555.

NOTICES OF COMMUNICATIONS TO THE BRITISH ASSOCIATION FOR THE ADVANCEMENT OF SCIENCE.

[From the *Reports of the British Association for the Advancement of Science*, 1865 to 1873, *Notices and Abstracts of Miscellaneous Communications to the Sections.*]

1. *On the Problem of the in-and-circumscribed Triangle.* Report, 1870, pp. 9, 10.

I HAVE recently accomplished the solution of this problem, which I spoke of at the Meeting in 1864. The problem is as follows: required the number of the triangles the angles of which are situate in a given curve or curves, and the sides of which touch a given curve or curves. There are in all 52 cases [see 514] of the problem, according as the curves which contain the angles and are touched by the sides are distinct curves, or are any or all of them the same curve. The first and easiest case is when the curves are all of them distinct; the number of triangles is here $=2aceBDF$, where a, c, e are the *orders* of the curves containing the angles (or, say, of the angle-curves) respectively; and B, D, F are the *classes* of the curves touched by the sides (or, say, of the side-curves) respectively. An interesting case is when the angle-curves are one and the same curve; or, say, $a=c=e$ (where the sign $=$ is used to denote the identity of the curves); the number of triangles is here $=\{2a(a-1)(a-2)+A\}\,BDF$, where a, A are the order and class of the curve $a=c=e$. In the reciprocal case, where the side-curves are one and the same curve, say $B=D=F$, we have of course a like formula, viz. the number of triangles is here $=\{2B(B-1)(B-2)+b\}\,ace$, where B, b are the class and order of the curve $B=D=F$. The last and most difficult case is when the six curves are all of them one and the same curve, say $a=c=e=B=D=F$; the number of triangles is here $=$ one-sixth of

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\begin{aligned}
& A^4(\;.\quad\;\; .\quad\;\; .\quad\;\; .\;\; +1),\\
&+A^3(\;.\quad 2a^3-18a^2+52a-46)\\
&+A^2(\;.\; -18a^3+162a^2-420a+221)\\
&+A\;(\;.\quad 52a^3-420a^2+704a+172)\\
&+\quad a^4-46a^3+221a^2+172a\\
&+\alpha\left\{\begin{array}{l} A^2(\;.\quad .\quad .\quad -9)\\ +A\;(\;.\quad .\;-12a+135)\\ \quad -9a^2+135a-600\end{array}\right\},
\end{aligned}
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where a is the order, A the class of the curve; α is the number, three times the class + the number of cusps, or (what is the same thing) three times the order + the number of inflexions.

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2. *On a Correspondence of Points and Lines in Space.* Report, 1870, p. 10.

NINE points in a plane may be the intersection of two (and therefore of an infinite series of) cubic curves; say, that the nine points are an "ennead": and similarly nine lines through a point may be the intersection of two (and therefore of an infinite series of) cubic cones; say, the nine lines are an ennead. Now, imagine (in space) any 8 given points; taking a variable point P, and joining this with the 8 points, we have through P 8 lines, and there is through P a ninth line completing the ennead; this is said to be the corresponding line of P. We have thus to any point P a single corresponding line through the point P; this is the correspondence referred to in the heading, and which I would suggest as an interesting subject of investigation to geometers. Observe that, considering the whole system of points in space, the corresponding lines are a triple system of lines, *not* the whole system of lines in space. It is thus, not any line whatever, but only a line of the triple system, which has on it a corresponding point. But as to this some explanation is necessary; for starting with an arbitrary line, and taking upon it a point P, it would seem that P might be so determined that the given line and the lines from P to the eight points should form an ennead,—that is, that the arbitrary line would have upon it a corresponding point or points.

The question of the foregoing species of correspondence was suggested to me by the consideration of a system of 10 points, such that joining any one whatever of them with the remaining nine points, the nine lines thus obtained form an ennead; or, say, that each of the 10 points is the "enneadic centre" of the remaining nine. I have been led to such a system of 10 points by my researches upon Quartic Surfaces; but I do not as yet understand the theory.

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3. *On the Number of Covariants of a Binary Quantic.* Report, 1871, pp. 9, 10.

THE author remarked [see 462] that it had been shown by Prof. Gordan that the number of the covariants of a binary quantic of any order was finite, and, in particular, that the numbers for the quintic and the sextic were 23 and 26 respectively. But the demonstration is a very complicated one, and it can scarcely be doubted that a more simple demonstration will be found. The question in its most simple form is as follows: viz. instead of the covariants we substitute their leading coefficients, each of which is a "seminvariant" satisfying a certain partial differential equation; say, the quantic is $(a, b, c, \ldots, k \between x, y)^n$, then the differential equation is $(a\partial_b + 2b\partial_c \ldots + nj\partial_k)\, u = 0$, which *quà* equation with $n+1$ variables admits of n independent solutions: for instance, if $n = 3$, the equation is $(a\partial_b + 2b\partial_c + 3c\partial_d)\, u = 0$, and the solutions are a, $ac - b^2$,

$a^2d - 3abc + 2b^3$; the general value of u is $u =$ any function whatever of the last-mentioned three functions. We have to find the rational non-integral functions of these functions which are rational and integral functions of the coefficients; such a function is

$$\frac{1}{a^2}\{(a^2d - 3abc + 2b^3)^2 + 4(ac - b^2)^3\},$$

$$= a^2d^2 + 4ac^3 + 4b^3d - 3b^2c^2 - 6abcd,$$

and the original three solutions, together with the last-mentioned function $a^2d^2 + \&c.$, constitute the complete system of the seminvariants of the cubic function; viz. every other seminvariant is a rational *and integral* function of these. And so, in the general case, the problem is to complete the series of the n solutions a, $ac - b^2$, $a^2d - 3abc + 2b^3$, $a^3e - 4a^2bd + 6ab^2c - 3b^4$, &c. by adding thereto the solutions which, being rational but non-integral functions of these, are rational and integral functions of the coefficients; and thus to arrive at a series of solutions such that every other solution is a rational and integral function of these.

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4. *Note on certain Families of Surfaces.* Report, 1871, pp. 19, 20.

SEE the paper numbered 538, of which this Note is a duplicate.

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5. *On the Mercator's Projection of a Surface of Revolution.* Report, 1873, p. 9.

THE theory of Mercator's projection is obviously applicable to any surface of revolution; the meridians and parallels are represented by two systems of parallel lines at right angles to each other, in such wise that for the infinitesimal rectangles included between two consecutive arcs of meridian and arcs of parallel the rectangle in the projection is similar to that on the surface. Or, what is the same thing, drawing on the surface the meridians at equal infinitesimal intervals of angular distance, we may draw the parallels at such intervals as to divide the surface into infinitesimal squares; the meridians and parallels are then in the projection represented by two systems of equidistant parallel lines dividing the plane into squares. And if the angular distance between two consecutive meridians instead of being infinitesimal is taken moderately small (5° or even 10°), then it is easy on the surface or *in plano*, using only the curve which is the meridian of the surface, to lay down graphically the series of parallels which are in the projection represented by equidistant parallel lines. The method is, of course, an approximate one, by reason that the angular distance between the two consecutive meridians is finite instead of infinitesimal.

I have in this way constructed the projection of a skew hyperboloid of revolution: viz. in one figure I show the hyperbola, which is the meridian section, and by means of it (taking the interval of the meridians to be $= 10°$) construct the positions of the successive parallels; I complete the figure by drawing the hyperbolas which are the orthographic projections of the meridians, and the right lines which are the orthographic projections of the parallels; the figure thus exhibits the orthographic projection (on the plane of a meridian) of the hyperboloid divided into squares as above. The other figure, which is the Mercator's projection, is simply two systems of equidistant parallel lines dividing the paper into squares. I remark that in the first figure the projections of the right lines on the surface are the tangents to the bounding hyperbola; in particular, the projection of one of these lines is an asymptote of the hyperbola. This I exhibit in the figure, and by means of it trace the Mercator's projection of the right line on the surface; viz. this is a serpentine curve included between the right lines which represent two opposite meridians and having these lines for asymptotes. It is sufficient to show one of these curves, since obviously for any other line of the surface belonging to the same system the Mercator's projection is at once obtained by merely displacing the curve parallel to itself, and for any line of the other system the projection is a like curve in a reversed position.

A Mercator's projection might be made of a skew hyperboloid not of revolution; viz. the curves of curvature might be drawn so as to divide the surface into squares, and the curves of curvature be then represented by equidistant parallel lines as above; and the construction would be only a little more difficult. The projection presented itself to me as a convenient one for the representation of the geodesic lines on the surface, and for exhibiting them in relation to the right lines of the surface; but I have not yet worked this out. In conclusion, it may be remarked that a surface in general cannot be divided into squares by its curves of curvature, but that it may be in an infinity of ways divided into squares by two systems of curves on the surface, and any such system of curves gives rise to a Mercator's projection of the surface.