551.

TWO SMITH'S PRIZE DISSERTATIONS.

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WRITE dissertations on the following subjects:

1. The theory of interpolation, with a determination of the limits of error in the value of a function obtained by interpolation.

2. Determinants.

1. The general problem is to find y a function of x having given values for given values of x. The problem thus stated is of course indeterminate; in practice, we assume a certain form for the function y, the coefficients of which form are determined by the given conditions, viz. either y is known to be of the form in question, the actual value being then determined as above, or it is assumed that y is approximately equal to a function of the form in question, and the value is then approximately determined in such wise that, for the given values of x, the function y shall have its given values.

The ordinary case is when we have the values of y corresponding to n given values of x, and y is taken to be a function of the form $A + Bx + \dots + Kx^{n-1}$.

Suppose to fix the ideas $n = 4$, and that y_1, y_2, y_3, y_4 are the values of y corresponding to the values a, b, c, d of x. We may at once write down the expression

$$\begin{aligned} y = {} & \frac{(x-b)(x-c)(x-d)}{(a-b)(a-c)(a-d)} y_1 \\ & + \frac{(x-c)(x-d)(x-a)}{(b-c)(b-d)(b-a)} y_2 \\ & + \frac{(x-d)(x-a)(x-b)}{(c-d)(c-a)(c-b)} y_3 \\ & + \frac{(x-a)(x-b)(x-c)}{(d-a)(d-b)(d-c)} y_4, \end{aligned}$$

for clearly this is of the form in question $A+Bx+Cx^2+Dx^3$, and y becomes $=y_1$ for $x=a$; $=y_2$ for $x=b$, &c. And the like for any value of n. This is known as Lagrange's interpolation formula.

The given values of x may be equidistant, say they are $0, 1, 2, \ldots, n-1$, and the corresponding values of y are $y_0, y_1, \ldots, y_{n-1}$; then writing down the expression

$$y_x = y_0 + \frac{x}{1}\Delta y_0 + \frac{x\,.\,\overline{x-1}}{1\,.\,2}\Delta^2 y_0 + \ldots + \frac{x\,.\,\overline{x-1}\ldots\overline{x-n+2}}{1\,.\,2\ldots n-1}\Delta^{n-1}y_0,$$

where, as usual,

$$\Delta y_0 = y_1 - y_0, \quad \Delta^2 y_0 = y_2 - 2y_1 + y_0, \text{ \&c.};$$

then for $x=0, 1, 2$, &c. the values of y are

$$\begin{aligned} &y_0, \\ &y_0 + \Delta y_0, \quad = y_1, \\ &y_0 + 2\Delta y_0 + \Delta^2 y_0, \quad = y_2, \\ &\text{\&c.}, \end{aligned}$$

or the required conditions are satisfied.

As regards the determination of the limits of error, taking a particular case $n=4$, suppose that we have the values y_0, y_1, y_2, y_3 of y corresponding to the values 0, 1, 2, 3 of x, and that the true value of y is known to be

$$= A + Bx + Cx^2 + Dx^3 + Kx^4,$$

where K is a function of x, which for any value of x within the given values (i.e. from $x=0$ to $x=3$) is known to be at least $=P$ and at most $=Q$, i.e., $K > P < Q$, where to fix the ideas P and Q are each positive, Q being the greater. Here calculating the interpolation value of $y - Kx^4$ (the last term Kx^4 by Lagrange's formula), we have

$$\begin{aligned} y = y_0 + \frac{x}{1}\Delta y_0 + \frac{x\,.\,\overline{x-1}}{1\,.\,2}\Delta^2 y_0 + \frac{x\,.\,\overline{x-1}\,\overline{x-2}}{1\,.\,2\,.\,3}\Delta^3 y_0 \\ + Kx^4 \qquad\qquad\qquad\qquad \\ - \tfrac{1}{2}K_1 x(x-2)(x-3) \\ + 8\,K_2 x(x-1)(x-3) \\ - \tfrac{27}{2}K_3 x(x-1)(x-2), \end{aligned}$$

viz. this is the true value of y. Hence using the approximate formula as given by the first line, the last four lines give the error, viz. this is

$$= Kx^4 + K_3\tfrac{81}{2}x^2 + K_2(8x^3+24x) + K_1\tfrac{5}{2}x^2 - K_3(\tfrac{27}{2}x^3+27x) + K_2(32x^2) - K_1(\tfrac{1}{2}x^3+3x).$$

But K_1, K_2, K_3 being each $>P$ and $<Q$, this is

$$\begin{aligned} > P\,(x^4 + \;\; 8x^3 + 43x^2 + 24x) \\ - Q\,(\quad 14x^3 + 32x^2 + 30x), \end{aligned}$$

and it is

$$
\begin{aligned}
&< Q\,(x^4 + 8\,x^3 + 43x^2 + 24x) \\
&- P(\qquad 14x^3 + 33x^2 + 30x),
\end{aligned}
$$

the difference of these limits being

$$= (Q - P)\,(x^4 + 22x^3 + 75x^2 + 54x).$$

2. A determinant is a function of n^2 letters; viz. arranging these in the form of a square, the determinant

$$\begin{vmatrix} a_1, & b_1, & c_1 \\ a_2, & b_2, & c_2 \\ a_3, & b_3, & c_3 \end{vmatrix}$$

is a function linear in regard to each of the n^2 letters, and such that interchanging any two entire lines, or any two entire columns, the sign of the determinant is reversed, its absolute value being unaltered.

The above definition leads to a rule for calculating the actual value of the determinant, which rule may be taken as a definition, viz. the determinant is the sum of $1 \,.\, 2 \,.\, 3 \ldots n$ terms obtained as follows: starting from the term

$$+ a_1 b_2 c_3 \ldots,$$

we permute in every possible way the suffixes 1, 2, 3, ..., and give to the term a sign, $\pm$, which is that compounded of as many $-$ signs as there are cases in which an inferior number succeeds a superior number. Or, what is the same thing, any arrangement may be obtained by a succession of interchanges of two letters; and then taking for each interchange the sign $-$, we obtain the sign $\pm$ of the term in question. The positive and the negative terms are each of them $\frac{1}{2}(1 \,.\, 2 \,.\, 3 \ldots n)$ in number.

To show the connexion of the two definitions, it is sufficient to observe that in the second definition, attending for instance only to the first and second columns, to any terms Ma_1b_2, Na_1b_3, &c., there always correspond other terms $-Ma_2b_1$, $-Na_3b_1$, &c., so that taking the pairs together, these are $M(a_1b_2 - a_2b_1)$, $N(a_1b_3 - a_3b_1)$, &c., terms which change their sign, but remain unaltered as to their absolute values by the interchange of the first and second columns.

Among the fundamental properties of determinants are as follows:

The properties are the same as regards lines and columns.

A determinant vanishes if any line vanishes (that is, if each term of the line is $= 0$).

A determinant vanishes if two lines are identical.

A determinant is a linear function of its lines.

Whence—

Determinant having a line sA is $= s$ times the determinant having the line A (sA is here used to denote the line each term of which is s times the corresponding term of the line A).

Determinant having a line $A + A'$ = determinant with line A + determinant with line A'.

It follows that, if any line of a determinant is the sum of the other lines, each multiplied by an arbitrary coefficient, or, what is the same thing, if we can with any of the lines, each multiplied by an arbitrary coefficient, compose a line 0, then the determinant is $= 0$.

The same principle leads to a theorem for the product of two determinants of the same order n, viz. it is found that the product is a determinant of the same order n, each term thereof being a sum of the products of the terms of a line of one of the factors into the corresponding terms of a line of the other factor. Starting with this expression of the product, we decompose it into a series of determinants each of which is either $= 0$, or it is a product of a single term of the one factor into the other factor, and the sum of all these products is equal to the product of the two factors.

If we have n quantities $x, y, \ldots$ connected by as many linear equations

$$a_1x + b_1y + c_1z + \ldots = 0,$$

then the determinant

$$\begin{vmatrix} a_1, & b_1, & c_1, \ldots \\ a_2, & b_2, & c_2, \ldots \\ a_3, & b_3, & c_3, \ldots \\ \vdots & & \end{vmatrix} \text{ is } = 0\,;$$

and so, if we have n linear equations

$$a_1x + b_1y + c_1z + \ldots = u,$$

then each of the quantities $x, y, z, \ldots$ is given as the quotient of two determinants, the denominator being in each case

$$\begin{vmatrix} a_1, & b_1, & c_1, \ldots \\ a_2, & b_2, & c_2, \ldots \\ a_3, & b_3, & c_3, \ldots \\ \vdots & & \end{vmatrix},$$

and the numerators being (save as to their signs) that for x

$$\begin{vmatrix} u_1, & b_1, & c_1, \ldots \\ u_2, & b_2, & c_2, \ldots \\ u_3, & b_3, & c_3, \ldots \\ \vdots & & \end{vmatrix},$$

and the like for $y, z, \ldots$.

A determinant remains unaltered when the lines and columns are interchanged, the dexter diagonal (\\) remaining unaltered.

A determinant

$$\begin{vmatrix} a_1, & b_1, & c_1, & d_1, \ldots \\ a_2, & b_2, & c_2, & d_2, \ldots \\ a_3, & b_3, & c_3, & d_3, \ldots \\ a_4, & b_4, & c_4, & d_4, \ldots \\ \vdots & & & \end{vmatrix},$$

is a sum of products of complementary determinants

$$\Sigma \pm \begin{vmatrix} a_1, & b_1 \\ a_2, & b_2 \end{vmatrix} \begin{vmatrix} c_3, & d_3, \ldots \\ c_4, & d_4, \ldots \\ \vdots & \end{vmatrix},$$

or, say of products of complementary minors.

In particular, it is a sum

$$\Sigma \pm a_1 \begin{vmatrix} b_2, & c_2, & d_2, \ldots \\ b_3, & c_3, & d_3, \ldots \\ b_4, & c_4, & d_4, \ldots \\ \vdots & & \end{vmatrix},$$

of products of first minors into single terms or $(n-1)^{\text{th}}$ minors.

This last theorem affords a convenient rule for the development of a determinant.