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ON A PENULTIMATE QUARTIC CURVE.

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I HAVE had occasion to consider with some particularity the form of a curve about to degenerate into a system of multiple curves; a simple instance is a trinodal quartic curve about to degenerate into the form $x^2y^2=0$, or say a "penultimate" of $x^2y^2=0$. To fix the ideas, take x, y, z to denote the perpendiculars on the sides of an equilateral triangle, altitude $=1$ (so that $x+y+z=1$), and let the curve be symmetrical in regard to the coordinates x, y, its equation being thus

$$(a, a, 1, f, f, h)(x, y, z)^2=0,$$

where a, f, h are ultimately all indefinitely small in regard to unity: to diminish the number of cases I further assume

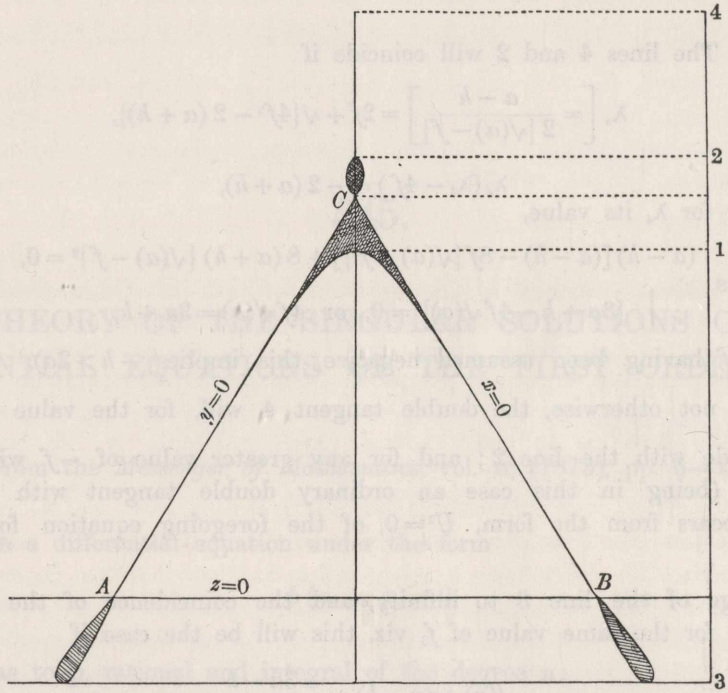
$$a=+,\ \ f \text{ and } h=-,$$
$$h^2>a^2,\ \text{ that is, } a+h=-,$$
$$f>a,\ \ ,,\ \ ,,\ \ \sqrt{(a)}+f=-,$$

but I do not in the first instance take a, f, h to be indefinitely small. Then if $-f$ is not too large, the curve is as shown in the figure[1], viz. it is a triloop curve, with two horizontal double tangents, 3 touching the curve in two real points, 4 touching it in two imaginary points. Imagine $-f$ increased: the new curve will have the same general form, intersecting the first curve at A and B but touching it at C, viz. it will pass inside the loop C but outside the loops A, B; and outside the remainder of the curve; and the 4 will also move downwards as shown. The new position of 4 will be below the first position.

Supposing that a, h have given values, and that $-f$ continually is increased in regard to $\sqrt{(a)}$; two things may happen. First, the double tangent 3 may move down

[1] The figure is drawn with very small values of a, f, h, in order to exhibit as nearly as may be one of the penultimate forms of the curve; but this is not in anywise assumed in the reasoning of the text. Observe in the figure that the points A, B are ordinary double points, and that there are at each of them two distinct tangents inclined at a small angle to each other.

to $z=-\infty$, the lower loops lengthening out and finally becoming each of them a pair of parabolic branches parallel at infinity; and then reappearing at $z=+\infty$, again move downwards, each loop becoming in this case a pair of hyperbolic branches touching two asymptotes at $z=-\infty$, and then again on the opposite sides thereof at $z=+\infty$,



and coming down as a single branch to touch the double tangent 3 which is now above 4. Secondly, the double tangent 4 may come to coincide with the horizontal tangent 2, at the instant of coincidence being a tangent of four-pointic contact; and becoming afterwards (being as before above 2) an ordinary double tangent with two real points of contact; viz. instead of a simple loop at C we have a heart-shaped loop.

But to investigate whether the two cases actually happen, and in what order of succession, we require the expressions of z for the several lines in question; we find, without difficulty,

$$\text{for line 1,}\quad z_1=\frac{1}{1+2\lambda_1},\quad \text{where}\quad \lambda_1=-2f+\sqrt{\{4f^2-2(a+h)\}},$$

$$\text{,, 2,}\quad z_2=\frac{1}{1-2\lambda_2},\quad \text{,,}\quad \lambda_2=\quad 2f+\sqrt{\{4f^2-2(a+h)\}},$$

$$\text{,, 3,}\quad z_3=\frac{1}{1-2\lambda_3},\quad \text{,,}\quad \lambda_3=\frac{a-h}{2\{-\sqrt{(a)}-f\}},$$

$$\text{,, 4,}\quad z_4=\frac{1}{1-2\lambda_4},\quad \text{,,}\quad \lambda_4=\frac{a-h}{2\{\sqrt{(a)}-f\}},$$

where λ_1, λ_2, λ_3, λ_4 are all positive. Observe that in the limiting case $-f=\sqrt{(a)}$, where, instead of the loops at A, B, we have cusps; z_1, z_2, and z_4 are (in general) positive; $\lambda_3 = \infty$, and therefore $z_4 = 0$; that is, the line 3 coincides with AB, ceasing to be a double tangent; there is in this case the one double tangent 4.

First. z_3 becomes infinite for $1-2\lambda_3=0$; that is, $a-h=-\sqrt{(a)}-f$, or $-f=\sqrt{(a)}+(a-h)$; viz. for $-f=\sqrt{(a)}+(a-h)-\epsilon$, we have $z_3=-\infty$, and for $-f=\sqrt{(a)}+(a-h)+\epsilon$, we have $z_3=+\infty$.

Secondly. The lines 4 and 2 will coincide if

$$\lambda_4 \left[= \frac{a-h}{2\{\sqrt{(a)}-f\}}\right] = 2f + \sqrt{\{4f^2 - 2(a+h)\}},$$

that is, if

$$\lambda_4(\lambda_4 - 4f) = -2(a+h),$$

or, substituting for λ_4 its value,

$$(a-h)[(a-h) - 8f\{\sqrt{(a)}-f\}] + 8(a+h)\{\sqrt{(a)}-f\}^2 = 0,$$

the condition is

$$\{3a+h-4f\sqrt{(a)}\}^2 = 0, \text{ or } 4f\sqrt{(a)} = 3a+h,$$

(observe that, f having been assumed negative, this implies $-h>3a$). That is, $3a+h$ being $=-$ but not otherwise, the double tangent 4 will, for the value $-f=\frac{-3a-h}{4\sqrt{(a)}}$, come to coincide with the line 2; and for any greater value of $-f$ will be as before above line 2, (being in this case an ordinary double tangent with real points of contact) as appears from the form, $U^2=0$, of the foregoing equation for the determination of f.

The passage of the line 3 to infinity, and the coincidence of the lines 4 and 2 may take place for the same value of f, viz. this will be the case if

$$\sqrt{(a)} + (a-h) = \frac{-3a-h}{4\sqrt{(a)}},$$

that is, if

$$7a + 4\sqrt{(a)} + h\{1-4\sqrt{(a)}\} = 0 \text{ or } -h = \frac{a\{7+4\sqrt{(a)}\}}{1-4\sqrt{(a)}},$$

or, a being small, for the value $-h=7a$ approximately. If $-h$ is less than the above value, then $\frac{-3a-h}{4\sqrt{(a)}}$ is less than $\sqrt{(a)}+a-h$, or $-f$ increasing from $\sqrt{(a)}$, the coincidence of the lines 2, 4 takes place before the line 3 goes off to infinity: contrarywise, if $-h$ is greater than the above value.

In any form of the curve (i.e. whatever be the value of f in regard to a, h), if we imagine a, h indefinitely diminished, the lines 1, 2 and 4 will continually approach C, and the curve will gather itself up into certain definite portions of the lines $x=0$, $y=0$. Thus any secant through A (not being indefinitely near to the line AC), which meets the curve in real points, will meet it in two points tending to coincide at the intersection of the secant with the line $x=0$; analytically there are always two intersections real or imaginary which (the secant not being indefinitely near the line AC) tend to coincide at the intersection of the secant with the line $x=0$; and we thus see how we ultimately arrive at the line $x=0$ twice repeated; and similarly for the line $y=0$.