530.

SOLUTION OF A SENATE-HOUSE PROBLEM.

[From the *Oxford, Cambridge and Dublin Messenger of Mathematics,* vol. v. (1869), pp. 24—27.]

THE Problem, proposed January 7, 1869, is "If θ_1 and θ_2 are two values of θ which satisfy the equation

$$1+\frac{\cos\theta\cos\phi}{\cos^2\alpha}+\frac{\sin\theta\sin\phi}{\sin^2\alpha}=0,$$

show that θ_1 and θ_2, if substituted for θ and ϕ in this equation, will satisfy it."

That is, writing

$$\frac{\cos\theta_1\cos\phi}{a^2}+\frac{\sin\theta_1\sin\phi}{b^2}+1=0,$$

$$\frac{\cos\theta_2\cos\phi}{a^2}+\frac{\sin\theta_2\sin\phi}{b^2}+1=0,$$

where $a^2+b^2=1$, it is to be shown that

$$\frac{\cos\theta_1\cos\theta_2}{a^2}+\frac{\sin\theta_1\sin\theta_2}{b^2}+1=0.$$

From the given equations, we have

$$\frac{\cos\phi}{a^2}:\frac{\sin\phi}{b^2}:1=\sin\theta_1-\sin\theta_2:\cos\theta_2-\cos\theta_1:\sin(\theta_2-\theta_1),$$

which are

$$=\cos\tfrac{1}{2}(\theta_1+\theta_2):\sin\tfrac{1}{2}(\theta_1+\theta_2):-\cos\tfrac{1}{2}(\theta_1-\theta_2).$$

Whence eliminating ϕ, we have

$$a^4 \cos^2 \tfrac{1}{2}(\theta_1+\theta_2) + b^4 \sin^2 \tfrac{1}{2}(\theta_1+\theta_2) - \cos^2 \tfrac{1}{2}(\theta_1-\theta_2) = 0,$$

that is,

$$a^4\{1+\cos(\theta_1+\theta_2)\} + b^4\{1-\cos(\theta_1+\theta_2)\} - \{1+\cos(\theta_1-\theta_2)\} = 0,$$

or, what is the same thing,

$$a^4+b^4-1+(a^4-b^4-1)\cos\theta_1\cos\theta_2+(-a^4+b^4-1)\sin\theta_1\sin\theta_2=0.$$

But from the equation $a^2+b^2=1$, we have

$$a^4+b^4-1=-2a^2b^2,$$

$$a^4-b^4-1=-2b^2,$$

$$-a^4+b^4-1=-2a^2,$$

and the equation is thus

$$\frac{\cos\theta_1\cos\theta_2}{a^2}+\frac{\sin\theta_1\sin\theta_2}{b^2}+1=0,$$

which is the required equation.

Stopping at the result obtained previous to the use of the relation $a^2+b^2=1$, but making some obvious substitutions in the formulæ, the theorem may be presented in a more general form as follows; viz.:

If we have

$$\frac{xx_1}{a^2}+\frac{yy_1}{b^2}-1=0,$$

$$\frac{xx_2}{a^2}+\frac{yy_2}{b^2}-1=0,$$

where

$$\frac{x^2}{\alpha^2}+\frac{y^2}{\beta^2}-1=0,\quad \frac{x_1^{\,2}}{\alpha^2}+\frac{y_1^{\,2}}{\beta^2}-1=0,\quad \frac{x_2^{\,2}}{\alpha^2}+\frac{y_2^{\,2}}{\beta^2}-1=0,$$

then

$$\left(\frac{a^4}{\alpha^4}+\frac{b^4}{\beta^4}-1\right)+\left(\frac{a^4}{\alpha^4}-\frac{b^4}{\beta^4}-1\right)\frac{x_1x_2}{\alpha^2}+\left(-\frac{a^4}{\alpha^4}+\frac{b^4}{\beta^4}-1\right)\frac{yy_2}{\beta^2}=0,$$

a relation, the geometrical signification of which is: If (x, y) be a point on the conic $\frac{x^2}{\alpha^2}+\frac{y^2}{\beta^2}-1=0$, and if the polar hereof in regard to the conic $\frac{x^2}{a^2}+\frac{y^2}{b^2}-1=0$ meet the first-mentioned conic in the points (x_1, y_1) and (x_2, y_2), then these points are harmonics in regard to the conic

$$\left(\frac{a^4}{\alpha^4}+\frac{b^4}{\beta^4}-1\right)+\left(\frac{a^4}{\alpha^4}-\frac{b^4}{\beta^4}-1\right)\frac{x^2}{\alpha^2}+\left(-\frac{a^4}{\alpha^4}+\frac{b^4}{\beta^4}-1\right)\frac{y^2}{\beta^2}=0;$$

and since the theorem is projective, it is seen that the first two conics may be any conics whatever, the third conic being a conic having with the other two a common system of conjugate points.

If to fix the ideas we write $\alpha=\beta=c$; then the theorem is, if the polar of a point on the circle $x^2+y^2=c^2$ in regard to the conic $\frac{x^2}{a^2}+\frac{y^2}{b^2}-1=0$, meet the circle in two points, these are harmonics in regard to the conic

$$(a^4+b^4-c^4)+(a^4-b^4-c^4)\frac{x^2}{c^2}+(-a^4+b^4-c^4)\frac{y^2}{c^2}=0.$$

This last conic will be similar to the conic $\frac{x^2}{a^2}+\frac{y^2}{b^2}-1=0$, if $c^4=(a^2+b^2)^2$; viz. if $c^2=a^2+b^2$, then the conic is $\frac{x^2}{a^2}+\frac{y^2}{b^2}+1=0$; but if $c^2=-a^2-b^2$, the conic is not only similar to, but is the conic $\frac{x^2}{a^2}+\frac{y^2}{b^2}-1=0$. Considering the given conic $\frac{x^2}{a^2}+\frac{y^2}{b^3}-1=0$ to be an ellipse, the first case $(c^2=a^2+b^2)$ gives the two points (x_1, y_1), (x_2, y_2) harmonics in regard to the imaginary conic $\frac{x^2}{a^2}+\frac{y^2}{b^2}+1=0$, but this is at once transformed into a real theorem, for we have $(-x_1, -y_1)$ and (x_2, y_2), or, what is the same thing, (x_1, y_1), $(-x_2, -y_2)$ harmonics in regard to $\frac{x^2}{a^2}+\frac{y^2}{b^2}-1=0$; and the theorem is: "Given the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}-1=0$, and the circle $x^2+y^2=a^2+b^2$ (which is the locus of the intersection of a pair of orthotomic tangents of the ellipse), if the polar in regard to the ellipse of a point on the circle meet the circle in the points Q, R, and if the *opposite* points to these be Q_1, R_1, then (Q, R_1), or what is the same thing (Q_1, R) are harmonics in regard to the ellipse."

The second case $(c^2=-a^2-b^2)$ gives a real theorem if b^2 be negative; viz. writing $-b^2$ for b^2 we have the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, $b^2=a^2+c^2$, an obtuse-angled hyperbola; the circle $x^2+y^2=a^2-b^2$, which is the locus of the intersection of a pair of orthotomic tangents of the hyperbola, is consequently imaginary; but the concentric orthotomic circle hereof, viz. the circle of the theorem, $x^2+y^2=b^2-a^2$, is a real circle; and the theorem is: "Given the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}-1=0\,(b^2>a^2)$ and the circle $x^2+y^2=b^2-a^2$ (the concentric orthotomic circle of the imaginary circle which is the locus of the intersection of a pair of orthotomic tangents of the hyperbola), if the polar in regard to the hyperbola of a point on the circle meet the circle in two points Q, R, then these are harmonics in regard to the hyperbola."

Of course, if reality be disregarded, the two theorems may each of them be stated of a conic generally; and observe, that in the first theorem the circle is the locus of the intersection of orthotomic tangents, and we have the opposites of the points Q, R; in the second theorem the circle is the concentric orthotomic circle of the circle, which is the locus of the intersection of orthotomic tangents, but we have the points Q, R themselves.