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A "SMITH'S PRIZE" PAPER[1]; SOLUTIONS.

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1. *Mention what form of given relation $\phi(a, b, c, \ldots)=0$ between the roots of a given equation will in general serve for the rational determination of the roots; explain the case of failure; and state what information as to the roots is furnished by a given relation not of the form in question.*

In the given relation, $\phi(a, b, c, \ldots)$ must be a wholly unsymmetrical function of the roots; that is, a function altered by any permutation whatever of the roots; or, what is the same thing, by any interchange whatever of two roots.

For this being so, if $\alpha, \beta, \gamma, \ldots$ be the values of the roots, then for some one order, say $\alpha, \beta, \gamma, \ldots$, of these values the given relation $\phi(a, b, c, \ldots)=0$ will be satisfied by writing therein $a=\alpha$, $b=\beta$, $c=\gamma$, &c.; but it will in general be satisfied for this order only, and not for any other order whatever (viz. it will not be satisfied by writing $a=\beta$, $b=\alpha$, $c=\gamma$, &c., or by any other such system). The given equation determines that the roots are equal to $\alpha, \beta, \gamma, \ldots$ in some order or other, but the given equation combined with the given relation $\phi(a, b, c, \ldots)=0$, determines that a is $=\alpha$ and not equal to any other value, $b=\beta$ and not equal to any other value, &c.; and it thus appears *a priori*, that the two together must rationally determine each of the roots $a, b, c, \ldots$; the *a posteriori* verification, and actual rational determination of the values of $a, b, c, \ldots$ respectively, is a separate question which is not here considered.

The function $\phi(a, b, c, \ldots)$ may be of the proper form, and yet the particular values $\alpha, \beta, \gamma, \ldots$ be such that the given relation $\phi(a, b, c, \ldots)=0$ is satisfied, not only for the single arrangement $a=\alpha$, $b=\beta$, $c=\gamma$, &c., but for some other arrangement,

[1] Set by me for the Master of Trinity, Feb. 3, 1870.

$a=\delta$, $b=\gamma$, $c=\beta, \ldots$ or for more than one such other arrangement. (For instance, if the given relation be $a+2b+3c-32=0$, and the roots are 3, 5, 7; the relation is satisfied by $a=5$, $b=3$, $c=7$, and also by $a=3$, $b=7$, $c=5$.) Here the given equation and relation do not completely determine each root, they only determine that a is $=\alpha$ or $=\delta$ (or as the case may be $=$ some other one value); and similarly that b is $=\beta$ or $=\gamma$ (or as the case may be $=$ some other one value), and so for the other roots $c, d, \ldots$; and it thus appears *a priori*, that in such a case each root is determined, not rationally, but by means of an equation, the order of which is equal to the number of the values of such root; we have here the case of failure of the general theorem.

When the given relation $\phi(a, b, c, \ldots)=0$ is not of the required form; that is, when $\phi(a, b, c, \ldots)$ is a partially symmetrical function, there will be in general several arrangements of $\alpha, \beta, \gamma, \ldots$, such that equating $a, b, c, \ldots$ to $\alpha, \beta, \gamma, \ldots$ according to each of these arrangements, the given relation $\phi(a, b, c, \ldots)=0$ will be satisfied; and it follows that each of the roots $a, b, c, \ldots$ is determined not rationally, but by means of an equation of a certain order (not necessarily the same order for each of the roots). Thus, if the relation be symmetrical as regards a pair of roots a and b; then if it be satisfied, suppose by $a=\alpha$, $b=\beta$, $c=\gamma, \ldots$, it will also be satisfied by $a=\beta$, $b=\alpha$, $c=\gamma, \ldots$, but not in general in any other manner; each of the roots a, b has here either of the values α, β, and the two roots a, b in question will be given, not rationally, but by means of the same quadratic equation. And observe, moreover, that any other function $\psi(a, b, c, \ldots)$ of the same form as ϕ, that is, symmetrical in regard to the two roots a, b, will for the two arrangements $a=\alpha$, $b=\beta$, $c=\gamma \ldots$, and $a=\beta$, $b=\alpha$, $c=\gamma, \ldots$ acquire not two distinct values, but one and the same value, that is, the value of $\psi(a, b, c, \ldots)$ will be determined *rationally*; and so in general.

There is for the partially symmetrical function $\phi(a, b, c, \ldots)$ a case of failure similar to that which arises for the completely unsymmetrical function, viz. the particular values $\alpha, \beta, \gamma \ldots$ may be such as to give more ways of satisfying the given relation $\phi(a, b, c, \ldots)=0$, than there would be but for such particular values of $\alpha, \beta, \gamma, \ldots$; and there is then a corresponding elevation of the order of the equation for the determination of the roots $a, b, c, \ldots$ or some of them.

2. *If the roots* $(\alpha, \beta, \gamma, \delta)$ *of the equation*

$$(a, b, c, d, e)(u, 1)^4=0$$

are no two of them equal; and if there exist unequal magnitudes θ, ϕ *such that*

$$(\theta+\alpha)^4 : (\theta+\beta)^4 : (\theta+\gamma)^4 : (\theta+\delta)^4=(\phi+\alpha)^4 : (\phi+\beta)^4 : (\phi+\gamma)^4 : (\phi+\delta)^4;$$

show that the cubinvariant $ace-ad^2-b^2e-c^3+2bcd$ *is* $=0$; *and find the values of* θ, ϕ.

We have

$$\left(\frac{\theta+\alpha}{\phi+\alpha}\right)^4=\left(\frac{\theta+\beta}{\phi+\beta}\right)^4=\left(\frac{\theta+\gamma}{\phi+\gamma}\right)^4=\left(\frac{\theta+\delta}{\phi+\delta}\right)^4;$$

and we cannot have any two of the fourth roots, say $\dfrac{\theta+\alpha}{\phi+\alpha}$ and $\dfrac{\theta+\beta}{\phi+\beta}$ equal to each other; for this would imply $(\theta-\phi)(\alpha-\beta)=0$, that is, $\theta=\phi$, or else $\alpha=\beta$.

Hence assuming $\dfrac{\theta+\alpha}{\phi+\alpha}=\lambda$, we may write

$$\frac{\theta+\alpha}{\phi+\alpha}=\lambda,\quad \frac{\theta+\beta}{\phi+\beta}=-\lambda,\quad \frac{\theta+\gamma}{\phi+\gamma}=i\lambda,\quad \frac{\theta+\delta}{\phi+\delta}=-i\lambda,$$

$$\{i=\sqrt{(-1)}\text{ as usual}\},$$

viz. this is one of three systems of equations; the other two may be obtained therefrom by writing γ, δ, β and δ, β, γ successively in place of β, γ, δ. Hence assuming

$$v=\frac{\theta+u}{\phi+u},$$

the four values of u are $\alpha, \beta, \gamma, \delta$, and the corresponding four values of v are $\lambda, -\lambda, i\lambda, -i\lambda$; and v, u are linearly related to each other; the anharmonic ratio of $(\alpha, \beta, \gamma, \delta)$ is therefore equal to that of $(1, -1, i, -i)$, viz. we have

$$\frac{(\alpha-\gamma)(\beta-\delta)}{(\alpha-\delta)(\beta-\gamma)}=\frac{(1-i)(-1+i)}{(1+i)(-1-i)},\ =\frac{(1-i)^2}{(1+i)^2},\ =-1,$$

that is,

$$(\alpha-\gamma)(\beta-\delta)+(\alpha-\delta)(\beta-\gamma)=0,$$

or, what is the same thing,

$$2(\alpha\beta+\gamma\delta)-(\alpha+\beta)(\gamma+\delta)=0,$$

viz. we have this relation, or else one of the like relations

$$2(\alpha\gamma+\delta\beta)-(\alpha+\gamma)(\delta+\beta)=0,$$

$$2(\alpha\delta+\beta\gamma)-(\alpha+\delta)(\beta+\gamma)=0,$$

that is, the product of the three functions $2(\alpha\beta+\gamma\delta)-(\alpha+\beta)(\gamma+\delta)$ is $=0$.

But the product in question is (save as to a numerical factor) the cubinvariant J of the quartic function; or the equation in question is the required equation $J=0$. More simply, the linear transformation $v=\dfrac{\theta+u}{\phi+u}$, gives for v the equation $v^4-\lambda^4=0$; which is $(1, 0, 0, 0, -\lambda^4 \between v, 1)^4=0$; the cubinvariant hereof is $=0$, and therefore also the cubinvariant of the original function $(a, b, c, d, e \between u, 1)^4$.

Reverting to the equations

$$\frac{\theta+\alpha}{\phi+\alpha}=\lambda,\quad \frac{\theta+\beta}{\phi+\beta}=-\lambda,\quad \frac{\theta+\gamma}{\phi+\gamma}=i\lambda,\quad \frac{\theta+\delta}{\phi+\delta}=-i\lambda,$$

(which, as we have seen, give $2(\alpha\beta+\gamma\delta)=(\alpha+\beta)(\gamma+\delta)$), the same equations give

$$\frac{\theta+\alpha}{\phi+\alpha}+\frac{\theta+\beta}{\phi+\beta}=0;\quad \frac{\theta+\gamma}{\phi+\gamma}+\frac{\theta+\delta}{\phi+\delta}=0,$$

that is,

$$2\theta\phi + 2\alpha\beta - (\theta + \phi)(\alpha + \beta) = 0,$$
$$2\theta\phi + 2\gamma\delta - (\theta + \phi)(\gamma + \delta) = 0,$$

or, what is the same thing,

$$\begin{array}{rcccc} 2\theta\phi & : & 2 : \theta + \phi = & -\alpha\beta(\gamma + \delta) + \gamma\delta(\alpha + \beta) \\ & : & \gamma + \delta & - \alpha - \beta \\ & : & \gamma\delta & - \alpha\beta, \end{array}$$

viz. we have thus the values of $\theta\phi$, $\theta + \phi$ (and thence of θ, ϕ) corresponding to the relation $2(\alpha\beta + \gamma\delta) = (\alpha + \beta)(\gamma + \delta)$ of the roots. And by cyclically permuting β, γ, δ as before, we have the values of $\theta\phi$, $\theta + \phi$ corresponding to the other two forms respectively of the relation between the roots.

3. *If in a plane A, B, C, D are fixed points and P a variable point, find the linear relation*

$$\alpha . PAB + \beta . PBC + \gamma . PCD + \delta . PDA = 0$$

which connects the areas of the triangles PAB, &c.

Taking $(x, y, 1)$, $(x_1, y_1, 1)$, &c. for the coordinates of P, A, B, C, D respectively, we have

$$PAB = \begin{vmatrix} x, & y, & 1 \\ x_1, & y_1, & 1 \\ x_2, & x_2, & 1 \end{vmatrix}, = 012, \text{ suppose,}$$

$$PBC = 023, \text{ \&c.}$$

(where the values of the several determinants fix the signs of the several triangles). The identical equation then is

$$\alpha . 012 + \beta . 023 + \gamma . 034 + \delta . 041 = 0;$$

(that such an equation exists appears at once by the consideration that α, β, γ, δ can be determined so that the coefficients of x, y, and the constant term shall severally vanish); and in order actually to find the values we may make P coincide with the points A, B, C, D successively. We thus have

$$\beta . 123 + \gamma . 134 = 0,$$
$$\gamma . 234 + \delta . 241 = 0,$$
$$\delta . 341 + \alpha . 312 = 0,$$
$$\alpha . 412 + \beta . 423 = 0,$$

or, what is the same thing,

$$\beta . 123 + \gamma . 341 = 0,$$
$$\gamma . 234 + \delta . 412 = 0,$$
$$\delta . 341 + \alpha . 123 = 0,$$
$$\alpha . 412 + \beta . 234 = 0,$$

and these are at once seen to give

$$\alpha : \beta : \gamma : \delta = 234.341 : -341.412 : 412.123 : -123.341,$$

so that the required identical relation is

$$012.234.341 - 023.341.412 + 034.412.123 - 041.123.341 = 0,$$

in which 012, 023, 034, 041 stand for the triangles *PAB*, *PBC*, *PCD*, *PDA*, and 234, 341, 412, 123 for the triangles *BCD*, *CDA*, *DAB*, *ABC* respectively.

4. *Find at any point of a plane curve the angle between the normal and the line drawn from the point to the centre of the chord parallel and indefinitely near to the tangent at the point.*

Examine whether a like question applies to a point on a surface and the indicatrix section at such point.

Taking the origin at the point on the curve, the axis of x coinciding with the tangent and that of y with the normal; the equation of the curve taken to terms of the third order in x will by

$$y = bx^2 + cx^3,$$

and if, considering x as a small quantity of the first order, and therefore y as a small quantity of the second order, we regard y as given, and find the two values x_1, x_2, each of the order $\sqrt{(y)}$, which satisfy the equation, then, as will appear, $x_1 + x_2$ is a small quantity of the order x^2, and consequently $\frac{x_1 + x_2}{y}$ will have a finite value. And if ϕ be the required angle, then obviously $\tan\phi = \frac{\frac{1}{2}(x_1 + x_2)}{y}$.

We have as a first approximation $bx^2 = y$, or say $x = \frac{y^{\frac{1}{2}}}{b^{\frac{1}{2}}}$, whence to a second approximation $bx^2 = y - \frac{cy^{\frac{3}{2}}}{b^{\frac{3}{2}}}$, $x^2 = \frac{y}{b}\left(1 - \frac{cy^{\frac{1}{2}}}{b^{\frac{3}{2}}}\right)$, whence $x = \frac{y^{\frac{1}{2}}}{b^{\frac{1}{2}}}\left(1 - \frac{cy^{\frac{1}{2}}}{2b^{\frac{3}{2}}}\right)$, $= \frac{y^{\frac{1}{2}}}{b^{\frac{1}{2}}} - \frac{cy}{2b^2}$; say we have

$$x_1 = \frac{y^{\frac{1}{2}}}{b^{\frac{1}{2}}} - \frac{cy}{2b^2},$$

$$x_2 = -\frac{y^{\frac{1}{2}}}{b^{\frac{1}{2}}} - \frac{cy}{2b^2},$$

and thence

$$\tfrac{1}{2}(x_1 + x_2) = -\frac{cy}{2b^2};$$

whence

$$\tan\phi = -\frac{c}{2b^2},$$

which gives the value of the angle ϕ; it would be easy to express b, c in terms of the differential coefficients

$$d_x y,\ d_x{}^2 y,\ d_x{}^3 y.$$

It would at first sight appear that a like question might be asked as to a surface; viz. that it might be proposed to determine the angle between the normal and a line drawn from the point to the centre of the indicatrix conic. But this is not so; in fact, taking the origin at a point on the surface, the axes of x, y being in the tangent plane, and the axis of z coinciding with the normal: then to the third order we have

$$z = (A,\ B,\ C \between x,\ y)^2 + (a,\ b,\ c,\ d \between x,\ y)^3;$$

but here, regarding z as a given constant, if we take account of the terms of the third order, the section is not a conic but a cubic; and it has not in general any centre; and if (as in the ordinary theory) we neglect the terms of the third order, thus obtaining an indicatrix conic, the centre of this conic lies *on* the normal, and there is no angle corresponding to the angle ϕ of the plane problem.

The only case where there is such an angle is when the cubic terms $(a,\ b,\ c,\ d \between x,\ y)^3$ contain as a factor the quadric terms $(A,\ B,\ C \between x,\ y)^2$ (one relation between the coefficients A, B, C, a, b, c, d). For then we have

$$z = (A,\ B,\ C \between x,\ y)^2 (1 + 2lx + 2my), \text{ viz.}$$

$$z = (A,\ B,\ C \between x,\ y)^2 + 2(lxz + myz),$$

approximately to the third order; and then regarding z as a given constant, this last equation represents a conic having for the coordinates of its centre, say $x = \alpha z$, $y = \beta z$, and there is an angle $\phi = \tan^{-1}\sqrt{(\alpha^2 + \beta^2)}$; this is, in fact, what happens in the case of a quadric surface, for the section by a plane parallel and indefinitely near to the tangent plane is then a conic, the centre of which is not on the normal; and the angle ϕ (in the case of a central surface) is in fact the inclination of the normal to the radius from the centre.

I take the opportunity of adding a remark that the indicatrix is never a parabola, but in the separating case between the ellipse and the hyperbola it is a pair of parallel lines. The indicatrix, a parabola, is commonly obtained as follows: viz. taking the axes as before, but starting from an equation $U = 0$, the equation presents itself in the form

$$z = (A,\ B,\ C,\ F,\ G,\ H \between x,\ y,\ z)^2,$$

which, considering z as a given constant, represents a conic which, it is said, *may be a parabola*. But observe that z is of the order $(x,\ y)^2$, the terms $2Fyz + Gzx$, are consequently of the order $(x,\ y)^3$, but they are not all the terms of this order which would be obtained by the expansion of z as a function of $(x,\ y)$; there is consequently no meaning in retaining them, and they ought to be rejected; similarly the term in z^2 which is of the order $(x,\ y)^4$ ought to be rejected; the equation is thus reduced to

$$z = Ax^2 + 2Hxy + By^2,$$

which, when $AB - H^2 = 0$, represents not a parabola but a pair of parallel lines. On referring to Dupin's *Développements de Géométrie, &c.* (see p. 49) I find that he is quite accurate; his expression is, "elle peut cependant être une parabole; alors elle se présente sous la forme de deux droites parallèles *équidistantes de leur centre*": and he afterwards examines in particular "ce cas remarquable."

5. *Shew that a cubic surface has at most four conical points; and a quartic surface at most sixteen conical points.*

If a cubic surface has two conical points, then the line joining these has with the surface two intersections at each of the conical points, and therefore lies wholly in the surface. Hence, for a cubic surface with three conical points A, B, C, the lines AB, BC, CA lie wholly in the surface, and these three lines form the complete section of the surface by the plane ABC; it is clear that there cannot be in this plane a fourth conical point: but there may be, not in this plane, a fourth conical point D. Suppose that this is so, there cannot be a fifth conical point E; for if there were, the line DE would lie wholly in the surface, and would therefore meet the plane ABC at some point in the section of the surface by this plane; that is, at some point in one of the lines AB, AC, BC; say at a point in AB: but then the lines AB, DE would intersect, or the four conical points A, B, D, E would lie in a plane. Hence there cannot be any fifth conical point E.

For a quartic surface; suppose this has k conical points, and let any one of these be made the vertex of a cone circumscribing the surface; each generating line is a tangent of the surface; and considering any section by a plane through the vertex, and observing that from a double point of a quartic curve we may draw six tangents to the curve, it appears that the order of the cone is $=6$. It is easy to see that the lines from the vertex to the remaining $(k-1)$ conical points are each of them a double line of the cone, and that the cone has not any other double lines; the cone is therefore a cone of the order 6, with $(k-1)$ double lines. A proper cone of the order 6 has at most 10 double lines, but the cone need not be a proper one; it may, in fact, break up into 6 planes, and in this case the double lines are the 15 lines of intersections of the several pairs of planes. Hence $k-1$ is $=15$ at most: or k is $=16$ at most.

6. *Find the differential equation of the parallel surfaces of an ellipsoid.*

Let (x, y, z) be the coordinates of a point on the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$; (X, Y, Z) the coordinates of a point on the normal at a distance $=k$ from the first-mentioned point. We have

$$\frac{X-x}{\frac{x}{a^2}}=\frac{Y-y}{\frac{y}{b^2}}=\frac{Z-z}{\frac{z}{c^2}}, =\rho \text{ suppose};$$

that is,

$$X=x\left(1+\frac{\rho}{a^2}\right),\quad Y=y\left(1+\frac{\rho}{b^2}\right),\quad Z=z\left(1+\frac{\rho}{c^2}\right),$$

and thence

$$k^2=\rho^2\left(\frac{x^2}{a^4}+\frac{y^2}{b^4}+\frac{z^2}{c^4}\right).$$

Moreover

$$x=\frac{a^2X}{a^2+\rho},\quad y=\frac{b^2Y}{b^2+\rho},\quad z=\frac{c^2Z}{c^2+\rho},$$

substituting these values in the equation of the ellipsoid, we have

$$1 = \frac{a^2X^2}{(a^2+\rho)^2} + \frac{b^2Y^2}{(b^2+\rho)^2} + \frac{c^2Z^2}{(c^2+\rho)^2},$$

which determines ρ as a function of X, Y, Z. The tangent plane of the ellipsoid at the point (x, y, z) and of the parallel surface at the point (X, Y, Z), are parallel to each other (or what is the same thing, the parallel surface cuts at right angles the normal of the ellipsoid), we have therefore

$$\frac{x}{a^2}dX + \frac{y}{b^2}dY + \frac{z}{c^2}dZ = 0,$$

or substituting for x, y, z their values, this is

$$\frac{XdX}{a^2+\rho} + \frac{YdY}{b^2+\rho} + \frac{ZdZ}{c^2+\rho} = 0,$$

where ρ denotes as above a function of (X, Y, Z) given by the equation

$$1 = \frac{a^2X^2}{(a^2+\rho)^2} + \frac{b^2Y^2}{(b^2+\rho)^2} + \frac{c^2Z^2}{(c^2+\rho)^2}.$$

We have thus the differential equation of the parallel surfaces. It may be remarked, that the integral equation (involving k as the constant of integration), is found by the elimination of x, y, z, ρ from the foregoing equations

$$x = \frac{a^2X}{a^2+\rho},\quad y = \frac{b^2Y}{b^2+\rho},\quad z = \frac{c^2Z}{c^2+\rho},$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1,\quad k^2 = \rho^2\left(\frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4}\right),$$

or, what is the same thing, by the elimination of ρ from the equations

$$\frac{k^2}{\rho^2} = \frac{X^2}{(a^2+\rho)^2} + \frac{Y^2}{(b^2+\rho)^2} + \frac{Z^2}{(c^2+\rho)^2},$$

$$1 = \frac{a^2X^2}{(a^2+\rho)^2} + \frac{b^2Y^2}{(b^2+\rho)^2} + \frac{c^2Z^2}{(c^2+\rho)^2};$$

these may be replaced by

$$\frac{X^2}{a^2+\rho} + \frac{Y^2}{b^2+\rho} + \frac{Z^2}{c^2+\rho} - \frac{k^2}{\rho} - 1 = 0,$$

$$\frac{X^2}{(a^2+\rho)^2} + \frac{Y^2}{(b^2+\rho)^2} + \frac{Z^2}{(c^2+\rho)^2} - \frac{k^2}{\rho^2} = 0,$$

or, since here the second equation is the derived equation of the first in regard to the parameter ρ, the parallel surface is the envelope of the quadric surface

$$\frac{X^2}{a^2+\rho} + \frac{Y^2}{b^2+\rho} + \frac{Z^2}{c^2+\rho} - \frac{k^2}{\rho} = 0,$$

where ρ is the variable parameter. Or analytically, we find the equation by equating to zero the discriminant in regard to ρ, of the quartic function

$$\rho\,(a^2+\rho)(b^2+\rho)(c^2+\rho)\left(1 + \frac{k^2}{\rho} - \frac{X^2}{a^2+\rho} - \frac{Y^2}{b^2+\rho} - \frac{Z^2}{c^2+\rho}\right).$$

7. *Explain wherein consists the peculiarity of the following problem, and solve it by geometrical considerations:—*

Determine the least circle inclosing three given points.

The peculiarity of the problem is that the variable parameters upon which the circle depends, (say α, β the coordinates of the centre and k the radius), are not subject to any equations, but only to the inequalities

$$k^2 > (\alpha - \alpha_1)^2 + (\beta - \beta_1)^2,$$
$$k^2 > (\alpha - \alpha_2)^2 + (\beta - \beta_2)^2,$$
$$k^2 > (\alpha - \alpha_3)^2 + (\beta - \beta_3)^2,$$

(α_1, β_1; α_2, β_2; α_3, β_3, the coordinates of the given points, and the sign > including =). The problem therefore cannot be solved by the ordinary analytical method, but it is easily solved geometrically as follows: Let A, B, C be the three points; consider all the circles inclosing the three points, viz. O a circle not passing through any of them; A a circle through the point A, B a circle through the point B, AB a circle through the points A and B, &c. Then for any circle O, if the centre be fixed and the radius gradually diminish, the circle will at last pass through one of the points ABC; that is, every circle O is greater than some circle A, B, or C; and the circle O is therefore not a minimum. Taking next a circle A, we may imagine the centre to move from its original position in a straight line towards the point A, the circle thus gradually diminishing until it passes through one of the points B or C; that is, every circle A is greater than some circle AB or AC, and therefore no circle A is a minimum; and in like manner no circle B or C is a minimum. There remain the circles AB, AC, BC; if the triangle ABC is acute-angled, then in each series, the least circle is the circle ABC circumscribed about the triangle; and this is then the minimum circle inclosing the three points. But if the triangle is obtuse-angled, say at C, then the least circle CA or CB is the circle ABC circumscribed about the triangle; but this is not the least circle AB, viz. the circle AB, being diminished to ABC, may be further diminished until it becomes the circle on the diameter AB; but below this it cannot be diminished; and consequently the minimum circle inclosing the three points is in this case the circle on the diameter AB.

8. *A particle describes an ellipse under the simultaneous action of given central forces, each varying as (distance)$^{-2}$, at the two foci respectively: find the differential relation between the time and the eccentric anomaly.*

Taking the equation of the ellipse to be $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, and the absolute forces at the two foci $(ae, 0)$, $(-ae, 0)$ to be μ, μ' respectively, the differential equations of motion will be

$$\frac{d^2x}{dt^2} = -\mu \frac{x - ae}{(a - ex)^3} - \mu' \frac{(x + ae)}{(a + ex)^3},$$

$$\frac{d^2y}{dt^2} = -\mu \frac{y}{(a - ex)^3} - \mu' \frac{y}{(a + ex)^3}.$$

But if u be the eccentric anomaly, then

$$x = a \cos u, \quad y = b \sin u, \quad = a\sqrt{(1-e^2)} \sin u,$$

and the equations become

$$-\sin u \frac{d^2u}{dt^2} - \cos u \left(\frac{du}{dt}\right)^2 = -\frac{\mu}{a^3} \frac{\cos u - e}{(1 - e\cos u)^3} - \frac{\mu'}{a^3} \frac{\cos u + e}{(1 + e\cos u)^3}$$

$$+\cos u \frac{d^2u}{dt^2} - \sin u \left(\frac{du}{dt}\right)^2 = -\frac{\mu}{a^3} \frac{\sin u}{(1 - e\cos u)^3} - \frac{\mu'}{a^3} \frac{\sin u}{(1 + e\cos u)^3},$$

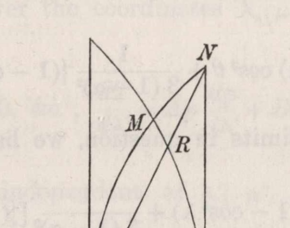
and multiplying by $-\cos u$, $-\sin u$ respectively, and adding, we have

$$\left(\frac{du}{dt}\right)^2 = \frac{\mu}{a^3} \frac{1}{(1 - e\cos u)^2} + \frac{\mu'}{a^3} \frac{1}{(1 + e\cos u)^2},$$

which is the required differential relation.

9. *Show that the attraction of an indefinitely thin double-convex lens on a point at the centre of one of its faces is equal to that of the infinite plate included between the tangent plane at the point and the parallel tangent plane of the other face of the lens.*

The figure represents the upper half only of the lens, but in speaking of any portion thereof, such as PRQ, we include the symmetrically situate portion of the under-half of the lens.



Let α, $= PQ$, be the thickness of the lens, $\angle NPQ = \lambda$, which angle is ultimately $= \frac{\pi}{2}$. Then it is at once seen that the attraction of the cone NPQ is $= 2\pi\alpha(1 - \cos\lambda)$: and from this it follows that the attraction of the infinite plate is $= 2\pi\alpha$. The attraction of the whole infinite plate except the cone NPQ is $= 2\pi\alpha\cos\lambda$, which is indefinitely small in regard to $2\pi\alpha$; and, *a fortiori*, the attraction of the portion MPR of the lens is indefinitely small in regard to $2\pi\alpha$. We have then only to show that the attraction of the solid NRQ is indefinitely small in regard to $2\pi\alpha$; for, this being so, the attraction of the lens may be taken to be equal to that of the cone NPQ, and will therefore ultimately be $= 2\pi\alpha$, the attraction of the infinite plate.

Let the position of an element of the solid in question be determined by r its distance from P, θ the inclination of r to the axis PQ, and ϕ the azimuth in regard to any fixed plane through the axis; then $dm = r^2 \sin\theta\, dr\, d\theta\, d\phi$, and the attraction in the direction PQ is $= \int \sin\theta \cos\theta\, dr\, d\theta\, d\phi, = 2\pi \int \left(\frac{\alpha}{\cos\theta} - r\right) \sin\theta\cos\theta\, d\theta$, the integral in regard to ϕ having been taken from $\phi = 0$ to $\phi = 2\pi$, and that in regard to r from $r = r$ (value at the face MQ of the lens) to $r = \frac{\alpha}{\cos\theta}$ (value at the tangent plane QN). Taking the radius of the surface QM of the lens to be $=1$, we have

$$(1 - \alpha + r\cos\theta)^2 + r^2 \sin^2\theta = 1,$$

that is,

$$r^2 + 2r\cos\theta\,(1-\alpha) = 2\alpha - \alpha^2,$$

$$\{r + (1-\alpha)\cos\theta\}^2 = (1-\alpha)^2\cos^2\theta + 2\alpha - \alpha^2,$$

or

$$r = -(1-\alpha)\cos\theta + \sqrt{\{(1-\alpha)^2\cos^2\theta + 2\alpha - \alpha^2\}},$$

which is the value of r to be substituted in the formula

$$\frac{1}{2\pi} A = \int (\alpha \sin\theta - r\sin\theta\cos\theta)\, d\theta,$$

and the integral is to be taken from $\theta = 0$ to $\theta = \lambda$; viz. this is

$$\int [\alpha\sin\theta + (1-\alpha)\sin\theta\cos^2\theta - \sin\theta\cos\theta\sqrt{\{(1-\alpha)^2\cos^2\theta + 2\alpha - \alpha^2\}}]\, d\theta,$$

$$= -\alpha\cos\theta - \tfrac{1}{3}(1-\alpha)\cos^3\theta + \frac{1}{3(1-\alpha)^2}\{(1-\alpha)^2\cos^2\theta + 2\alpha - \alpha^2\}^{\frac{3}{2}};$$

so that taking this between the limits in question, we have

$$\frac{1}{2\pi} A = \alpha(1-\cos\lambda) + \tfrac{1}{3}(1-\alpha)(1-\cos^3\lambda) + \frac{1}{3(1-\alpha)^2}[\{(1-\alpha)^2\cos^2\lambda + 2\alpha - \alpha^2\}^{\frac{3}{2}} - 1]$$

or writing for greater convenience $\lambda = \frac{\pi}{2} - \mu$, $(\mu = \angle PNQ)$, this is

$$\frac{1}{2\pi} A = \alpha(1-\sin\mu) + \tfrac{1}{3}(1-\alpha)(1-\sin^3\mu) + \frac{1}{3(1-\alpha)^2}[\{(1-\alpha)^2\sin^2\mu + 2\alpha - \alpha^2\}^{\frac{3}{2}} - 1]$$

$$= \alpha + \tfrac{1}{3}\left\{1 - \alpha - \frac{1}{(1-\alpha)^2}\right\} - \alpha\sin\mu$$

$$+ \frac{1}{3(1-\alpha)^2}[\{(1-\alpha)^2\sin^2\mu + 2\alpha - \alpha^2\}^{\frac{3}{2}} - (1-\alpha)^3\sin^3\mu]$$

$$= \frac{1}{3(1-\alpha)^2}(-3\alpha^2 + 2\alpha^3) - \alpha\sin\mu$$

$$+ \frac{1}{3(1-\alpha)^2}[\{(1-\alpha)^2\sin^2\mu + 2\alpha - \alpha^2\}^{\frac{3}{2}} - (1-\alpha)^3\sin^3\mu];$$

$\sin\mu$ is here an indefinitely small quantity of the order $\alpha^{\frac{1}{2}}$, all the terms are therefore at least of the order $\alpha^{\frac{3}{2}}$, and are to be neglected in comparison with α; or neglecting such terms we have $A=0$ (that is, the attraction of the solid NRQ is indefinitely small in regard to α); and the theorem is thus proved.

10. *Indicate in what manner the Lagrangian equations of motion*

$$\frac{d}{dt}\frac{dT}{d\xi'}-\frac{dT}{d\xi}=\frac{dV}{d\xi},\ \&c.$$

lead to the equations

$$A\frac{dp}{dt}+(C-B)\,qr=0,\ \&c.$$

for the motion of a solid body about a fixed point.

The expression of the *vis viva* function T is

$$T=\tfrac{1}{2}(Ap^2+Bq^2+Cr^2),$$

but this expression will not by itself lead to the equations of motion; we require to know also the expressions of p, q, r in terms of certain coordinates λ, μ, ν, which determine the position of the body in regard to axes fixed in space, and of the differential coefficients λ', μ', ν' of these coordinates in regard to the time; each of the quantities p, q, r will be a linear function of λ', μ', ν' ($p=a\lambda'+b\mu'+c\nu'$, &c.), containing in any manner whatever the coordinates λ, μ, ν. This being so, the equations of motion will be

$$\frac{d}{dt}\frac{dT}{d\lambda'}-\frac{dT}{d\lambda}=0,\ \&c.;\ \frac{dT}{d\lambda'}=Ap\frac{dp}{d\lambda'}+Bq\frac{dq}{d\lambda'}+Cr\frac{dr}{d\lambda'},$$

where $\frac{dp}{d\lambda'}$, $\frac{dq}{d\lambda'}$, $\frac{dr}{d\lambda'}$ are each independent of λ', μ', ν'; hence, in the equation, the only terms containing the differential coefficients of p, q, r, are the terms

$$\frac{dp}{d\lambda'}\cdot A\frac{dp}{dt}+\frac{dq}{d\lambda'}\cdot B\frac{dq}{dt}+\frac{dr}{d\lambda'}\cdot C\frac{dr}{dt}$$

of $\frac{d}{dt}\cdot\frac{dT}{d\lambda'}$; and hence, assuming that the equations of motion are the known equations $A\frac{dp}{dt}+(C-B)\,qr=0$, it appears that the equation $\frac{d}{dt}\cdot\frac{dT}{d\lambda'}-\frac{dT}{d\lambda}=0$ will assume the form

$$\frac{dp}{d\lambda'}\left\{A\frac{dp}{dt}+(C-B)\,qr\right\}+\frac{dq}{d\lambda'}\left\{B\frac{dq}{dt}+(A-C)\,rp\right\}+\frac{dr}{d\lambda'}\left\{C\frac{dr}{dt}+(B-A)\,pq\right\}=0;$$

there are of course two other equations only differing from this in that in place of λ', they contain μ' and ν' respectively; and since p, q, r regarded as functions of λ', μ', ν' are independent functions, the determinant formed with the differential coefficients

$\dfrac{dp}{d\lambda'}, \dfrac{dq}{d\lambda'}, \dfrac{dr}{d\lambda'}$, &c. is not $=0$; and the three equations are therefore equivalent (as they should be) to the equations

$$A\frac{dp}{dt}+(C-B)\,qr=0,\ \&c.$$

What precedes is a complete answer to the question, but in regard to the actual expressions of p, q, r, it may be remarked, that these quantities may be expressed very symmetrically in terms of the quantities

$$\lambda,\ \mu,\ \nu=\tan\tfrac{1}{2}\theta\cos f,\quad \tan\tfrac{1}{2}\theta\cos g,\quad \tan\tfrac{1}{2}\theta\cos h,$$

which determine the positions of the principal axes in regard to the axes fixed in space, by means of the angles of position ($\cos f$, $\cos g$, $\cos h$) of the resultant axis, and the rotation θ about this axis; viz. writing $\kappa=1+\lambda^2+\mu^2+\nu^2$, we then have

$$\kappa p=2\,(\quad\lambda'+\nu\mu'-\mu\nu'),$$
$$\kappa q=2\,(-\nu\lambda'+\mu'+\lambda\nu'),$$
$$\kappa r=2\,(\quad\mu\lambda'-\lambda\mu'+\nu'),$$

and the above result may be verified *a posteriori* without any difficulty. See *Camb. Math. Jour.*, vol. III. (1843), [6], p. 224, [*Coll. Math. Papers*, vol. I. p. 33].

11. *Find in the Hamiltonian form,*

$$\frac{d\eta}{dt}=\frac{dH}{d\varpi},\quad \frac{d\varpi}{dt}=-\frac{dH}{d\eta},\ \&c.$$

the equations for the motion of a particle acted on by a central force.

Taking as coordinates r the radius vector, v the longitude, y the latitude, the equation of the *vis viva* function is

$$T=\tfrac{1}{2}\{r'^2+r^2(\cos^2 y\,.\,v'^2+y'^2)\},$$

hence

$$\frac{dT}{dr'}=\qquad r'\qquad=\mathrm{r}\ \text{suppose},$$

$$\frac{dT}{dv'}=r^2\cos^2 y\,.\,v'=\mathrm{v}\qquad ,, \quad ,$$

$$\frac{dT}{dy'}=\qquad r^2y'\qquad=\mathrm{y}\qquad ,, \quad ,$$

and the expression of T in terms of r, v, y, and of the new coordinates r, v, y is

$$T=\tfrac{1}{2}\left(\mathrm{r}^2+\frac{\mathrm{v}^2}{r^2\cos^2 y}+\frac{\mathrm{y}^2}{r^2}\right);$$

whence writing

$$H=\tfrac{1}{2}\left(\mathrm{r}^2+\frac{\mathrm{v}^2}{r^2\cos^2 y}+\frac{\mathrm{y}^2}{r^2}\right)-V,$$

the equations are

$$\frac{dH}{dr}=\frac{d\text{r}}{dt},\quad \frac{dH}{d\text{v}}=\frac{dv}{dt},\quad \frac{dH}{d\text{y}}=\frac{dy}{dt},$$

$$\frac{dH}{d\text{r}}=-\frac{d\text{r}}{dt},\quad \frac{dH}{dv}=-\frac{d\text{v}}{dt},\quad \frac{dH}{dy}=-\frac{d\text{y}}{dt}.$$

We have

$$\frac{dH}{d\text{r}}=\text{r},\quad \frac{dH}{d\text{v}}=\frac{\text{v}}{r^2\cos^2 y},\quad \frac{dH}{d\text{y}}=\frac{\text{y}}{r^2},$$

$$\frac{dH}{dr}=-\frac{1}{r^3}\left(\frac{\text{v}^2}{\cos^2 y}+\text{y}^2\right)-\frac{dV}{dr},\quad \frac{dH}{dv}=0,\quad \frac{dH}{dy}=-\frac{\text{v}^2\sin y}{r^2\cos^3 y};$$

and, substituting these values, the equations of motion present themselves as six equations of the first order between r, v, y, r, v, y, and t in the form

$$dt=\frac{dr}{\text{r}}=\frac{dv}{\dfrac{\text{v}}{r^2\cos^2 y}}=\frac{dy}{\dfrac{\text{y}}{r^2}}=\frac{d\text{r}}{-\dfrac{1}{r^3}\left(\dfrac{\text{v}^2}{\cos^2 y}+\text{y}^2\right)-\dfrac{dV}{dr}}=\frac{d\text{v}}{0}=\frac{d\text{y}}{-\dfrac{\text{v}^2\sin^2 y}{r^2\cos^3 y}}.$$

12. *An unclosed polygon of* $(m+1)$ *vertices is constructed as follows: viz. the abscissæ of the several vertices are* $0, 1, 2 \ldots m$, *and, corresponding to the abscissa* k, *the ordinate is equal to the chance of* $(m+k)$ *heads in* $2m$ *tosses of a coin; and* m *then continually increases up to any very large value: what information in regard to the successive polygons, and to the areas of any portions thereof, is afforded by the general results of the Theory of Probabilities?*

It is somewhat more convenient to take account also of the abscissæ $-1, -2, \ldots, -m$, thereby obtaining a polygon of $2m+1$ vertices, symmetrical in regard to the axis of y. In such a polygon, the sum of the $2m+1$ ordinates is $=1$; the central ordinate is the largest, and the ordinates continually diminish as k increases: moreover for any large value of m the area of the whole polygon is very nearly, and may be regarded as being, $=1$; and the area between the ordinates corresponding to the abscissæ $+k$, $-k$ as being equal to the probability of a number of heads between $m+k$, $m-k$, in the $2m$ tosses of the coin. A general result of the Theory of Probabilities is that in a great number of trials the several events tend to happen in the proportion of their respective probabilities; viz. in the case of the $2m$ tosses there is a tendency to an equal number of heads and tails. But observe that this does not mean that the probability of m heads and m tails increases with the number $2m$ of the trials; or even that, α being any given number, the probability of a number of heads between $m+\alpha$ and $m-\alpha$ increases with the number $2m$ of trials; on the contrary, it diminishes; what it does mean is that taking the limit of deviation to vary with m, say a number of heads between $m+\alpha m$, $m-\alpha m$, the probability of such a number increases with m; viz. that taking α a fraction however small, m can be taken so large that the probability of a number of heads between $m+\alpha m$, $m-\alpha m$ in the $2m$ trials, shall be as nearly as we please $=1$.

The conclusion in regard to the areas of the polygons is that, taking k any given value whatever, however large, the ratio (m being of course $>k$) which the area between the ordinates to the abscissæ $m+k$, $m-k$ bears to the area of the whole polygon (or to unity) continually decreases as $2m$ increases, and ultimately vanishes; but contrarywise, taking α any given fraction whatever, however small, the ratio which the area between the ordinates to the abscissæ $m+\alpha m$, $m-\alpha m$ bears to the area of the whole polygon (or to unity) continually increases as $2m$ increases, and ultimately becomes $=1$.

13. *Show that for the quadric cones which pass through six given points the locus of the vertices is a quartic surface having upon it twenty-five right lines; and, thence or otherwise, that for the quadric cones passing through seven given points the locus of the vertices is a sextic curve.*

Suppose $U=0$, $V=0$, $W=0$, $S=0$ are any particular four quadric surfaces passing through the six points, say

$$(U=(a, \ldots)(x, y, z, w)^2, \quad V=(b, \ldots)(x, y, z, w)^2, \text{ \&c.});$$

then the equation of the general quadric surface through the six points will be

$$\alpha U+\beta V+\gamma W+\delta S=0,$$

and this surface will be a cone, having (x, y, z, w) for the coordinates of its vertex, if only we have simultaneously

$$\alpha\frac{dU}{dx}+\beta\frac{dV}{dx}+\gamma\frac{dW}{dx}+\delta\frac{dS}{dx}=0,$$

$$\alpha\frac{dU}{dy}+\text{\&c.} \qquad\qquad =0,$$

$$\alpha\frac{dU}{dz}+\text{\&c.} \qquad\qquad =0,$$

$$\alpha\frac{dU}{dw}+\text{\&c.} \qquad\qquad =0.$$

Eliminating $(\alpha, \beta, \gamma, \delta)$ we have an equation $\nabla=0$, where ∇ is the Jacobian or functional determinant $\dfrac{d(U, V, W, S)}{d(x, y, z, w)}$ formed with the differential coefficients of the four functions (U, V, W, S): the locus of the vertex is thus a quartic surface.

Calling the six points 1, 2, 3, 4, 5, 6, then taking as vertex any point in the line 12, the lines from such point to the points 1 and 2 coincide with the line 12, and we can through this line and the lines to the remaining points 3, 4, 5, 6 describe a quadric cone; the quartic surface therefore passes through the line 12; and similarly it passes through each of the fifteen lines 12, 13, ..., 56.

Again, taking the vertex anywhere in the line of intersection of the planes 123 and 456, we have an improper quadric cone, viz. the plane-pair formed by these two planes; the line in question is therefore a line of the quartic surface; and similarly the quartic surface contains each of the ten lines 123.456, 124.356, ..., 156.234. We have thus in all 25 lines on the quartic surface.

In the case of seven points 1, 2, 3, 4, 5, 6, 7, the locus is the curve of intersection of the quartic surfaces which correspond to the points 1, 2, 3, 4, 5, 6 and the points 1, 2, 3, 4, 5, 7 respectively: these have in common the ten lines 12, 13, 14, 15, 23, 24, 25, 34, 35, 45 (which it is easy to see do not form part of the required locus), and they have therefore, as a residual intersection, a curve of the order $16-10$, $=6$, or sextic curve, which is the locus of the vertices of the cones which pass through the seven given points.

14. *Show that the envelope of a variable circle having its centre on a given conic and cutting at right angles a given circle is a bicircular quartic; which, when the given conic and circle have double contact, becomes a pair of circles; and, by means of the last-mentioned particular case of the theorem, connect together the porisms arising out of the two problems:*

(1) *given two conics, to find a polygon of n sides inscribed in the one and circumscribed about the other;*

(2) *given two circles, to find a closed series of n circles each touching the two given circles and the two adjacent circles of the series.*

The equation of the given circle is taken to be

$$(x-\alpha)^2+(y-\beta)^2=\gamma^2,$$

and that of the conic $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$. This being so, we have $a\cos\theta$, $b\sin\theta$ as the coordinates of a point on the conic, which point may be taken to be the centre of the variable circle, and introducing the condition that the two circles cut at right angles, the equation of the variable circle is

$$(x-a\cos\theta)^2+(y-b\cos\theta)^2=(\alpha-a\cos\theta)^2+(\beta-b\sin\theta)^2-\gamma^2,$$

that is,

$$x^2+y^2-\alpha^2-\beta^2+\gamma^2-2ax\cos\theta-2by\sin\theta=0,$$

where θ is the variable parameter; and the equation of the envelope therefore is

$$(x^2+y^2-\alpha^2-\beta^2+\gamma^2)^2-4a^2x^2-4b^2y^2=0,$$

which is a quartic curve; and writing herein $\dfrac{x}{z}$, $\dfrac{y}{z}$ in place of x, y the equation would be of the second order in regard to x^2+y^2, z, and it thus appears that the curve has double points at each of the points $x^2+y^2=0$, $z=0$, viz. that the envelope is a bicircular quartic.

If the fixed circle touches the conic, then by a consideration of the figure it at once appears that the point of contact is a double point on the curve; and so if there is a double contact, then each of the points of contact is a double point on the curve. But in this case the curve is a bicircular quartic with *four* double points; viz. it is a pair of circles.

The porism in regard to the two conics is, that in general it is not possible to find any polygon of n sides satisfying the conditions; but that the conics may be such that there exists an infinity of polygons; viz. any point whatever of the one conic may then be taken as a vertex of the polygon, and then constructing the figure, the $(n+1)^{\text{th}}$ vertex will coincide with the first vertex, and there will be a polygon of n sides.

Now imagine that the conic touched by the sides is a circle having double contact with the other conic. Describe any one of the polygons, and with each vertex as centre describe the orthotomic circle, which will, it is clear, be a circle passing through the points of contact with the fixed circle of the sides through the vertex. We have thus a closed series of n circles, each touching the two adjacent circles of the series. And by considering any other polygon, we have a like series of n circles: and by what precedes the envelope of all the circles of the several series is a pair of circles; that is, the circles of every series touch these two circles. We have consequently two circles, such that there exists an infinity of closed series of n circles, each circle touching the two fixed circles, and also the two adjacent circles of the series; which is the porism arising out of the second problem.