But then writing $b = \dfrac{q}{3a}$, we have, as may be shown, $a+b$ a three-valued function of the coefficients; it would have been wrong to complete the solution by writing $b = \sqrt[3]{\{\tfrac{1}{2}r \pm \sqrt{(\tfrac{1}{4}r^2 - \tfrac{1}{27}q^3)}\}}$, since here $(a+b)$ would be given as a 9-valued function, having only 3 of its values roots, and the other 6 values being irrelevant. An interesting variation of the solution is to write $x = ab\,(a+b)$, giving $a^3b^3(a^3+b^3) = r$ and $3a^3b^3 = q$, or say $\tfrac{1}{2}(a^3+b^3) = \tfrac{3}{2}\dfrac{r}{q}$, $a^3b^3 = \tfrac{1}{3}q$; whence

$$\{\tfrac{1}{2}(a^3 - b^3)\}^2 = \frac{9}{q^2}(\tfrac{1}{4}r^2 - \tfrac{1}{27}q^3),$$

and therefore

$$a = \sqrt[3]{\left\{\frac{3}{2}\frac{r}{q} \pm \frac{3}{q}\sqrt{(\tfrac{1}{4}r^2 - \tfrac{1}{27}q^3)}\right\}},\quad b = \sqrt[3]{\left\{\frac{3}{2}\frac{r}{q} \mp \frac{3}{q}\sqrt{(\tfrac{1}{4}r^2 - \tfrac{1}{27}q^3)}\right\}},$$

and here although a, b are each of them a 6-valued function, yet, as may be shown, $ab\,(a+b)$ is only a 3-valued function.

In the case of a numerical cubic, even when the coefficients are real, substituting their values in the expression

$$x = \sqrt[3]{\{\tfrac{1}{2}r \pm \sqrt{(\tfrac{1}{4}r^2 - \tfrac{1}{27}q^3)}\}} + [\tfrac{1}{3}q \div \sqrt[3]{\{\tfrac{1}{2}r \pm \sqrt{(\tfrac{1}{4}r^2 - \tfrac{1}{27}q^3)}\}}],$$

this *may* depend on an expression of the form $\sqrt[3]{(\gamma + \delta i)}$, where γ and δ are real numbers (viz. it will do so if $\tfrac{1}{4}r^2 - \tfrac{1}{27}q^3$ is a negative number), and here we *cannot* by the extraction of any root or roots of real numbers reduce $\sqrt[3]{(\gamma + \delta i)}$ to the form $c + di$, c and d real numbers; hence, here the algebraical solution does not give the numerical solution. It is to be added that the case in question, called the "irreducible case," is that wherein the three roots of the cubic equation are all real; if the roots are one real and two imaginary, then, contrariwise, the quantity under the cube root is real, and the algebraical solution gives the numerical one.

The irreducible case is solvable by a trigonometrical formula, but this is not a solution by radicals; it consists, in effect, in reducing the given numerical cubic (not to a cubic of the form $z^3 = a$, solvable by the extraction of a cube root, but) to a cubic of the form $4x^3 - 3x = a$, corresponding to the equation $4\cos^3\theta - 3\cos\theta = \cos 3\theta$ which serves to determine $\cos\theta$ when $\cos 3\theta$ is known.

A quartic equation is solvable by radicals; and it may be remarked, that the existence of such a solution depends on the existence of 3-valued functions such as $ab + cd$, of the four roots (a, b, c, d); by what precedes, $ab + cd$ is the root of a cubic equation, which equation is solvable by radicals; hence $ab + cd$ can be found by radicals; and since $abcd$ is a given value, ab and cd can each be found by radicals. But by what precedes, if ab be known, then any similar function, say $a + b$, is obtainable rationally; and, consequently, from the values of $a + b$ and ab we may by radicals

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obtain the value of a or b, that is, an expression for a root of the given quartic expression; the expression finally obtained is 4-valued, corresponding to the different values of the several radicals which enter therein, and we have therefore the expression by radicals of each of the four roots of the quartic equation. But when the quartic is numerical, the same thing arises as in the cubic: the algebraical expression does not in every case give the numerical one.

It will be understood from the foregoing explanation as to the quartic, how in the next following case, that of a quintic equation, the question of the solvability by radicals depends on the existence or non-existence of k-valued functions of the five roots (a, b, c, d, e); a fundamental theorem on the subject is that a rational function of 5 letters, if it has less than 5, cannot have more than 2 values; viz. that there are no 3-valued, or 4-valued, functions of 5 letters; and by reasoning, depending in part upon this theorem, Abel showed that a general quintic equation is not solvable by radicals: and *à fortiori* the general equation of any order higher than 5 is not solvable by radicals.

The general theory of the solvability of an equation by radicals depends very much on Vandermonde's remark, that supposing an equation is solvable (by radicals) and that we have therefore an algebraical expression of x in terms of the coefficients, then substituting for the coefficients their values in terms of the roots, the resulting value of the expression must reduce itself to any one at pleasure of the roots $a, b, c, \ldots$; thus in the case of the quadric equation where the solution is $x = +\tfrac{1}{2}p \pm \sqrt{(\tfrac{1}{4}p^2 - q)}$, writing for p, q their values $a+b$, ab, this is $x = \tfrac{1}{2}[(a+b) \pm \sqrt{\{(a-b)^2\}}]$, $= a$ or b according to the value of the radical. But it is not considered necessary in the present sketch to go further into the theory of the solvability of an equation by radicals. It may be proper to remark that, for quintic equations, there are solutions analogous to the trigonometrical solution of a cubic equation, viz. the quintic equation is here in effect reduced to some special form of quintic equation; for instance, to Jerrard's form $x^5 + ax + b = 0$ or to some form presenting itself in the theory of elliptic functions; but the solutions in question are not solutions by radicals. And there are various other interesting parts of the theory which have been excluded from consideration.

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