

Determinant having a line $a + b =$ determinant with line a unconnected with line b .

It follows that, if any line of a determinant is the sum of the other lines, each multiplied by an arbitrary coefficient, or, similarly, the same thing if we can with any of the lines, each multiplied by an arbitrary coefficient compose a line 0, then the determinant is = 0.

The same principle leads to a determinantal corollary to the above to that of the same order n , viz. it is found that the product is a determinant of the same order n , each term thereof being a sum of products of all terms of a line of one of the factors into the corresponding line of the other factor. Starting with the expression of the product, we can easily deduce a series of determinants each of which

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ON A DIFFERENTIAL FORMULA CONNECTED WITH THE THEORY OF CONFOCAL CONICS.

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THE following transformations present themselves in connexion with the theory of confocal conics.

The coordinates x, y of a point are considered as functions of the parameters h, k where

$$\frac{x^2}{a+h} + \frac{y^2}{b+h} = 1,$$

$$\frac{x^2}{a+k} + \frac{y^2}{b+k} = 1;$$

and then assuming $\xi = x + iy, \eta = x - iy$ ($i = \sqrt{(-1)}$ as usual), and writing $c = a - b$, we find

$$h = \frac{1}{2}(-a - b + \xi\eta) + \frac{1}{2}\sqrt{\{(\xi^2 - c)(\eta^2 - c)\}},$$

$$k = \frac{1}{2}(-a - b + \xi\eta) - \frac{1}{2}\sqrt{\{(\xi^2 - c)(\eta^2 - c)\}},$$

whence if

$$H = (a+h)(b+h), \quad K = (a+k)(b+k),$$

we have

$$H = \frac{1}{4}\{\xi\sqrt{(\eta^2 - c)} + \eta\sqrt{(\xi^2 - c)}\}^2,$$

$$K = \frac{1}{4}\{\xi\sqrt{(\eta^2 - c)} - \eta\sqrt{(\xi^2 - c)}\}^2,$$

or, say

$$\sqrt{(H)} = \frac{1}{2}\{\xi\sqrt{(\eta^2 - c)} + \eta\sqrt{(\xi^2 - c)}\},$$

$$\sqrt{(K)} = \frac{1}{2}\{\xi\sqrt{(\eta^2 - c)} - \eta\sqrt{(\xi^2 - c)}\},$$

and thence

$$\begin{aligned} h + \frac{1}{2}(a+b) + \sqrt{(H)} &= \frac{1}{2}\{\xi + \sqrt{(\xi^2 - c)}\} \{\eta + \sqrt{(\eta^2 - c)}\}, \\ k + \frac{1}{2}(a+b) + \sqrt{(K)} &= \frac{1}{2}\{\xi + \sqrt{(\xi^2 - c)}\} \{\eta - \sqrt{(\eta^2 - c)}\} \\ &= \frac{1}{2}c \frac{\xi + \sqrt{(\xi^2 - c)}}{\eta + \sqrt{(\eta^2 - c)}}. \end{aligned}$$

These also follow from the known differential formula

$$4(dx^2 + dy^2) = (h-k)\left(\frac{dh^2}{H} - \frac{dk^2}{K}\right),$$

that is,

$$\frac{4d\xi d\eta}{\sqrt{(\xi^2 - c)} \sqrt{(\eta^2 - c)}} = \frac{dh^2}{H} - \frac{dk^2}{K},$$

implying

$$\frac{2\alpha d\xi}{\sqrt{(\xi^2 - c)}} = \frac{dh}{\sqrt{(H)}} + \frac{dk}{\sqrt{(K)}},$$

$$\frac{2d\eta}{\alpha \sqrt{(\eta^2 - c)}} = \frac{dh}{\sqrt{(H)}} - \frac{dk}{\sqrt{(K)}},$$

where α is a constant. The foregoing integral formulæ give at once

$$\frac{dh}{\sqrt{(H)}} = \frac{d\xi}{\sqrt{(\xi^2 - c)}} + \frac{d\eta}{\sqrt{(\eta^2 - c)}},$$

$$\frac{dk}{\sqrt{(K)}} = \frac{d\xi}{\sqrt{(\xi^2 - c)}} - \frac{d\eta}{\sqrt{(\eta^2 - c)}},$$

and substituting these values we find $\alpha=1$, and the differential formulæ are then satisfied.

We thence have

$$\text{const.} = \sqrt{\{(a+h)(b+h)\}} \pm \sqrt{\{(a+k)(b+k)\}},$$

as the integral of the differential equation

$$\frac{dh}{\sqrt{(H)}} \pm \frac{dk}{\sqrt{(K)}} = 0.$$