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THEOREMS IN RELATION TO CERTAIN SIGN-SYMBOLS.

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I FIND the following among my papers:

Let the latin letters $a, b, \ldots$ denote *lines* of n signs $\pm$, and the greek letters $\alpha, \beta, \ldots$ *columns* of the same number n of signs $\pm$; two symbols of the same kind are multiplied together by multiplying their corresponding terms, the product being thus a symbol of the same kind; in particular, the product of a symbol by itself, or square of a symbol, is a line (or column as the case may be) of +'s: and the symbol itself is thus a square root of a line (or column) of +'s. Thus n being $= 5$, we say that the latin letters denote roots of + + + + + and the greek letters roots of

$$\begin{array}{c} + \\ + \\ + \\ + \\ +. \end{array}$$

The roots a, b, c, d, e will be *independent* if no one of them is equal to the product of all or any of the others; and, this being so, the 32 roots are the terms of

$$(1+a)(1+b)(1+c)(1+d)(1+e);$$

it follows that, for any other system of independent roots a', b', c', d', e', we have

$$(1+a')(1+b')(1+c')(1+d')(1+e') = (1+a)(1+b)(1+c)(1+d)(1+e):$$

and conversely if either system be independent and this equation is satisfied, then the other system is also independent.

In particular a, b, c, d, e being independent, then a, b, c, bd, e (viz. any term d is replaced by its product by some other term b) is also independent; and by a similar transformation on the new series a, b, c, bd, e, and so on in succession we can pass from a given independent system a, b, c, d, e to *any other independent system whatever*.

A similar but more general theorem is the following: let a, b, c, d, e be independent, and l be equal to the product of all or any of these roots, but so that as regards, suppose (b, c, e), the number of these factors contained in l is even (or may be $=0$), e.g. l is $=a$, or abc, &c., but it is not $=ab$, or bce, &c.

Then a, lb, lc, d, le is an independent system: to show this we must show that

$$\overline{1+a}\,\overline{1+b}\,\overline{1+c}\,\overline{1+d}\,\overline{1+e}=\overline{1+a}\,\overline{1+lb}\,\overline{1+lc}\,\overline{1+d}\,\overline{1+le},$$

that is,

$$\overline{1+a}\,\overline{1+d}\,(\overline{1+lb}\,\overline{1+lc}\,\overline{1+le}-\overline{1+b}\,\overline{1+c}\,\overline{1+e})=0,$$

that is,

$$\overline{1+a}\,\overline{1+d}\,(\overline{l-1}\,\overline{b+c+e}+\overline{l^2-1}\,\overline{bc+be+ce}+\overline{l^3-1}\,bce)=0,$$

or, since $l^2=1$, $l^3=l$, this is

$$(1+a)(1+d)(l-1)(b+c+e+bce)=0,$$

which is easily verified under the assumed conditions as to l, e.g. $l=abc$,

$$(1+a)\,l=abc+a^2bc=abc+bc=(1+a)\,bc,$$

$$(1+a)(l-1)=(1+a)(bc-1),$$

and the equation is

$$(1+a)(1+d)(bc-1)(b+c+e+bce)=0;$$

and we in fact have

$$bc\,(b+c+e+bce)=c+b+ebc+e;$$

that is,

$$(bc-1)(b+c+e+bce)=0.$$

The proof is obviously quite general.

All that precedes applies also to the columns.

Now consider a square of 5×5 signs $\pm$; I say that, if this is *independent as to its lines*, it will be also *independent as to its columns*.

To prove this consider any particular square, say

	α	β	γ	δ	ϵ
a					
b	+	−	−	+	−
c					
d					
e					

independent as to its lines, and also independent as to its columns: I derive from this the square

	α	$\theta\beta$	$\theta\gamma$	δ	$\theta\epsilon$
a					
b					
c					
bd					
e					

viz. in the new square the line d is replaced by bd, the designation of the columns will be presently explained. This new square is, by what precedes, independent as to its lines; we have to show that it is also independent as to its columns.

As regards the columns, any column is either unchanged or it is changed in its fourth place only, according as the sign in b is for that column $+$ or $-$; that is, if we write $\theta = +$, the columns of the new square are (as above written down) α, $\theta\beta$, $\theta\gamma$, δ, $\theta\epsilon$;

$$\begin{array}{c} + \\ + \\ - \\ + \end{array}$$

and θ is a product of all or some of the original columns α, β, γ, δ, ϵ: but as regards β, γ, ϵ it contains an even number (or it may be 0) of these factors; for otherwise the sign in the second line of θ instead of being $+$ would be $-$. But these are the very conditions that show that the columns α, $\theta\beta$, $\theta\gamma$, δ, $\theta\epsilon$ are independent.

Hence starting from the square

−	+	+	+	+
+	−	+	+	+
+	+	−	+	+
+	+	+	−	+
+	+	+	+	−

which obviously is independent as to its lines and also as to its columns; and transforming as above any number of times in succession, we obtain ultimately a square which has for its lines *any system* whatever of independent roots, and by what precedes each of the new squares is also independent as to its columns; that is, every square independent as to its lines is also independent as to its columns. Q.E.D.