548.

ON LISTING'S THEOREM.

[From the *Messenger of Mathematics*, vol. II. (1873), pp. 81—89.]

LISTING'S theorem, (established in his Memoir*, *Die Census räumlicher Gestalten*), is a generalisation of Euler's theorem $S+F=E+2$, which connects the number of summits, faces, and edges in a polyhedron; viz. in Listing's theorem we have for a figure of any sort whatever

$$A-B+C-D-(p-1)=0,$$

or, what is the same thing,

$$A+C=B+D+(p-1),$$

where

$$A=a,$$
$$B=b-\kappa',$$
$$C=c-\kappa''+\pi,$$
$$D=d-\kappa''',$$

in which theorem a relates to the points; b, κ' relate to the lines; c, κ'', π to the surfaces; d, κ''' to the spaces; and p relates to the detached parts of the figure, as will be explained.

a is the number of points; there is no question of multiplicity, but a point is always a single point. A point is either detached or situate on a line or surface.

b is the number of lines (straight or curved). A line is always finite, and if not reentrant there must be at each extremity a point: no attention is paid to cusps, inflexions, &c., and if the line cut itself there must be at each intersection a point;

* *Gött. Abh.* t. x. (1862).

and in general a point placed on a line constitutes a termination or boundary of the line. Thus a line is either an oval (that is, a non-intersecting closed curve of any form whatever), a punctate oval (oval with a single point upon it), or a biterminal (line terminated by two distinct points). For instance, a figure of eight is taken to be two punctate ovals; an oval, placing upon it two points, is thereby changed into two biterminals.

κ'. The definition, analogous to the subsequent definitions of κ'' and κ''', would be that κ' is the sum, for all the lines, of the number of circuits for each line; but inasmuch as for an oval the number of circuits is $=1$, and for any other line (punctate oval, or biterminal) it is $=0$, κ' is in fact the number of ovals.

c is the number of surfaces. A surface is always finite, and if not reentrant there must be at every termination thereof a line: no attention is paid to cuspidal lines, &c., and if the surface cut itself there must be at each intersection a point or a line; and in general a point or a line placed on a surface constitutes a termination or boundary thereof. It may be added that if a line intersects a surface there must be at the intersection a point, constituting a termination or boundary as well of the line as of the surface. Thus a surface is either an ovoid (simple closed surface, such as the sphere or the ellipsoid), a ring (surface such as the torus or anchor-ring), or other more complicated form of reentrant surface; or else it is a surface in part bounded by a point or points, line or lines. We may in particular consider a blocked surface having upon it one or more blocks: where by a block is meant a point, line, or connected superficial figure composed of points and lines in any manner whatever, the superficial area (if any) included within the block being disregarded as not belonging to the surface, or being, if we please, cut out from the surface. Thus an ovoid having upon it a point, and a segment or incomplete ovoid bounded by an oval, are each of them to be regarded as a one-blocked ovoid; the boundary being in the first case the point, and in the second case the oval; and so in general the blocked surface is bounded by the boundary or boundaries of the block or blocks. It will be understood from what precedes, and it is almost needless to mention, that for any surface we can pass along the surface from each point to each point thereof; any line which would prevent this would divide the surface into two or more distinct surfaces.

κ'' is the sum, for the several surfaces, of the number of circuits on each surface. The word circuit here signifies a path on the surface from any point to itself: all circuits which can by continuous variation be made to coincide being regarded as identical; and the evanescible circuit reducible to the point itself being throughout disregarded. Moreover, we count only the simple circuits, disregarding circuits which can be obtained by any repetition or combination of these. Thus for an ovoid, or for a one-blocked ovoid, there is only the evanescible circuit, that is, no circuit to be counted; but for a two-blocked ovoid there is besides one circuit, or we count this as one; and so for a n-blocked ovoid we count $n-1$ circuits. For a ring it is easy to see that (besides the evanescible circuit) there are, and we accordingly count, two circuits; and so in other cases.

π. It might be possible to find an analogous definition, but the most simple one is that π denotes the number of ovoids (unblocked ovoids) or other surfaces not bounded by any point or line.

d is the number of spaces, reckoning as one of them infinite space.

κ''' is the sum, for the several spaces, of the number of circuits in each space: the word circuit here signifying a path in the space from a point to itself; all circuits which can by continuous variation be made to coincide being considered as identical, and the evanescible circuit reducible to the point itself being throughout disregarded. Moreover, we count only the simple circuits, disregarding circuits which can be obtained by a repetition or combination of these. Thus for infinite space, or for the space within an ovoid, there is only the evanescible circuit, or there is no circuit to be counted; and the same is the case if within such space we have any number of ovoidal blocks (the term will, I think, be understood without explanation); but if within the space we have an oval, ring, or other ring-block of any kind whatever, then there is (besides the evanescible circuit) a circuit interlacing the ring-block, and we count one circuit; and so if there are n ring-blocks, either separate or interlacing each other in any manner, then there are, and we accordingly count, n circuits. So for the space inside a ring there is (besides the evanescible circuit), and we count, one circuit; and the case is the same if we have within the ring any number of ovoidal blocks whatever; but if there is within the ring an oval ring or other ring-block, then there is one new circuit, and we count in all (for the space in question) two circuits

p is the number of detached parts of the figure; or, say the number of detached aggregations of points, lines, and surfaces. Observe, that rings interlacing each other in any manner (but not intersecting) are considered as detached; so also two closed surfaces, one within the other, are considered as detached. The figure may be infinite space alone; we have then $p=0$.

The examples which follow will further illustrate the meaning of the terms and nature of the theorem; and will also indicate in what manner a general demonstration of the theorem might be arrived at.

1. Infinite space.

$$\begin{array}{llll}
a=0, & & & A=0, \\
b=0, & \kappa'=0, & & B=\ \ 0 \\
c=0, & \kappa''=0, & \pi=0, & C=0, \\
d=1, & \kappa'''=0, & & D=\ \ 1 \\
p=0, & & & p-1=-1 \\
\hline
& & & 0 \quad\ = 0.
\end{array}$$

2. Spherical surface.

$$\begin{array}{llll}
a=0, & & & A=0, \\
b=0, & \kappa'=0, & & B=0 \\
c=1, & \kappa''=0, & \pi=1, & C=2, \\
d=2, & \kappa'''=0, & & D=2 \\
p=1, & & & p-1=0 \\
\hline
 & & 2 & \quad =2:
\end{array}$$

viz. the effect is to increase C by 2 and D and $p-1$ each by 1.

3. Spherical surface, with point upon it.

$$\begin{array}{llll}
a=1, & & & A=1, \\
b=0, & \kappa'=0, & & B=0 \\
c=1, & \kappa''=0, & \pi=0, & C=1, \\
d=2, & \kappa'''=0, & & D=2 \\
p=1, & & & p-1=0 \\
\hline
 & & 2 & \quad =2:
\end{array}$$

viz. the effect is to increase a and diminish π each by 1; that is, A is increased and C diminished each by 1.

4. Spherical surface with two points.

$$\begin{array}{llll}
a=2, & & & A=2, \\
b=0, & \kappa=0, & & B=0 \\
c=1, & \kappa''=1, & \pi=0, & C=0, \\
d=2, & \kappa'''=0, & & D=2 \\
p=1, & & & p-1=0 \\
\hline
 & & 2 & \quad =2:
\end{array}$$

viz. the second point increases a and κ'' each by 1, that is, it increases A and diminishes C each by 1.

And for each new point on the spherical surface there is this same effect; so that we have, for the next case:

5. Spherical surface with n points ($n \geqq 2$).

$$\begin{array}{llll}
a=n, & & & A=\quad n, \\
b=0, & \kappa=0, & & B=0 \\
c=1, & \kappa''=n-1, & \pi=0, & C=2-n, \\
d=2, & \kappa'''=0, & & D=2 \\
p=1, & & & p-1=0 \\
\hline
 & & 2 & \quad =2.
\end{array}$$

Imagine that besides the n points there is an aperture (bounded by a closed curve); the case is:

6. Spherical surface with n points ($n \overline{>} 2$) and aperture.

$$\begin{array}{llll} a = n, & & & A = \quad n, \\ b = 1, & \kappa = 1 & , & B = 0 \\ c = 1, & \kappa'' = n & , \ \pi = 0, & C = 1 - n, \\ d = 1, & \kappa''' = 0 & , & D = 1 \\ p = 1, & & & p - 1 = 0 \\ \hline & & & 1 \quad = 1: \end{array}$$

viz. b and κ are each increased by 1, and therefore B is unaltered; κ'' is increased, and therefore C is diminished, by 1; but d is diminished, and therefore D also diminished, by 1.

7. Spherical surface with n points ($n \overline{>} 2$) and two apertures.

$$\begin{array}{llll} a = n, & & & A = \quad n, \\ b = 2, & \kappa = 2 & , & B = 0 \\ c = 1, & \kappa'' = n + 1, & \pi = 0, & C = \ - n, \\ d = 1, & \kappa''' = 1 & , & D = 0 \\ p = 1, & & & p - 1 = 0 \\ \hline & & & 0 \quad = 0: \end{array}$$

viz. b and κ are each increased by 1, and thus B is still unaltered; κ'' is increased, and therefore C diminished, by 1; κ''' is increased, and therefore D diminished, by 1, and each new aperture produces the like effect. Thus we have:

8. Spherical surface with m apertures ($m \overline{>} 2$).

$$\begin{array}{llll} a = 0, & & & A = 0, \\ b = m, & \kappa' = m & , & B = 0 \\ c = 1, & \kappa'' = m - 1, & \pi = 0, & C = 2 - m, \\ d = 1, & \kappa''' = m - 1, & & D = 2 - m \\ p = 1, & & & p - 1 = 0 \\ \hline & & & 2 - m \quad = 2 - m; \end{array}$$

where, comparing with case 5, we see the different effects of a point and an aperture.

9. Spherical surface with m apertures ($m \overline{>} 2$) and a point or points on each or any of the bounding curves of the aperture.

If on the bounding curve of any aperture we place a point, this increases a, and therefore A, by 1; the bounding curve is no longer a simple closed curve, and we thus also have κ diminished, and therefore B increased by 1; and the balance still holds.

Placing on the same bounding curve a second point, a, and therefore A, is increased by 1; but the bounding curve is converted into two distinct curves; that is, b, and therefore B, is increased by 1; and the balance still holds. And the like for each new point on the same bounding curve.

10. Spherical surface with n points connected in any manner by lines.

Reverting to the cases 4 and 5, by joining any two points by a line, we increase b, and therefore B, by 1; but as regards κ'' the two united points take effect as a single point; that is, κ'' is diminished, and therefore C increased, by 1; the balance is therefore undisturbed.

The case is the same for each new line, if only we do not thereby produce on the surface a closed polygon, or partition an existing closed polygon; in each of these cases we still increase b, and therefore B, by 1; and instead of diminishing κ'', we increase c, by 1, and therefore still increase C by 1; and the balance continues to subsist.

By continuing to join the several points we at last arrive at a spherical surface partitioned into polygons in any manner whatever; or, what is the same thing, we have:

11. Closed polyhedral surface. Here, if S is the number of summits, F the number of faces, E the number of edges; then

$$\begin{array}{llll}
a=S, & & A=S, & \\
b=E, & \kappa\ =0, & & B=E \\
c=F, & \kappa''=0,\ \ \pi=0, & C= & F, \\
d=2, & \kappa'''=0, & & D=\quad 2 \\
p=1, & & & p-1=\quad 0 \\
\hline
 & & S+F & =E+2,
\end{array}$$

so that we have Euler's theorem. Observe that this theorem (Euler's) does not apply to annular polyhedral surfaces, or to polyhedral shells. For instance, consider a shell, the exterior and interior surfaces of which are each of them a closed polyhedral surface; $S=S'+S''$, $F=F'+F''$, $E=E'+E''$, where $S'+F'=E'+2$, $S''+F''=E''+2$, and therefore $S+F=E+4$. Listing's theorem, of course, applies, viz. we have

$$\begin{array}{llll}
12.\quad a=S'+S'', & A=S'+S'', & & \\
b=E'+E'', & B= & & E'+E'' \\
c=F'+F'', & C=F'+F'', & & \\
d=3, & D= & & 3 \\
p=2, & p-1= & & 1 \\
\hline
 & S+F & & =E'+E''+4.
\end{array}$$

As another group of examples, consider a plane rectangle, for instance, a sheet of paper bounded by its four edges; here

$$\begin{array}{llllll}
13. & a=4, & & & A=4, & \\
& b=4, & \kappa'=0, & & & B=4 \\
& c=1, & \kappa''=0, & \pi=0, & C=1, & \\
& d=1, & \kappa'''=0, & & & D=1 \\
& p=1, & & & & p-1=0 \\
\hline
& & & & 5 & =5.
\end{array}$$

Let the paper be formed into a tube by uniting two opposite sides, the suture not being obliterated, but continuing as a line drawn lengthwise from one extremity of the tube to the other: here

$$\begin{array}{llllll}
14. & a=2, & & & A=2, & \\
& b=3, & \kappa'=0, & & & B=3 \\
& c=1, & \kappa''=0, & \pi=0, & C=1, & \\
& d=1, & \kappa'''=1, & & & D=0 \\
& p=1, & & & & p-1=0 \\
\hline
& & & & 3 & =3.
\end{array}$$

Let the suture be obliterated, so that we have simply a tube open at each end; here

$$\begin{array}{llllll}
15. & a=0, & & & A=0, & \\
& b=2, & \kappa'=2, & & & B=0 \\
& c=1, & \kappa''=1, & \pi=0, & C=0, & \\
& d=1, & \kappa'''=1, & & & D=0 \\
& p=1, & & & & p-1=0 \\
\hline
& & & & 0 & =0.
\end{array}$$

Let the tube be formed into an annulus by bending it round and joining the two extremities, the suture not being obliterated, but continuing as a closed curve round the tube; here

$$\begin{array}{llllll}
16. & a=0, & & & A=0, & \\
& b=1, & \kappa'=1, & & & B=0 \\
& c=1, & \kappa''=1, & \pi=0, & C=0, & \\
& d=2, & \kappa'''=2, & & & D=0 \\
& p=1, & & & & p-1=0 \\
\hline
& & & & 0 & =0.
\end{array}$$

Let the suture be obliterated, so that we have simply a tubular annulus; here

17.
$$\begin{array}{lllll} a=0, & & & A=0, & \\ b=0, & \kappa'=0, & & & B=0 \\ c=1, & \kappa''=2, & \pi=1, & C=0, & \\ d=2, & \kappa'''=2, & & & D=0 \\ p=1, & & & & p-1=0 \\ \hline & & & 0 & =0. \end{array}$$

We may compare herewith the case of a simple annulus or closed curve.

18.
$$\begin{array}{lllll} a=0, & & & A=0, & \\ b=1, & \kappa'=1, & & & B=0 \\ c=0, & \kappa''=0, & \pi=0, & C=0, & \\ d=1, & \kappa'''=1, & & & D=0 \\ p=1, & & & & p-1=0 \\ \hline & & & 0 & =0. \end{array}$$

Add to such an annulus, for instance, three radii meeting in the centre; then

19.
$$\begin{array}{lllll} a=4, & & & A=4, & \\ b=6, & \kappa'=0, & & & B=\ \ 6 \\ c=0, & \kappa''=0, & \pi=0, & C=0, & \\ d=1, & \kappa'''=3, & & & D=-2 \\ p=1, & & & & p-1=\ \ 0 \\ \hline & & & 4 & =\ \ 4. \end{array}$$

Let the last-mentioned figure become tubular, all sutures being obliterated; then

20.
$$\begin{array}{lllll} a=0, & & & A=\ \ 0, & \\ b=0, & \kappa'=0, & & & B=\ \ 0 \\ c=1, & \kappa''=6, & \pi=1, & C=-4, & \\ d=2, & \kappa'''=6, & & & D=-4 \\ p=1, & & & & p-1=\ \ 0 \\ \hline & & & -4 & =-4. \end{array}$$

And so if instead of the tubular figure, annulus with three radii, we had a tubular figure, annulus with diameter, then

21.
$$\begin{array}{lllll} a=0, & & & A=\ \ 0, & \\ b=0, & \kappa'=0, & & & B=\ \ 0 \\ c=1, & \kappa''=4, & \pi=1, & C=-2, & \\ d=2, & \kappa'''=4, & & & D=-2 \\ p=1, & & & & p-1=\ \ 0 \\ \hline & & & -2 & =-2; \end{array}$$

and the like in other cases.